Observer Design for Non-Integrable Systems [1]

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Abstract

Systems for which no observer with linear error dynamics exists are termed non-integrable. Generically, nonlinear systems are non-integrable. Using results developed for approximate feedback linearization, we propose two approaches to observer design for non-integrable, unforced, single-output systems. An example illustrates the design procedure.

1. Introduction

A state-feedback-linearizing controller for the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}) \cdot u, \tag{1}$$

where $\boldsymbol{x} \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}$ is the input, and $\boldsymbol{f}$, $\boldsymbol{g}$ are C^∞-vector fields, exists if and only if a certain non-generic involutivity condition is satisfied [9]. Unfortunately, many systems of engineering interest fail this condition, e.g. the ball-and-beam [8] and the acrobot [7, 3]. Thus, several researchers have considered the problem of approximating such a system by a feedback linearizable one for purposes of controller synthesis. This approach is know as *approximate feedback linearization*, and is described in, for example, [1, 2, 5, 6, 8, 11] and the references therein.

The "dual" problem is observer synthesis for the single-output system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}), & (2)\\ y &= h(\boldsymbol{x}), & (3) \end{aligned}$$

where $y \in \mathcal{R}$ is the measured output, and h is a C^∞ function satisfying $h(0) = 0$. *Exact error linearization* [12] is a method to construct an observer whose error dynamics are exactly linear in a particular set of coordinates. Again, the approach requires a certain codistribution to be integrable — a non-generic property.

In this paper, we show how recently introduced results on approximating non-integrable distributions [1] can be used to synthesize an observer for *non-integrable* systems. The observer synthesis may be considered "dual" to results on approximate state-feedback linearization [1, 2], because the resulting estimate error dynamics is *approximately* linear, described by a stable, linear time-invariant system that is regularly perturbed. The result extends our previous work on observers with perturbed error dynamics [4, 14].

The paper is organized as follows. Section 2 briefly reviews the synthesis in the integrable case. The non-integrable synthesis follows in Section 3, followed by an example in Section 4. Section 5 concludes the paper by describing our ongoing research.

2. The Integrable Case

The following two conditions are necessary and sufficient for the existence of the exact error linearizing observer for the system (2)-(3) [12]:

Condition C1 (Observability): The codistribution

$$\boldsymbol{\Theta} := \text{span}\ \{dh,\, d\mathcal{L}_f h, \ldots,\, d\mathcal{L}_f^{n-1}h\}$$

is nonsingular at $\boldsymbol{x} = 0$; and

Condition C2 (Integrability): The vector fields $\boldsymbol{\pi}_i$ $(1 \le i \le n)$ defined by

$$\begin{aligned} (d\mathcal{L}_f^{i-1}h)(\boldsymbol{\pi}_1) &= \delta_{in}, \quad 1 \le i \le n & (4)\\ \boldsymbol{\pi}_j &= -\text{ad}_f\, \boldsymbol{\pi}_{j-1}, \quad 2 \le j \le n, & (5) \end{aligned}$$

where δ_{ij} is the Kronecker delta, satisfy $[\boldsymbol{\pi}_r\,,\, \boldsymbol{\pi}_s] = 0$ for all $1 \le r,\, s \le n$, in a neighborhood of $\boldsymbol{x} = 0$.

If system (2)-(3) satisfies **C1** and **C2**, then a local state coordinate transformation $\boldsymbol{z} = \boldsymbol{\phi}(\boldsymbol{x})$ exists such that (2)-(3) can be expressed in observer form

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{A}\boldsymbol{z} + \boldsymbol{a}(y), & (6)\\ y &= \boldsymbol{C}\boldsymbol{z}, & (7) \end{aligned}$$

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where the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, and $\boldsymbol{a}$ is a C^∞ output injection vector field. Synthesis of an observer for (6)-(7) follows naturally: Denoting $\widehat{\boldsymbol{z}}$ as the estimate of the state $\boldsymbol{z}$, an observer with linear error dynamics is

$$\dot{\widehat{\boldsymbol{z}}} = \boldsymbol{A}\widehat{\boldsymbol{z}} + \boldsymbol{a}(y) + \boldsymbol{G}(y - \boldsymbol{C}\widehat{\boldsymbol{z}}). \tag{8}$$

