ROBUST SWING-UP CONTROL FOR A ROTATIONAL DOUBLE PENDULUM

Scott A. Bortoff[1]

Department of Electrical and Computer Engineering, University of Toronto
10 King's College Place, Toronto, Ontario M4E 2M8 Canada
bortoff@control.toronto.edu

Abstract. A variant of the double-inverted pendulum is introduced, and a robust feedback control is presented that transfers the state between the pendant and the inverted equilibrium manifolds. The control is computed using partial feedback linearization in combination with a "limit-cycle inducing" control. It is shown to be robust with respect to parameter uncertainty. Experimental results support the claims of robustness, and computational simplicity.

Keywords. Nonlinear control systems, Nonlinear analysis, Robust control, Limit Cycles.

1. INTRODUCTION

For several decades, inverted pendulum systems have served as excellent test beds for control theory. Because they exhibit nonlinear, unstable, non-minimum phase dynamics, and because the full-state is not often fully measured, control objectives are always challenging. The literature abounds with results, which we briefly sample. An early example of stabilization and state-transfer control with experimental verification is (Mori *et al.* 1976). Furuta *et al.* (1980) present a double-inverted pendulum with disturbance rejection capability. In (Furuta *et al.* 1984), the same group presents a triple inverted pendulum, with two inputs. The *acrobot* (Bortoff 1994) and the *pendubot* (Spong and Block 1995, Bedrossian 1993) are similar under-actuated mechanical systems, and have similar control objectives: state transfer among equilibria, and stabilization.

[1] This research supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant OGP0138423.

Despite extensive study of such under-actuated systems, the design of robust controls to transfer the state among equilibrium configurations remains a challenge to this day. Many existing results are based on ad hoc feedforward control, which is often not robust with respect to parameter uncertainty, noise, etc. For example, Mori *et al.* (1976) compute an open-loop optimal control using the maximum principle. Indeed, achieving robustness is a primary motivation for continued study. For example, Furuta *et al.* (1992) refine their original open-loop technique, adding a bang-bang state feedback to improve robustness during state transfer between equilibrium configurations. The bang-bang approach also has been applied to a double pendulum (Yamakita *et al.* 1993), where a learning aspect is introduced. More recently, Lyapunov-based approaches have been used by Wiklund *et al.* (1993) to design a swing-up controller for the single, rotating inverted pendulum, and by Yamakita *et al.* (1995) for a double rotating inverted pendulum. Researchers interested in neural networks, fuzzy control, and learning control have also made contributions to the control of under-actuated systems, e.g., (Kawaji *et al.* 1994), although these approaches are beyond our

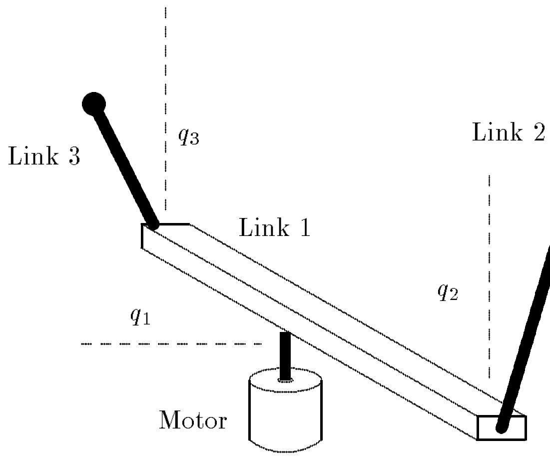

Fig. 1. Diagram of the double inverted pendulum.

scope.

Our results are similar in spirit to (Spong 1994), who presents a differential-geometric approach to state transfer, decomposing the dynamics into simplified but coupled subsystems. We present a robust state transfer control that is used to swing-up a novel rotational double pendulum system. A key step is the application of a partial feedback linearizing control which decouples the six-dimensional dynamics into three two-dimensional subsystems. Although two of these are nonlinear, they are considerably simplified when compared with the original dynamics. Our main contribution is to show that the physically-motived swing-up control induces a stable limit cycle that causes the unactuated links to swing from pendant to inverted positions. The control is applied in two stages, each of which swings-up one link. The overall controller switches between the individual swing-up controls, depending upon the state. The overall strategy is robust with respect to parameter uncertainty, and is inexpensive in terms of control effort and computational complexity. Experimental results from our rotational double pendulum affirm our assertions.

