Sketching Bode Magnitude and Phase Plots

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1 Introduction

Sketching asymptotic approximations of the magnitude and phase of a transfer function is an important skill. Why is this so? Certainly MATLAB or similar software can generate magnitude and phase plots much more precisely and quickly than any control engineer can do by hand. The reason it is important is this: So the designer can mentally picture how the magnitude and phase of a transfer function are affected by poles and zeros. By itself, MATLAB is only a fancy calculator. It lacks the intelligence required to design controllers. Good design is the product of both a competent designer (you) and good tools (MATLAB). By learning how to plot approximations to Bode magnitude and phase plots, you are mastering an important skill necessary to do good design.

This tutorial presents eight worked examples that illustrate how to construct asymptotic approximations of magnitude and phase plots. It is written using Latex and MATLAB, and each of the plots are in color. It is best viewed using ghostview on a workstation. You should first focus on stable, minimum phase plants with real poles and zeros. Once this is mastered, move on to plants that have complex conjugate poles and zeros. Plots for unstable and/or non-minimum phase plants are then a very straight-forward extension. Of course, being able to picture mentally the magnitude and phase, and being able to plot quickly approximations to magnitude and phase, will make sketching Nyquist plots much easier.

It is not enough simply to read what follows. You should try to sketch each the plots for each example on a separate piece of paper. Make up a few similar examples and sketch them too, until you are comfortable with each class of plant. Check your work with MATLAB. This last step is important, because it will show you where your straight-line approximation breaks down.

2 Minimum Phase, Stable Systems

Example 1. Draw the magnitude and phase of the system

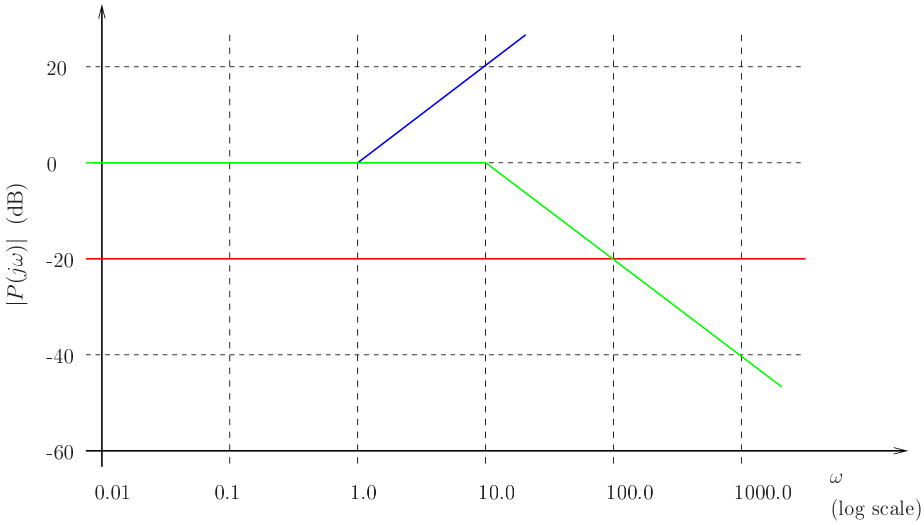
$$P(s) = \frac{s+1}{s+10}. \tag{1}$$

Step 0 Put P in bode form.

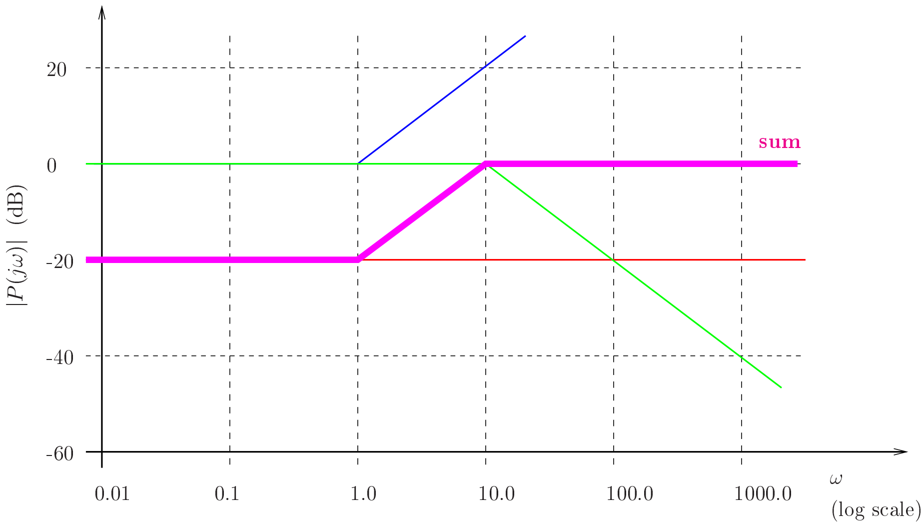
$$P(s) = \frac{1}{10}\frac{1+s/1}{1+s/10}$$