The estimation error $\widetilde{\boldsymbol{z}} := \boldsymbol{z} - \widehat{\boldsymbol{z}}$ is governed by $\dot{\widetilde{\boldsymbol{z}}} = (\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})\widetilde{\boldsymbol{z}}$, whose spectrum can be arbitrarily assigned. Since the purpose of our observer is to estimate $\boldsymbol{x}$ and not $\boldsymbol{z}$, which may have no physical importance, it is advantageous to implement (8) explicitly in the $\boldsymbol{x}$-coordinates:

$$\dot{\widehat{\boldsymbol{x}}} = \boldsymbol{f}(\widehat{\boldsymbol{x}}) + D\boldsymbol{\tau}(\boldsymbol{\phi}(\widehat{\boldsymbol{x}}))\Big(\boldsymbol{a}(y) - \boldsymbol{a}(h(\widehat{\boldsymbol{x}})) + \boldsymbol{G}\cdot(y - h(\widehat{\boldsymbol{x}}))\Big). \tag{9}$$

Here, $D\boldsymbol{\tau}(\boldsymbol{\phi}(\widehat{\boldsymbol{x}}))$ denotes the Jacobian of $\boldsymbol{\tau}$ evaluated at $\widehat{\boldsymbol{z}} = \boldsymbol{\phi}(\widehat{\boldsymbol{x}})$: $D\boldsymbol{\tau}(\boldsymbol{\phi}(\widehat{\boldsymbol{x}})) = \frac{\partial \tau(z)}{\partial z}|_{z=\phi(\hat{x})}$. Representation (9) has the advantage that neither $\boldsymbol{\phi}$ nor $\boldsymbol{\tau}$ explicitly appear. Only $D\boldsymbol{\tau}$, expressed in the $\boldsymbol{x}$-coordinates, and the vector field $\boldsymbol{a}$ need to be realized. On the other hand, (8) requires $\boldsymbol{a}$ and $\boldsymbol{\tau}$.

Two methods have been proposed in the literature for constructing the observer. In one method, $\boldsymbol{a}$ is calculated from the Generalized Characteristic Equation, and then differentiated to give $D\boldsymbol{\phi}$ which is then inverted to obtain $D\boldsymbol{\tau}$ [10]. A second method involves constructing $D\boldsymbol{\tau}$ first, then inverting and integrating to produce $\boldsymbol{\phi}$, and finally differentiating to get $\boldsymbol{a}$ [9]. Since our synthesis for the inexact case is based upon the latter procedure, we outline it here. Following [9], we define the matrix

$$\boldsymbol{\Pi}(\boldsymbol{x}) := [\boldsymbol{\pi}_1(\boldsymbol{x}), \boldsymbol{\pi}_2(\boldsymbol{x}), \ldots, \boldsymbol{\pi}_n(\boldsymbol{x})],$$

where the kth column of $\boldsymbol{\Pi}$ is $\boldsymbol{\pi}_k$ expressed in the $\boldsymbol{x}$-coordinates. Each entry of $\boldsymbol{\Pi}$ is a C^∞ function of $\boldsymbol{x}$, and by its very construction, $\boldsymbol{\Pi}$ satisfies the overdetermined system of partial differential equations

$$D\boldsymbol{\tau}(\boldsymbol{z})|_{z=\phi(x)} = \boldsymbol{\Pi}(\boldsymbol{x}). \tag{10}$$

Defining $\boldsymbol{\Omega}(\boldsymbol{x}) := (\boldsymbol{\Pi}(\boldsymbol{x}))^{-1}$, where $(\cdot)^{-1}$ denotes matrix inverse, (10) can be expressed compactly in the $\boldsymbol{x}$-coordinates:

$$D\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{\Omega}(\boldsymbol{x}). \tag{11}$$