2. CONTROLLER DERIVATION

Our version of the double inverted pendulum, shown in Figures 1 and 6, is a modification of the TIT pendulum introduced in (Furuta *et al.* 1992). Two unactuated pendulums, denoted *links 2* and *3*, are attached to the two opposite ends of a first, actuated link, denoted *link 1*. Thus, the mechanism is not an open chain of links. Defining q_1, q_2, and q_3 as the angles of links 1, 2 and 3, respectively, and $\boldsymbol{q} = [q_1 \quad q_2 \quad q_3]^T$, the Lagrangian equations of motion are

Table 1. Parameters for the experimental double rotating pendulum.

Param.	**Value**	**Param.**	**Value**
K_1	0.253	K_6	1.277
K_2	0.037	K_7	0.2953
K_3	0.0030	K_8	0.202
K_4	0.045	K_9	0.0016
K_5	0.0104	K_{10}	0.0010
		K_{11}	1.5

Table 2. Equilibrium manifolds.

Angles	**Denoted**
$q_2 = q_3 = \dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$	**D-D**
$q_2 = \dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0,\ q_3 = \pm\pi$	**D-U**
$q_2 = \pm\pi,\ q_3 = \dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$	**U-D**
$q_2 = \pm\pi,\ q_3 = \pm\pi,\ \dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$	**U-U**

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & 0 \\ m_{13} & 0 & m_{33} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} K_{11}u \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

where $\boldsymbol{q} = [q_1 \quad q_2 \quad q_3]^T$, u is the applied control,

$$\begin{aligned}
m_{11}(\boldsymbol{q}) &= K_1 + K_2 \sin^2(q_2) + K_3 \sin^2(q_3) \\
m_{12}(\boldsymbol{q}) &= K_4 \cos(q_2) \qquad m_{22}(\boldsymbol{q}) = K_2 \\
m_{13}(\boldsymbol{q}) &= K_5 \cos(q_3) \qquad m_{33}(\boldsymbol{q}) = K_3 \\
c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -K_4 \sin(q_2)\dot{q}_2^2 - K_5 \sin(q_3)\dot{q}_3^2 \\
&\qquad + K_2 \sin(2q_2)\dot{q}_1\dot{q}_2 + K_3 \sin(2q_3)\dot{q}_1\dot{q}_3 + K_8\dot{q}_1 \\
c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -0.5K_2 \sin(2q_2)\dot{q}_1^2 + K_6 \sin(q_2) + K_9\dot{q}_2 \\
c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -0.5K_3 \sin(2q_3)\dot{q}_1^2 + K_7 \sin(q_3) + K_{10}\dot{q}_3,
\end{aligned}$$

the parameters $K_1, \ldots, K_{11}$ are defined in Table 1, and the origin is the pendant (stable) configuration. The mechanism has four distinct equilibrium manifolds, as denoted in Table 2. Note that q_1 parameterizes each manifold, e.g., the mechanism's single open-loop stable equilibrium manifold satisfies $q_2 = q_3 = \dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$, while q_1 can assume an arbitrary value. For controllability reasons, link 3 is shorter than link 2, giving it approximately three times the natural frequency. (If both these links have the same natural frequency, then the system lacks controllability on all of its equilibrium manifolds.) Also note that links 2 and 3 are not inertially coupled owing to the mechanism's structure.