Now the DC gain is obvious: $1/10$.

Step 1 Plot the contribution for each pole, zero and the DC gain. The DC gain is drawn below in red, the zero at -1 is drawn below in blue. and the pole at -10 is drawn below in green.

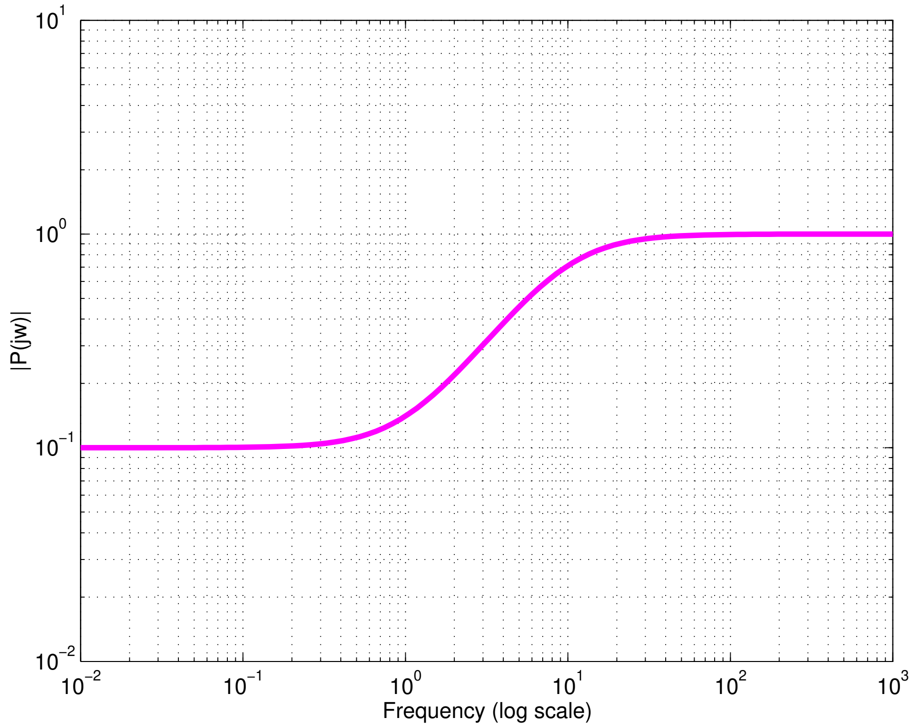


Step 2 Sum each of the three gains. The sum is shown below in magenta.

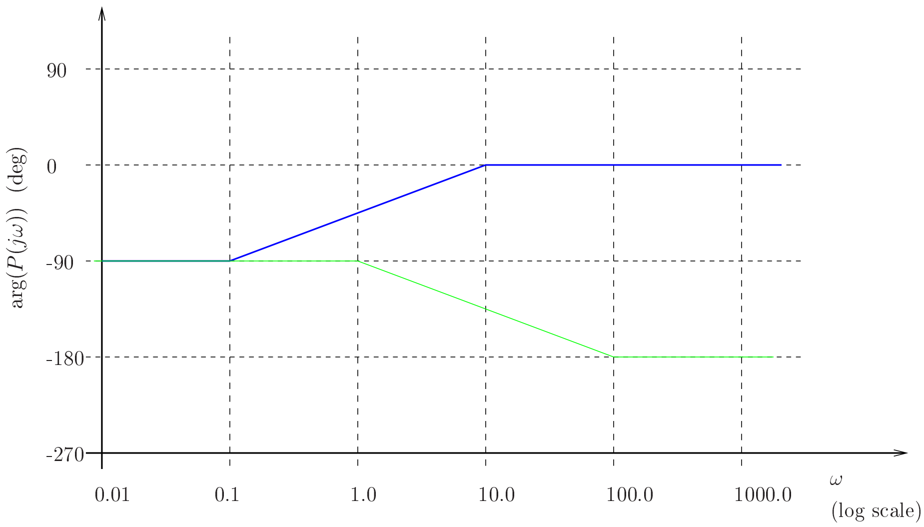


The MATLAB plot below is generated with the following commands.