Thus, $\boldsymbol{\Pi}(\boldsymbol{x}) = D\boldsymbol{\tau}(\boldsymbol{x})$, precisely what is required by (9). Moreover, $\boldsymbol{\Omega}$ is particularly simple to compute. Using (4)-(5), and slightly abusing the definition of $\boldsymbol{\Theta}$, we may define the "lower-triangular" matrix $\boldsymbol{\Gamma}$ to satisfy

$$\underbrace{\begin{bmatrix} dh \\ d\mathcal{L}_f h \\ \vdots \\ d\mathcal{L}_f^{n-1} h \end{bmatrix}}_{\boldsymbol{\Theta}(\boldsymbol{x})} \boldsymbol{\Pi}(\boldsymbol{x}) = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \star & \cdots & \star & \star \end{bmatrix}}_{\boldsymbol{\Gamma}(\boldsymbol{x})}. \tag{12}$$

Hence, $\boldsymbol{\Omega} = \boldsymbol{\Gamma}^{-1}\cdot\boldsymbol{\Theta}$, where $\boldsymbol{\Gamma}^{-1}$ is "upper-triangular," with determinant equal to 1. Furthermore, we have

$$\boldsymbol{\Omega} = \begin{bmatrix} d\mathcal{L}_f^{n-1}h + \alpha_{1,n-1}\, d\mathcal{L}_f^{n-2}h + \cdots + \alpha_{1,1}\, dh \\ d\mathcal{L}_f^{n-2}h + \alpha_{2,n-2}\, d\mathcal{L}_f^{n-3}h + \cdots + \alpha_{2,1}\, dh \\ \vdots \\ d\mathcal{L}_f h + \alpha_{n-1,1}\, dh \\ dh \end{bmatrix}, \tag{13}$$

where $\alpha_{k,i}$ $(2 \le k \le n, 1 \le i \le n-k)$ are C^∞ functions of $\boldsymbol{x}$. Condition **C1** implies that $\boldsymbol{\Pi}$ is invertible at 0. In this case, define the linearly-independent one-forms $\boldsymbol{\omega}_i$ $(1 \le i \le n-1)$ as $[\boldsymbol{\omega}_1^T, \ldots, \boldsymbol{\omega}_n^T]^T := \boldsymbol{\Omega}$. Then Conditions **C1** & **C2** are equivalent to $\boldsymbol{\omega}_i$ being exact, by the Frobenius Theorem. That is, **C1** & **C2** ensure the existence of the change of coordinates $\boldsymbol{z} = [z_1, \ldots, z_n]^T = [\phi_1(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x})]^T$ that satisfies

$$\phi_i(\boldsymbol{x}) = \int_N \boldsymbol{\omega}_i + \phi_i(0),$$

where N is any smooth path from 0 to $\boldsymbol{x}$ lying in a sufficiently small neighborhood of 0. We choose $\boldsymbol{\phi}(0) = 0$. Finally, the output injection vector $\boldsymbol{a} =: [a_1, \ldots, a_n]^T$ is calculated as

$$\begin{aligned} a_1 &= \mathcal{L}_f \phi_1, \\ a_i &= \mathcal{L}_f \phi_i - \phi_{i-1}, \quad 2 \le i \le n. \end{aligned}$$

3. Synthesis for Non-Integrable Systems

Although smooth systems (2)-(3) satisfy **C1** generically, they rarely satisfy **C2**. Throughout this section, we assume **C1** is satisfied but **C2** is not. In this case, some of the 1-forms $\boldsymbol{\omega}_i$ $(1 \le i \le n-1)$ may not be exact. (Note that $\boldsymbol{\omega}_n = dh$ is always exact.) Nevertheless, we show that a change of coordinates $\boldsymbol{\xi} = \boldsymbol{\phi}(\boldsymbol{x})$ that *approximately* satisfies the partial differential equation (11) can be used to construct an observer. The main idea is to decompose each $\boldsymbol{\omega}_i$ into an exact part $d\phi_i$, where ϕ_i is the ith component of a coordinate transformation used in the observer synthesis, and an inexact part $\boldsymbol{\epsilon}_i$, which will lead to nonlinear perturbations of the error dynamics:

$$\boldsymbol{\omega}_i = d\phi_i + \boldsymbol{\epsilon}_i, \quad 1 \le i \le n. \tag{14}$$

Assuming the decompositions (14) have been accomplished, it is of interest to determine when the exact parts, once integrated, define a local change of coordinates, and precisely how $\boldsymbol{\epsilon}$ appears in (2)-(3) expressed in these coordinates.