Our strategy for transferring the state from **D-D** (fully pendant) to **U-U** (fully inverted) consists of four steps:

(1) Apply a feedback control u_1 that swings link 3 from its pendant state (**D-D**) a neighborhood $\mathcal{N}_{\mathbf{U}}$ of its inverted state;

(2) Once the state has entered $\mathcal{N}_{\mathbf{U}}$, switch to a feedback control u_2 which stabilizes link 3 in its inverted state;
(3) With link 3 stabilized, apply a feedback control u_3 which swings link 2 into a neighborhood $\mathcal{N}_{\mathbf{U-U}}$ of the equilibrium manifold **U-U**;
(4) Once the state has entered $\mathcal{N}_{\mathbf{U-U}}$, switch to a feedback control u_4 which stabilizes both link 2 and link 3 on **U-U**.

Because of the switching nature of the control, the result is applicable to the single inverted pendulum, in either its rotational or its translational variants. Furthermore, we stress that each of these controls is nonlinear *feedback*, and not feed-forward. Indeed, with the switching logic, our control law can be considered a single, static nonlinear feedback.

We begin by defining a control $u_0(\boldsymbol{q}, \dot{\boldsymbol{q}}, \nu)$, that cancels the forces applied to link 1 by links 2 and 3, effectively decoupling the system. Solving (1) for $\ddot{q}_1$, equating the result with the new input ν, and solving for $u = u_0$ gives

$$u_0 = \frac{\Delta}{K_{11} m_{22} m_{33}} \cdot \nu - \frac{m_{12} c_2}{K_{11} m_{22}} - \frac{m_{13} c_3}{K_{11} m_{33}}, \quad (2)$$

where $\Delta = m_{11} m_{22} m_{33} - m_{12}^2 m_{33} - m_{13}^2 m_{22}$ is the determinate of the inertia matrix. Note that (2) is globally defined. Applying (2) to (1) and defining the states $\xi_1 = q_1$, $\xi_2 = \dot{q}_1$, $\eta_3 = q_3$, $\eta_4 = \dot{q}_3$, $\eta_3 = q_3$, $\eta_4 = \dot{q}_3$, we can rewrite (1) in normal form (Isidori 1989):

$$\dot{\xi}_1 = \xi_2 \quad (3)$$
$$\dot{\xi}_2 = \nu \quad (4)$$
$$\dot{\eta}_1 = \eta_2 \quad (5)$$
$$\dot{\eta}_2 = -\frac{K_6}{K_2} \sin \eta_1 - \frac{K_9}{K_2} \eta_2 + \nu \cdot \frac{K_4}{K_2} \cos \eta_1 + \sin(2\eta_1) \frac{\xi_2^2}{2} \quad (6)$$
$$\dot{\eta}_3 = \eta_3 \quad (7)$$
$$\dot{\eta}_4 = -\frac{K_7}{K_3} \sin \eta_3 - \frac{K_{10}}{K_3} \eta_4 + \nu \cdot \frac{K_5}{K_3} \cos \eta_3 + \sin(2\eta_3) \frac{\xi_2^2}{2}. \quad (8)$$

Here, the map from ν to $\xi_1 = q_1$ is made linear, and the dynamics representing link 2 (η_1, η_2) and link 3 (η_3, η_4) are decoupled from one-another. Note that the zero dynamics, defined by setting $\xi_1 = \xi_2 = 0$, are exactly those for two unforced pendulums.

2.1 *Swing-Up of Link 3*

In designing the link 3 swing-up control, u_1, we make use of the following two observations. First, because u_0 decouples the dynamics of link 2 from the the rest of the system, we may ignore (5) and (6). Of course, η_1 and η_2 will be excited by u_1, but their (almost) global exponential stability (the zero dynamics are exponentially stable) justifies this simplification. Secondly, because ξ_1 is rotational, and because each equilibrium manifold is invariant with respect to ξ_1, we can ignore the behavior of ξ_1 (3) during the swing-up operation. Once the system is stabilized on **U-U**, the control can move the system to the origin along **U-U**, should ξ_1 drift during swing-up. (This cannot be justified for the translational inverted pendulum system, because in that case, ξ_1 has limited travel.)