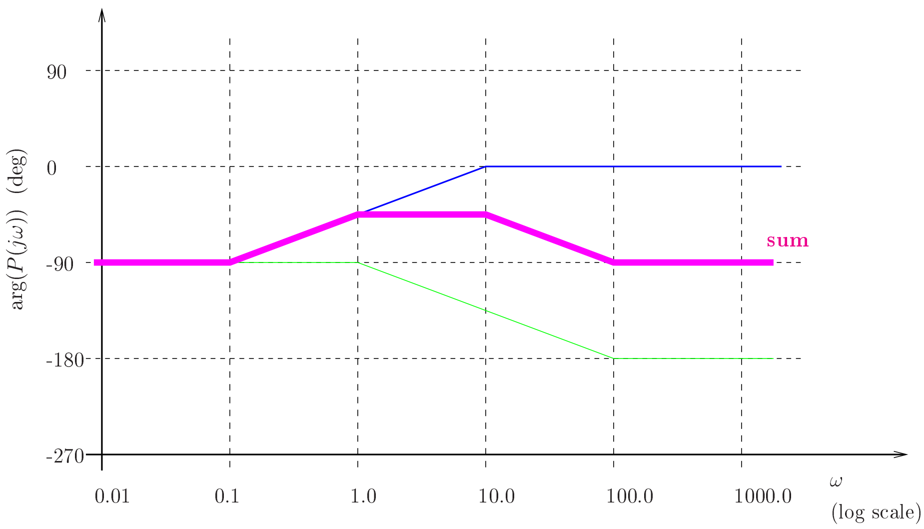
```
>>  num = [1  1];
>>  den = [1  10];
>>  w = logspace(-2,3,100);
>>  [mag, phase] = bode(num, den, w);
>>  loglog(w,mag,'m','linewidth',3);
>>  xlabel('Frequency (log scale)');  % axis labels
>>  ylabel('|P(jw)|');
>>  axis([0.01 1000 0.01 10]);        % set axes
>>  grid;
```



Step 3 Next, plot the phase contribution from each term. The contribution from the zero is shown in green, while the contribution from the pole is shown in blue.

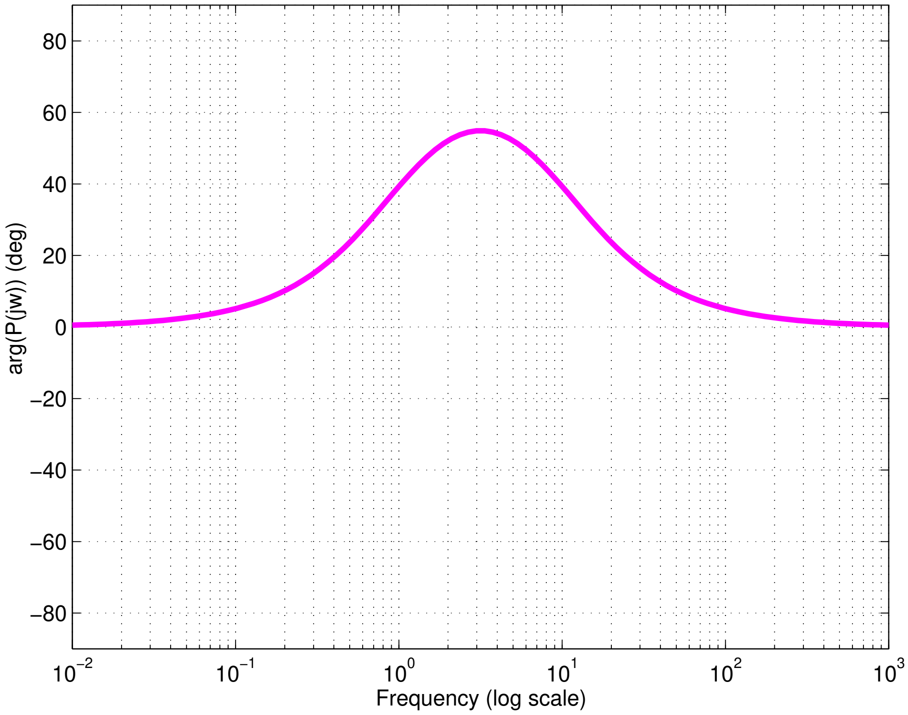


Step 4 Sum both terms to give the phase plot approximation.