Lemma 3.1 Given the smooth 1-forms $\boldsymbol{\omega}_i$ ($1 \le i \le n$) defined in (13), and suppose that $\boldsymbol{\epsilon}_i$ are smooth inexact parts of the decompositions $\boldsymbol{\omega}_i = d\boldsymbol{\phi}_i + \boldsymbol{\epsilon}_i$ such that $\boldsymbol{\epsilon}_i(0) = 0$ in the $\boldsymbol{x}$-coordinates. Then the mapping $\boldsymbol{\xi} = \boldsymbol{\phi}(\boldsymbol{x}) = [\phi_1(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x})]^T$ satisfying $\boldsymbol{\phi}(0) = 0$ is a local change of coordinates defined in some open neighborhood of 0.

Proof: Since $\boldsymbol{\epsilon}_i(0) = 0$ then $d\phi_i(0) = \boldsymbol{\omega}_i(0)$ ($1 \le i \le n$). Since by **C1** the $\boldsymbol{\omega}_i$ are linearly independent at $\boldsymbol{x} = 0$ so are the $d\phi_i$. By the Inverse Function Theorem, there exists an open neighborhood on which $\boldsymbol{\phi}$ is a change of coordinates. This $\boldsymbol{\phi}$ can be translated to ensure $\boldsymbol{\phi}(0) = 0$. ∎

Lemma 3.2 Let $\boldsymbol{\xi} = \boldsymbol{\phi}(\boldsymbol{x})$ be a change in coordinates as in Lemma 3.1. Then the system (2)-(3) written in $\boldsymbol{\xi}$-coordinates is of the form

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \bar{\boldsymbol{f}}(\boldsymbol{\xi}) = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{a}(y) + \boldsymbol{\eta}(\boldsymbol{\xi}), \\ y &= \boldsymbol{C}\boldsymbol{\xi}, \end{aligned} \tag{15}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}^T, \tag{16}$$

the derivatives of the vector field $\boldsymbol{\eta}$ with components $\eta_1, \ldots, \eta_n$ in $\boldsymbol{\xi}$-coordinates satisfy

$$\frac{\partial \eta_i}{\partial \xi_j} = \sum_{k=1}^{n-1} \frac{\partial \bar{f}_i}{\partial \xi_k} \boldsymbol{\epsilon}_k(\boldsymbol{\pi}_j) - \mathcal{L}_f(\boldsymbol{\epsilon}_i(\boldsymbol{\pi}_j)), \tag{17}$$

for $1 \le i, j \le n-1$, and

$$\frac{\partial \eta_n}{\partial \xi_j} = \sum_{k=1}^{n-1} \frac{\partial \bar{f}_n}{\partial \xi_k} \boldsymbol{\epsilon}_k(\boldsymbol{\pi}_j), \quad 1 \le j \le n-1. \tag{18}$$

Proof: Denoting the dual basis to $d\phi_i$ as $\frac{\partial}{\partial \xi_j}$ and defining the vector fields $\boldsymbol{\Delta}_j := \boldsymbol{\pi}_j - \frac{\partial}{\partial \xi_j}$ ($1 \le j \le n$), we note the following relation between $\boldsymbol{\Delta}_j$ and $\boldsymbol{\epsilon}_i$ ($1 \le i, j \le n$):

$$\begin{aligned} d\phi_i(\boldsymbol{\Delta}_j) &= d\phi_i(\boldsymbol{\pi}_j) - \delta_{ij} \\ &= \boldsymbol{\omega}_i(\boldsymbol{\pi}_j) - \boldsymbol{\epsilon}_i(\boldsymbol{\pi}_j) - \delta_{ij} \;=\; -\boldsymbol{\epsilon}_i(\boldsymbol{\pi}_j). \end{aligned}$$