In addition to these two observations, we will neglect the higher-order centripetal terms $\sin(2\eta_1)\frac{\xi_2^2}{2}$ and $\sin(2\eta_3)\frac{\xi_2^2}{2}$ during the *design* of the swing-up control; they are not ignored in our simulations & experiments, of course. To justify this simplification, we will show that ξ_2 remains small during swing-up. Note that this assumption has been made frequently by other researchers studying the swing-up problem for the rotational inverted pendulum, e.g., (Furuta *et al.* 1992, Wiklund *et al.* 1993).

The simplified dynamics that we consider when designing u_1 are therefore

$$\dot{\xi}_2 = \nu \quad (9)$$
$$\dot{\eta}_3 = \eta_4 \quad (10)$$
$$\dot{\eta}_4 = -\frac{K_7}{K_3} \sin \eta_3 - \frac{K_{10}}{K_3} \eta_4 + \nu \cdot \frac{K_5}{K_3} \cos \eta_3. \quad (11)$$

The key idea is to consider ν to be an input to the *internal dynamics* (Getz and Hedrick 1995) (10)-(11), an approach that has been applied to the swing-up control of the *acrobot* (Spong 1994). By essentially ignoring (9), we are left with a second-order nonlinear system, which can be analyzed using well-established phase plane methods.

Now, consider the following heuristic swing-up control for (10)-(11):

$$\nu_1 = \begin{cases} a_1(t) \cdot \cos(\eta_3) \cdot \arctan(\eta_4) & \text{if } |\eta_3| < \pi/2 \\ 0 & \text{if } |\eta_3| > \pi/2, \end{cases} \quad (12)$$

where $a_1(t) > 0$ is a time-varying quantity to be defined. The function ν is illustrated in Figure 2 for fixed a_1. The idea behind this control is to "pump" energy into link 3 by accelerating it in the same direction that it is rotating, but only when it is pendant, i.e., $|\eta_3| < \pi/2$. Note that this "desired acceleration" is smooth and bounded,

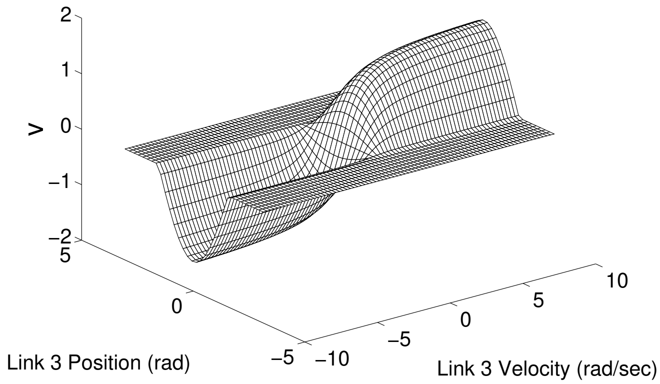

Fig. 2. Feedback function ν defined in (12).

$|\nu_1| < \pi/2$, which prevents ξ_2 from becoming too large, as we shall see.

This control will causes η_3 and η_4 in the simplified model (10)-(11) to enter a stable limit cycle, whose "amplitude" is a continuously increasing function of a_1 for sufficiently small a_1. This is not difficult to prove, since the internal dynamics (10)-(11) are second-order, and Bendixon's Theorem applies. As an illustration, we show the phase portrait of the internal dynamics for $a_1 = 3.0$ in Figure 3. Now, we may propose two the swing-up strategies for u_1. Either determine a fixed a_1 which causes the limit cycle to enter $\mathcal{N}_{\mathbf{U}}$, or to slowly increase $a_1(t)$ as a function of time until the limit cycle intersects $\mathcal{N}_{\mathbf{U}}$. For our system, the value of $a_1 = 3.0$ succeeds in this goal, as can be seen in Figure 3.