The MATLAB plot of phase shown below is generated with the following commands.

```
>>  num = [1  1];
>>  den = [1  10];
>>  w = logspace(-2,3,100);
>>  [mag, phase] = bode(num, den, w);
>>  semilogx(w,phase,'m','linewidth',3); % note the semilogx command
>>  xlabel('Frequency (log scale)');     % axis label !
>>  ylabel('arg(P(jw)) (deg)');          % axis label !
>>  axis([0.01 1000 -90 90]);            % set axes range
>>  grid;
```



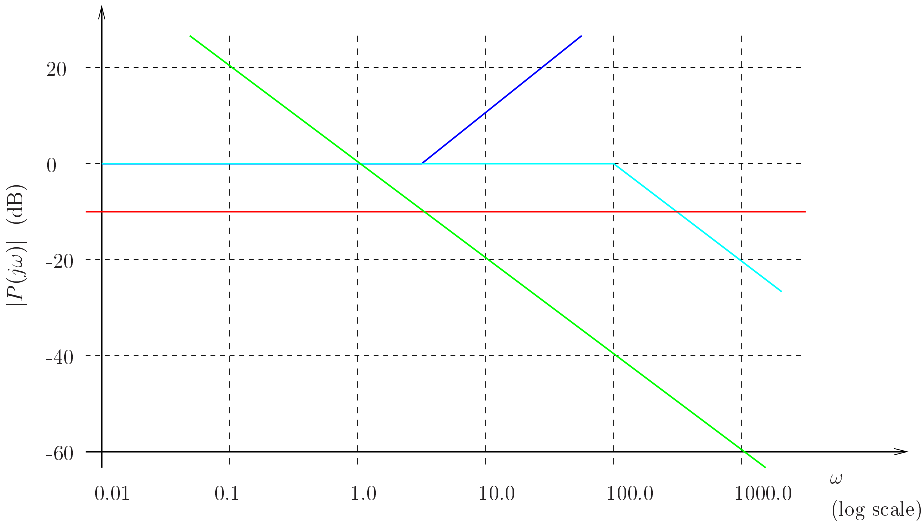
Example 2. Draw the magnitude and phase of the system