For $1 \le i \le n$, $1 \le j \le n-1$ we have

$$\begin{aligned} d\phi_i\left(\frac{\partial}{\partial \xi_{j+1}}\right) &= \delta_{ij+1} \;=\; d\phi_i(\boldsymbol{\pi}_{j+1}) - d\phi_i(\boldsymbol{\Delta}_{j+1}) \\ &= d\phi_i([-\boldsymbol{f}, \boldsymbol{\pi}_j]) + \boldsymbol{\epsilon}_i(\boldsymbol{\pi}_{j+1}) \\ &= (\mathcal{L}_f d\phi_i)(\boldsymbol{\pi}_j) - \mathcal{L}_f(d\phi_i(\boldsymbol{\pi}_j)) + \\ &\quad \boldsymbol{\epsilon}_i(\boldsymbol{\pi}_{j+1}) \\ &= d\bar{f}_i(\boldsymbol{\pi}_j) + \mathcal{L}_f(\boldsymbol{\epsilon}_i(\boldsymbol{\pi}_j)) + \boldsymbol{\epsilon}_i(\boldsymbol{\pi}_{j+1}) \\ &= d\bar{f}_i\left(\boldsymbol{\Delta}_j + \frac{\partial}{\partial \xi_j}\right) + \mathcal{L}_f(\boldsymbol{\epsilon}_i(\boldsymbol{\pi}_j)) + \\ &\quad \boldsymbol{\epsilon}_i(\boldsymbol{\pi}_{j+1}). \end{aligned}$$

Rearranging this last expression gives

$$\begin{aligned} d\bar{f}_i\left(\frac{\partial}{\partial \xi_j}\right) &= \delta_{ij+1} - d\bar{f}_i(\boldsymbol{\Delta}_j) - \mathcal{L}_f(\boldsymbol{\epsilon}_i(\boldsymbol{\pi}_j)) - \\ &\quad \boldsymbol{\epsilon}_i(\boldsymbol{\pi}_{j+1}) \\ &= \delta_{ij+1} - d\bar{f}_i(\boldsymbol{\Delta}_j) - (\mathcal{L}_f \boldsymbol{\epsilon}_i)(\boldsymbol{\pi}_j) \\ &= \delta_{ij+1} + \sum_{k=1}^{n-1} \frac{\partial \bar{f}_i}{\partial \xi_k} \boldsymbol{\epsilon}_k(\boldsymbol{\pi}_j) - (\mathcal{L}_f \boldsymbol{\epsilon}_i)(\boldsymbol{\pi}_j). \end{aligned}$$

Thus, $\bar{\boldsymbol{f}}$ takes the form (15) with $\boldsymbol{A}$ given by (16), the derivative of $\boldsymbol{\eta}$ satisfying (17)-(18). Finally, since $\xi_n = y$, we have $y = \boldsymbol{C}\boldsymbol{\xi}$. ∎

We note that (17)-(18) relates the partial of η_i with respect to ξ_j in terms of the partials of η_i with respect to ξ_k ($k \ne j$).

Given a coordinate transformation $\boldsymbol{\xi} = \boldsymbol{\phi}(\boldsymbol{x})$ satisfying the conditions of Lemma 3.1, we propose the following observer:

$$\dot{\widehat{\boldsymbol{\xi}}} = \boldsymbol{A}\widehat{\boldsymbol{\xi}} + \boldsymbol{a}(y) + \boldsymbol{\eta}(\widehat{\boldsymbol{\xi}}) + \boldsymbol{G} \cdot (y - \boldsymbol{C}\widehat{\boldsymbol{\xi}}). \tag{19}$$

In practice, (19) is implemented in the $\boldsymbol{x}$-coordinates, as noted in Section 2, and has exactly the same form as (9). Defining $\widetilde{\boldsymbol{\xi}} := \boldsymbol{\xi} - \widehat{\boldsymbol{\xi}}$, the resulting error dynamics satisfy

$$\dot{\widetilde{\boldsymbol{\xi}}} = (\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})\widetilde{\boldsymbol{\xi}} + \underbrace{\boldsymbol{\eta}(\boldsymbol{\xi}) - \boldsymbol{\eta}(\boldsymbol{\xi} - \widetilde{\boldsymbol{\xi}})}_{\widetilde{\boldsymbol{\eta}}(\boldsymbol{\xi}, \widetilde{\boldsymbol{\xi}})}.$$