Of course, since $\xi_2(t) = \int_{t=0}^{\tau=t} \nu(\tau)\, d\tau$, $|\xi_2(t)|$ might become large, contradicting our simplifying assumption of ignoring the centripetal force, or worse, pushing the state trajectory out of $\mathcal{N}_{\mathbf{U}}$. However, because ν_1 is bounded, and is an odd function of η_4, and is an even function of η_3, and because η_3 and η_4 converge to periodic functions, $\nu_1(t)$ as a function of time also converges to a periodic function whose "DC value" is zero. Physically, since η_3 and η_4 converge to a "symmetric" limit cycle, ν must also be periodic and symmetric about 0. Thus, ξ_2 converges to a periodic function with an offset, which would be due to the transient incurred during convergence of η_3 and η_4 to the stable limit cycle. Figure 4 shows the time history of $\xi_2(t)$ for the same simulation of (9)-(11), subject to (12) and $a_1(t) = 3$. The DC offset is very small, and $\xi_2(t)$ plainly converges to a periodic function.

Thus, the swing-up control for link 3 is

$$u_1 = u_0(\boldsymbol{q}, \dot{\boldsymbol{q}}, \nu_1)$$

where u_0 and ν_1 are defined in (2) and (12), respectively, and $a_1(t)$ is either a fixed constant, or a slowly increasing function with maximum value 3.

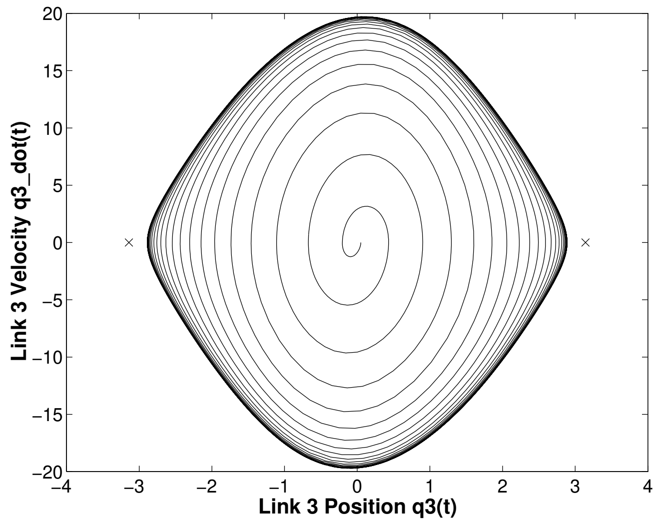

Fig. 3. Phase portrait for the external system (10)-(11) with control (12), initial condition $[\xi_2(0)\ \eta_3(0)\ \eta_4(0)]^T = [0\ 0.05\ 0]^T$ and $a = 3$. The origin is now unstable, and the limit cycle is stable, which can be shown with Bendixon's Theorem.

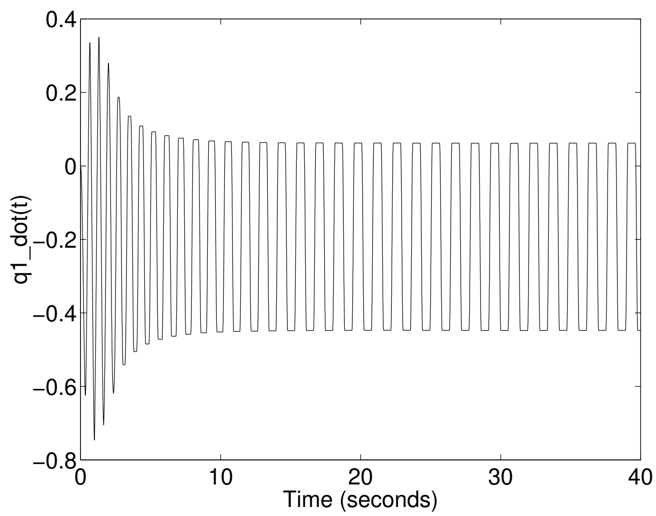

Fig. 4. Behavior of $\xi_2(t)$ during link 3 swing-up, for the control control (12), $a_1 = 3$, and initial condition $[0\ 0.05\ 0\ 0]^T$. Because $\nu(t)$ converges to a periodic function with 0 DC value, ξ_2 converges to a periodic function with small offset. Note that the assumption that $|\xi_2|$ remain small is satisfied.