$$P(s) = \frac{10(s+3)}{s(s+100)}. \tag{2}$$

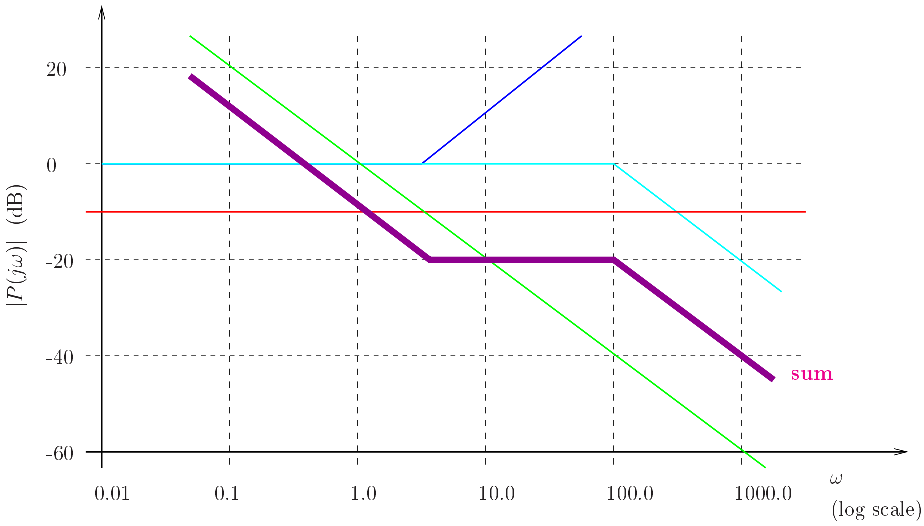
Step 0 Put into Bode form

$$P(s) = \frac{3}{10}\frac{1+s/3}{s(1+s/100)}.$$

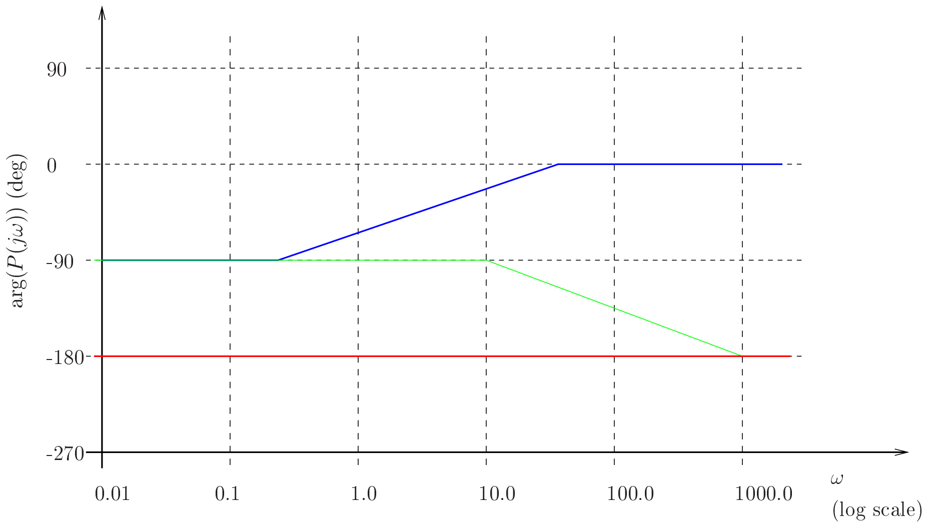
Step 1 Draw the contribution from the DC gain (3/10), the pole at 0, the zero at -3, and the pole at -100. The contribution due to the DC gain is in red, the contribution due to the zero is in blue, the contribution due to the pole at 0 is in green, and the contribution due to the pole at -100 is in cyan.



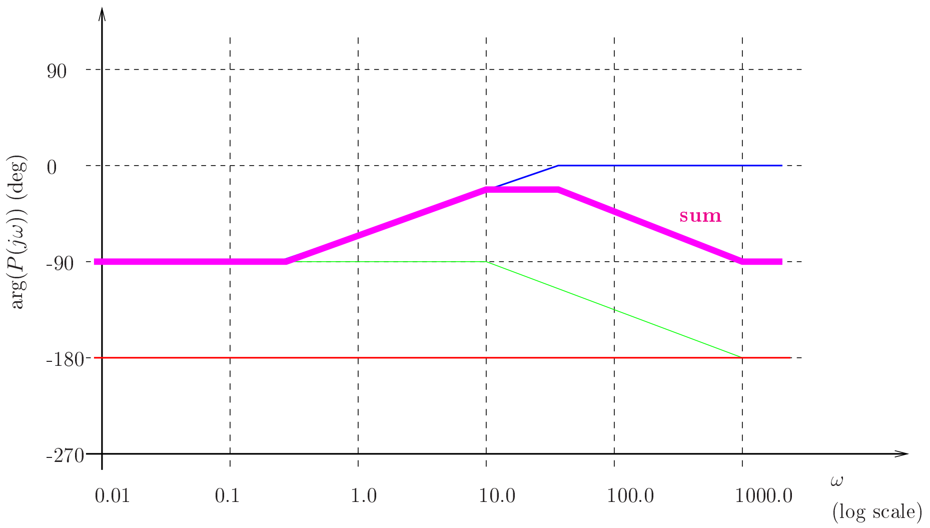
Step 2 Sum each contribution. The complete magnitude sketch is shown below in magenta.