Here, $\widetilde{\boldsymbol{\eta}}$ may be interpreted as a regular perturbation of a stable linear time-invariant system. To ensure local exponential stability of $\widetilde{\boldsymbol{\xi}} = 0$ in a neighborhood of $\boldsymbol{\xi} = 0$ for the perturbed system, the first-order part of the perturbation should be compensated, i.e., $\boldsymbol{G}$ should be designed for the linearized *perturbed* system. Expanding each component of $\widetilde{\boldsymbol{\eta}}$ in a Taylor series about $\boldsymbol{\xi}$, we have

$$\widetilde{\eta}_i(\boldsymbol{\xi}, \widetilde{\boldsymbol{\xi}}) = D\eta_i(\boldsymbol{\xi}) \cdot \widetilde{\boldsymbol{\xi}} + \frac{1}{2} \widetilde{\boldsymbol{\xi}}^T \cdot R\eta_i(\boldsymbol{\zeta}_i) \cdot \widetilde{\boldsymbol{\xi}},$$

where $R\eta_i$ is the Hessian of η_i, evaluated at appropriate $\boldsymbol{\zeta}_i \in [\boldsymbol{\xi},\ \widehat{\boldsymbol{\xi}}]$, i.e., $R\eta_i(\boldsymbol{\zeta}_i) = \frac{\partial^2 \eta_i}{\partial \xi^2}|_{\xi=\zeta_i}$, $(1 \le i \le n)$. Local exponential stabilization of $\widetilde{\boldsymbol{\xi}} = 0$ provided the trajectories of (2)-(3) remain sufficiently close to 0 can be accomplished by choosing a $\boldsymbol{G}$ to make $(\boldsymbol{A} - \boldsymbol{GC} + D\boldsymbol{\eta}(0))$ Hurwitz. Such a locally stabilizing $\boldsymbol{G}$ exists because of the observability assumption **C1**, and because observability is invariant under the change of coordinates $\boldsymbol{\xi} = \boldsymbol{\phi}(\boldsymbol{x})$.

A homotopy operator provides one way of obtaining the decompositions (14). We define W to be a compact, contractible subset of $\mathcal{R}^n$ containing 0 on which $\boldsymbol{\Theta}$ is nonsingular. A homotopy operator H maps k-forms on W to $(k-1)$-forms on W and satisfies

$$\boldsymbol{\beta} = dH\boldsymbol{\beta} + Hd\boldsymbol{\beta}, \quad \forall\ k\text{-forms } \boldsymbol{\beta}.$$

We restrict our attention to radial homotopy operators with centers at 0:

$$(H\boldsymbol{\beta})(\boldsymbol{x}) = \int_0^1 \sum_{k=1}^{n} \beta_k(\lambda \boldsymbol{x}) x_k d\lambda, \tag{20}$$

where β_k is the kth component of $\boldsymbol{\beta}$ in the $\boldsymbol{x}$-coordinates. Such an H guarantees that $(dH\boldsymbol{\beta})(\boldsymbol{X}) = \boldsymbol{\beta}(\boldsymbol{X})$, where $\boldsymbol{X}$ is a vector field in the radial direction from 0 in the $\boldsymbol{x}$-coordinates. Also, $dH\boldsymbol{\beta}$ and $\boldsymbol{\beta}$ are equal when restricted to 0. Hence, taking $\phi_i = H\boldsymbol{\omega}_i$ $(1 \le i \le n-1)$ we have $\boldsymbol{\epsilon}_i(0) = 0$. Since $\phi_n = h$, Lemma 3.1 implies $\boldsymbol{\phi}$ is a well-defined change of coordinates.