2.2 *Stabilization about* $\mathcal{N}_{\mathbf{U}}$

The control that stabilizes link 3 in its inverted position is simply an LQR-design,

$$
\begin{aligned}
\nu_2 &= -65.34(q_3 - s_3\pi) + 1.529\dot{q}_1 - 6.513\dot{q}_3 + \nu_3 \\
u_2 &= u_0(\boldsymbol{q}, \dot{\boldsymbol{q}}, \nu_2),
\end{aligned} \tag{13}
$$

where $s_3 = \pm 1$ indicates which direction link 3 swings up, and ν_3 is a new input, to be used to swing-up link 2 during the next phase. When the state becomes sufficiently close to the manifold **D-U**, which is determined empirically, then the controller switches to u_2, and commences the swing-up control of link 2. Note that q_1 does not appear in (13).

2.3 *Swing-up of Link 2.*

The swing-up control for link 2 is similar in spirit to that of link 3. We define a "desired acceleration" feedback

$$
\nu_3 = \begin{cases} a_2(t) \cdot \cos(\eta_1) \cdot \arctan(\eta_2) & \text{if } \ |\eta_1| < \pi/2 \\ 0 & \text{if } \ |\eta_1| > \pi/2, \end{cases} \tag{14}
$$

where $a_2(t) > 0$, which is substituted into (13), giving

$$
u_3 = u_0(\boldsymbol{q}, \dot{\boldsymbol{q}}, \nu_2). \tag{15}
$$

The controller can switch to (15) as soon as link 3 is stabilized in its inverted position. In contrast to the previous swing-up control, the theoretical proof of existence of a limit cycle is more complex, because (13) has feedback ν_2 to stabilize link 3. This not only couples the dynamics, but renders the map from ν_3 to ξ_1 non-minimum phase. Nonetheless, our simulations and experimental results indicate that

(1) ν_3 also induces a stable limit cycle in the (η_1, η_2) dynamics;
(2) that this limit cycle's "amplitude" is an increasing function of a_2, and
(3) that there exists an a_2 to force the trajectory through $\mathcal{N}_{\mathbf{U-U}}$.

Thus, we can adopt the same strategy, in which we slowly increase $a_2(t)$ until η_1 reaches $\pm\pi$, or we can a priori determine the value of a_2 at which this occurs, and use this fixed value.

2.4 *Stabilization about* **U-U**

When the full state passes within $\mathcal{N}_{\mathbf{U-U}}$, a final LQR controller is engaged that stabilizes the system about **U-U** and brings the state ξ_1 back to the origin, if desired:

$$
\begin{aligned}
u_4 = &-0.1(q_1 - q_1^d(t)) + 38.82(q_2 - s_2\pi) \\
&- 35.24(q_3 - s_3\pi) - 0.145\dot{q}_1 + 6.758\dot{q}_2 - 3.43\dot{q}_3,
\end{aligned} \tag{16}
$$

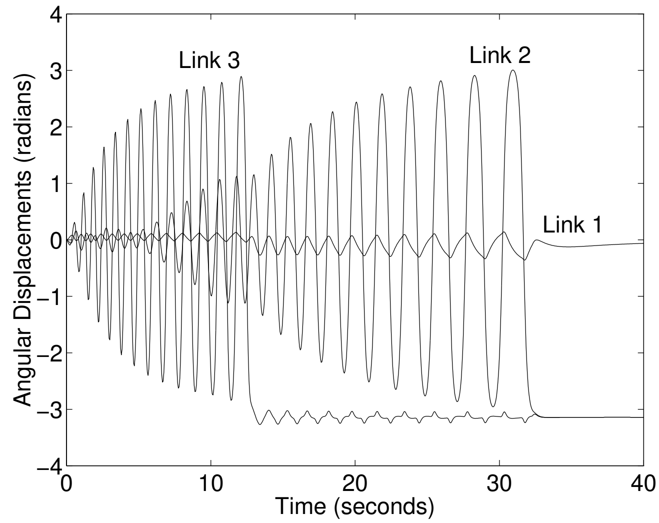

Fig. 5. Simulation of the complete swing-up controller.

where $s_2 = \pm 1$ indicates the direction that link 2 swings up, and $q_1^d(t)$ is a desired trajectory used to move the pendulum back to the origin after swing-up.