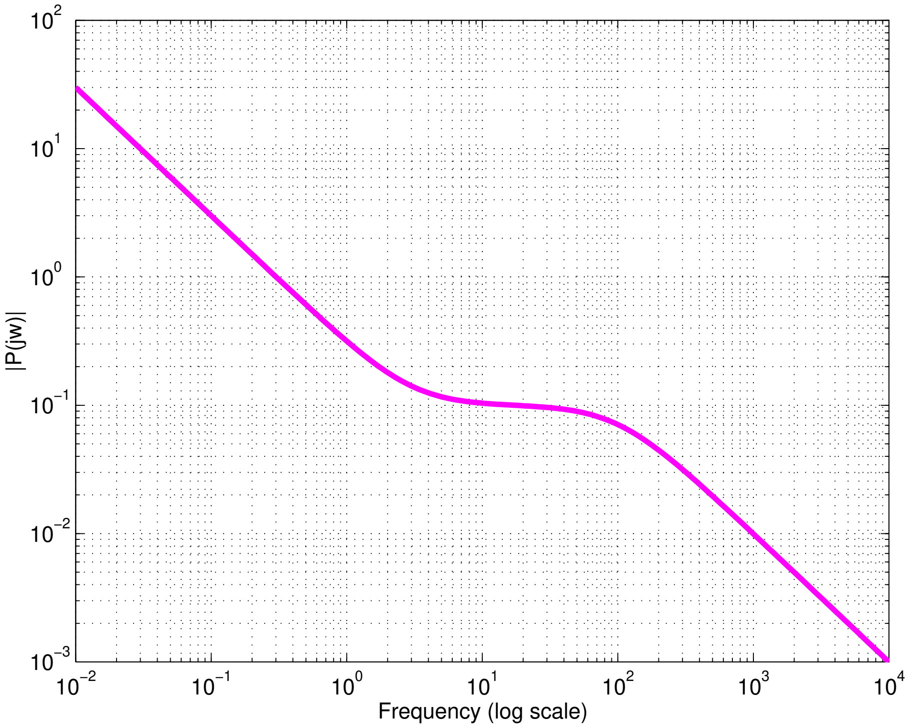
Step 3 Next, plot the phase contribution from each term. The contribution from the zero is shown in green, the contribution from the pole at -100 is shown in blue, and the contribution due to the pole at the origin is shown in red.

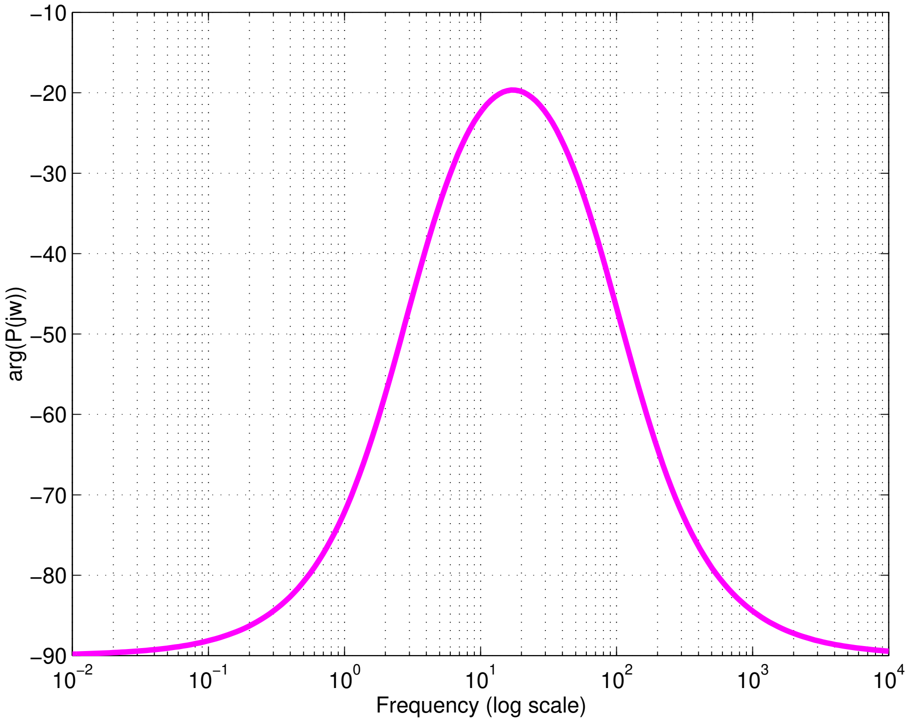


Step 4 Sum both terms to give the phase plot approximation.