Improved performance, in terms of a larger region of attraction for instance, may be achievable by *minimizing* a norm of $\widetilde{\boldsymbol{\eta}}$. Because $\mathcal{L}_f \boldsymbol{\epsilon}_k = (d\boldsymbol{\epsilon}_k)(\boldsymbol{f}) + d(\boldsymbol{\epsilon}_k(\boldsymbol{f}))$, the one-form $\boldsymbol{\epsilon}_k$ $(1 \le k \le n-1)$ appears in the first and second terms of $\frac{\partial \eta_i}{\partial \xi_j}$ in equation (17). This motivates us to consider decompositions of $\boldsymbol{\omega}_i$ that minimize $\|\boldsymbol{\epsilon}_i\|$, where $\|\cdot\|^2 := \int_W |\cdot|^2 \mu_x$, is the global $\boldsymbol{L}_2$ norm on forms, $|\cdot|$ is the pointwise norm on forms induced by the standard Euclidean metric (on vectors) in the $\boldsymbol{x}$-coordinates, and $\mu_x := dx_1 \wedge \cdots \wedge dx_n$ is a volume element. Because this decomposition is well-studied for the controller case, we only outline the approach — see [1, 2, 15] for details.

The problem of minimizing $\|\boldsymbol{\epsilon}_i\|$ over all C^∞ functions ϕ_i can be formulated as a calculus of variations problem. The optimal ϕ_i are obtained by solving the Euler-Lagrange equations consisting of second order partial differential equations. Specifically, the C^∞ functions ψ_i, $(1 \le i \le n-1)$ which minimize the costs

$$\int_W |\boldsymbol{\omega}_i - d\psi_i|^2 \mu_x, \qquad 1 \le i \le n-1,$$

are given by the unique (up to a constant) solutions of the Euler-Lagrange equations:

$$\frac{\partial^2 \psi_i}{\partial x_1^2} + \cdots + \frac{\partial^2 \psi_i}{\partial x_n^2} = \frac{\partial \omega_{i1}}{\partial x_1} + \cdots + \frac{\partial \omega_{in}}{\partial x_n} \quad \text{in } W, \tag{21}$$

$$(\boldsymbol{\omega}_i - d\psi_i)(\boldsymbol{n}) = 0 \quad \text{on } \partial W, \tag{22}$$

where $1 \le i \le n-1$, $\boldsymbol{n}$ is the outward normal vector to ∂W, the boundary of W, and the jth component of $\boldsymbol{\omega}_i$ in the $\boldsymbol{x}$-coordinates is denoted ω_{ij}. The Euler-Lagrange equations (21)-(22) are $n-1$ Poisson equations each subject to Neumann boundary conditions arising from natural boundary conditions.

It is important to note that for any W and $\boldsymbol{\Omega}$, letting $\phi_i = \psi_i$ $(1 \le i \le n-1)$ where ψ_i is the solution to (21)-(22) does not necessarily ensure $\boldsymbol{\epsilon}_i(0) = (\boldsymbol{\omega}_i - d\phi_i)(0)$. Hence, the sufficient conditions for $\boldsymbol{\phi}$ to be a change of coordinates are not necessarily satisfied. In fact it is an open issue as to whether $\boldsymbol{\phi}$ is well-defined about *any* point in W. It is reassuring to note however that in all the examples we have done involving spheres and cubes we obtained $\boldsymbol{\epsilon}_i(0) = 0$. Furthermore, the boundary conditions (22) and the smoothness of ϕ_i imply that if W is a cube centered about 0 then $\boldsymbol{\epsilon}_i = 0$ $(1 \le i \le n)$ at the corners.

4. Example

We consider the third-order system in observable form $\boldsymbol{f}(\boldsymbol{x}) = [x_2,\ x_3,\ x_2x_3 - 2x_1 - 3x_2 - 3x_3]^T$, $h(\boldsymbol{x}) = x_1$. This system satisfies **C1** at each point of $\mathcal{R}^3$. Furthermore, the origin is the only equilibrium solution, and is locally exponentially stable. We take W to be a closed unit ball centered at $\boldsymbol{x} = 0$. Using (13), we compute the 1-forms $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_3$ in $\boldsymbol{x}$-coordinates: $\boldsymbol{\omega}_1 = [3,\ 3-x_2,\ 1]$, $\boldsymbol{\omega}_2 = [3-x_2,\ 1,\ 0]$, $\boldsymbol{\omega}_3 = [1,\ 0,\ 0]$. Since $d\boldsymbol{\omega}_2 = -dx_2 \wedge dx_1$ is nonzero, the observer form (6)-(7) does not exist. An observer form would exist however if we allowed transformation of the output coordinates [13].