A simulation of the complete swing-up controller for the model of our experimental pendulum (1) is plotted in Figure 5. For this simulation, $a_1 = a_2 = 3$ are fixed. Note the small amount of drift in $q_1(t)$, and the small $|\dot{q}_1(t)|$ maintained throughout the swing-up phase.

3. EXPERIMENTAL RESULTS

The swing-up controller has been successfully applied to our experimental test-bed at the University of Toronto, photographed in Figure 6. The controller is implemented on a TMS320C30 DSP, which is hosted by a 486-based PC. Links 2 and 3 are sensed with analog pots and 12-bit A-D converters, while link 1 is sensed using an optical position encoder. The pendulum is actuated with a inexpensive DC motor driven by a voltage amplifier. The sample rate is set at 200Hz.

For the swing-up control, $a_1(t) = 3$ and $a_2(t) = 3$ work well, as the simulations have indicated. The joint trajectories are provided in Figure 7. Note that the stabilizing controllers ν_2 and u_4 are switched for larger deviations from **D-U** and **U-U**, respectively, when compared to the simulations in Figure 5. We see that q_1 drifts little from its initial condition, and that $\dot{q}_1$ remains small throughout the swing-up. The most difficult part of the experiment is the final balancing control (16), which yields a small region of attraction and lacks robustness to parameter uncertainty. For more experimental results, see `http://www.control.toronto.edu` on the WWW.

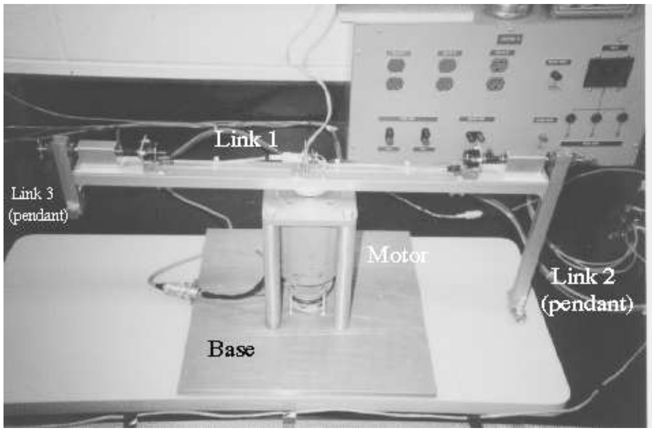

Fig. 6. Photograph of the double inverted pendulum.