The MATLAB plots for Example 2 are shown below. Note the axes units and labels.





Example 3. Draw the magnitude of the system

$$P(s) = \frac{s(s+100)}{(s^2+0.1s+4)}. \tag{3}$$

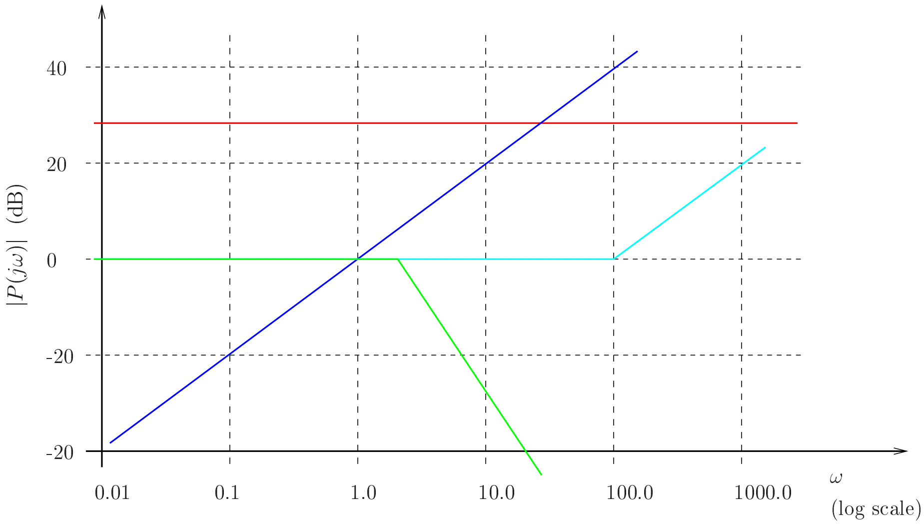
Step 0 Write in Bode form.

$$\frac{100}{4}\frac{s(1+s/100)}{\left(1+(0.1/4)\,s+(s/2)^2\right)}$$

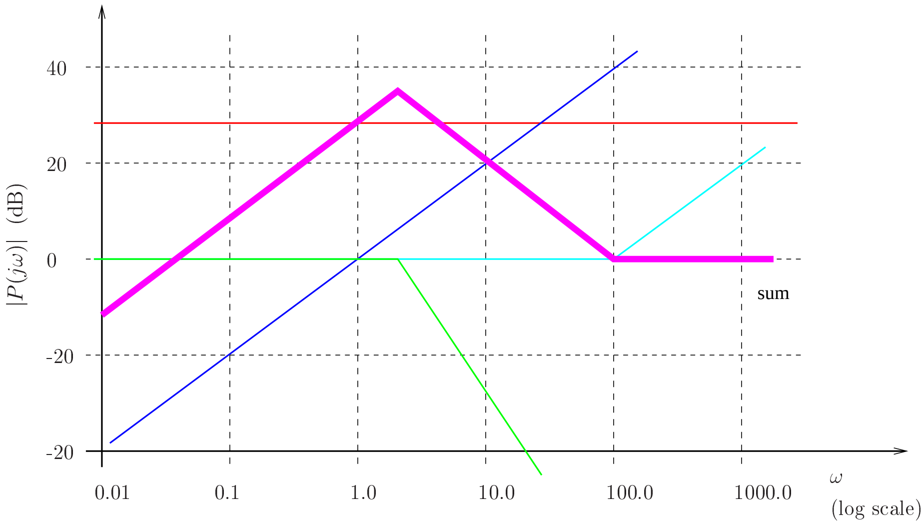
Thus, the DC gain is 25.

Step 1 Next, plot the contribution to each term. For the asymptotic plot, we first consider the second - order complex conjugate pole pair to be a real double pole at -2. (That is, assume the damping δ is one, for now.) The *asymptotic* plot is not a function of δ. Then, after the straight-line asymptotic plot is finished, we will take δ into account, refining the plot.

Below, the contribution due to the DC gain is in red, the contribution due to the zero at 0 is in blue, the contribution due to the zero at -100 is in cyan, the contribution due to the pole-pair at -2 is in green. Note the -40dB/decade slope of the latter.



Step 2 Sum the four terms, giving the plot below. In the next step, we will account for $\delta < 1$.