Since $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_3$ are exact, we need only compute the decomposition for $\boldsymbol{\omega}_2$. First, we use the homotopy operator (20) to obtain $\phi_2 = H\boldsymbol{\omega}_2 = 3x_1 + x_2 - x_1x_2/2$. Integrating $\boldsymbol{\omega}_1$ gives $\phi_1 = 3x_1 + 3x_2 + x_3 - x_2^2/2$ (where we have chosen the integration constant for ϕ_1 to ensure $\boldsymbol{\phi}(0) = 0$). As expected $\boldsymbol{\epsilon}_2$ restricted to the origin is 0 because the radial homotopy operator provides an exact approximation of $\boldsymbol{\omega}_2$ at its centre. By Lemma 3.1, $\boldsymbol{\phi}$ is a well-defined change of coordinates about 0. In fact, in this particular example $\boldsymbol{\phi}$ is well-defined on all of W, because $D\boldsymbol{\phi}$ is nonsingular and $\boldsymbol{\phi}$ is one-to-one on W.

Next, we decompose $\boldsymbol{\omega}_2$ optimally to minimize

$||\epsilon_2||$. Taking the outward normal vector as $\boldsymbol{n} = [x_1,\, x_2,\, x_3]^T$, and writing out (21)-(22) with ϕ_2 replacing ψ_i gives Laplace's equation with Neumann boundary conditions:

$$\frac{\partial^2 \phi_2}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_2^2} + \frac{\partial^2 \phi_2}{\partial x_3^2} = 0 \quad \text{on } W, \tag{23}$$

$$\sum_{k=1}^{3} \frac{\partial \phi_2}{\partial x_k} x_k = x_2 + x_1(3 - x_2) \quad \text{on } \partial W. \tag{24}$$

By inspection, the solution to (23)-(24), is the same ϕ_2 obtained above using the homotopy operator.

Writing the system in the new coordinates we obtain

$$\dot{\xi}_1 = -2x_1 = \underbrace{-2\xi_3}_{a_1} + \underbrace{0}_{\eta_1}, \tag{25}$$

$$\dot{\xi}_2 = 3x_2 + x_3 - \frac{x_2^2}{2} - \frac{x_1 x_3}{2} = \xi_1 - \underbrace{3\xi_3}_{-a_2} + \eta_2, \tag{26}$$

$$\dot{\xi}_3 = x_2 = \xi_2 - \underbrace{3\xi_3}_{-a_3} + \underbrace{\xi_3(3\xi_3 - \xi_2)/(\xi_3 - 2)}_{\eta_3}, \tag{27}$$

where $\eta_2 = \xi_3(12\xi_2 - 4\xi_1 - 2\xi_2^2 - 24\xi_3 + 4\xi_1\xi_3 + 6\xi_2\xi_3 - 12\xi_3^2 - \xi_1\xi_3^2 + 3\xi_3^3)/(2(\xi_3 - 2)^2)$. From (25)-(27) we note that $\boldsymbol{\eta}$ involves terms that are 2nd order and higher.

5. Conclusions

This paper presented an observer synthesis for a generic class of single-output nonlinear systems. The observer's error dynamics is governed by a stable LTI system that is subject to a nonlinear, time-varying, regular perturbation. Using recent results from approximate feedback linearization, we have generalized the *Exact Error Linearization* observer design method to non-integrable systems.

A number of important issues are currently under investigation. First, we are studying the incorporation of a change in output coordinates into the synthesis. For the non-integrable case, this can lead to considerable improvement in performance. We are also comparing several competing observer designs on an experimental test-bed called the wedge. Finally, development of symbolic software that will make these designs more accessible is underway.

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