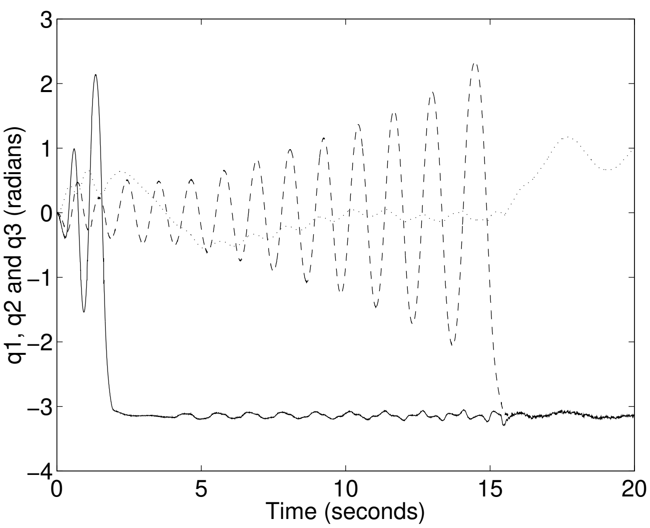

Fig. 7. Joint angle trajectories $q_1(t)$ $(\cdots)$, $q_2(t)$, $(- - -)$, and $q_3(t)$ (—) for the UofT Rotating Double Pendulum. Control u_1 is active for $t < 2$, u_3 for $2 < t < 16$, and u_4 for $t > 16$. The large deviations in q_1 for $t > 16$ are caused by friction and other model uncertainty.

4. ROBUSTNESS AND CONCLUSIONS

The control u_0 can be made robust with respect to uncertain parameters because all such uncertainty lies within the range of the actual control u. Thus, a number of techniques can be adopted to make u_0 robust, including high-gain, Lyapunov redesign, etc. The swing-up controls ν_1, ν_3 do not depend on any system parameters assuming they are implemented with the "slowly increasing" amplitude functions $a_1(t)$ and $a_2(t)$, respectively. It is only the stabilizing controllers ν_2 and ν_4, which are the most sensitive to parameter uncertainty. For the double rotating pendulum, "Getting to the top is easy part. Staying there is difficult."

5. REFERENCES

Bedrossian, Nazareth S. (1993). Nonlinear control of an underactuated two-link manipulator. In: *Proceedings of the American Control Conference.* pp. 2234–2238.

Bortoff, Scott A. (1994). Advanced nonlinear control using digital signal processing. *IEEE Transactions on Industrial Electronics* **41**(1), 32–39.

Furuta, Katsuhisa, Hiroyuki Kajiwara and Kazuhiro Kosuge (1980). Digital control of a double inverted pendulum on an inclined rail. *International Journal of Control* **32**(5), 907–924.

Furuta, Katsuhisa, M. Yamakita and S. Kobayashi (1992). Swing-up control of inverted pendulum using pseudo-state feedback. *Proceedings of the Institute of Mechanical Engineers, Part 1: Journal of Systems and Control Engineering* **206**, 263–269.

Furuta, Katsuhisa, T. Ochia and N. Ono (1984). Attitude control of a triple inverted pendulum. *International Journal of Control* **39**(6), 1351–1365.

Getz, Neil H. and J. Karl Hedrick (1995). An internal equilibrium manifold method of tracking for nonlinear nonminimum phase systems. pp. 2241–2245.

Isidori, Alberto (1989). *Nonlinear Control Systems, Second Edition.* Springer-Verlag. Berlin.

Kawaji, Shigeyasu, Ken'ichi Ogasawara and Hidenobu Honda (1994). Swing up control of a pendulum using genetic algorithms. In: *Proceedings of the* 33^{rd} *IEEE Conference on Decision and Control.*

Mori, Shozo, Hiroyoshi Nishihara and Katsuhish Furuta (1976). Control of unstable mechanical system: Control of pendulum. *International Journal of Control* **23**(5), 673–692.

Spong, Mark W. (1994). Partial feedback linearization of underactuated mechanical systems. In: *Proceedings of the IROS.*

Spong, Mark W. and Daniel Block (1995). The pendubot: A mechatronic system for control research and education. In: *Proceedings of the* 34^{th} *IEEE Conference on Decision and Control.* pp. 1–2.

Wiklund, Magnus, Anders Kristenson and Karl J. Åström (1993). A new strategy for swinging up an inverted pendulum. In: *Proceedings of the 1993 IFAC.* Vol. 9. pp. 151–154.

Yamakita, M., M. Iwashiro, Y. Sugahara and K. Furuta (1995). Robust swing-up control of double pendulum. In: *Proceedings of the American Control Conference.* pp. 290–294.

Yamakita, Masaki, Kenichi Nonaka and Katsuhisa Furuta (1993). Swing up control of a double pendulum. In: *Proceedings of the American Control Conference.* pp. 2229–2233.