Step 3 The magnitude plot sketched in Step 2 does not take into account the damping term, which is less than 1. The plot should be refined to take this into account. To do this, we refer to the plots below, which show the magnitude and the phase for the second-order system

$$N(s) = \frac{1}{s^2 + 2\delta s + 1},$$

for five different values of damping: $\delta = 0.01,\ 0.03,\ 0.1,\ 0.3,\ 0.7.$

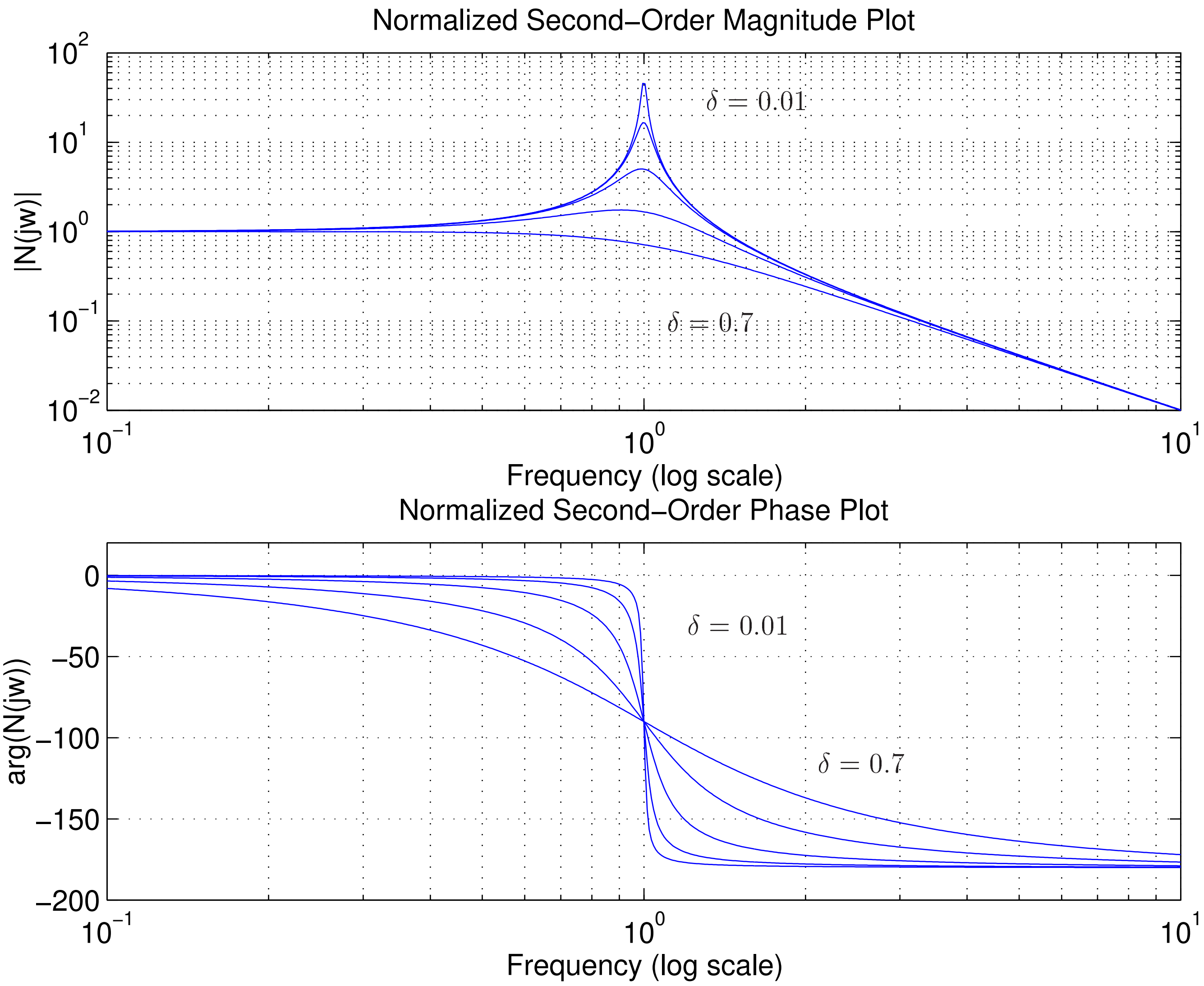
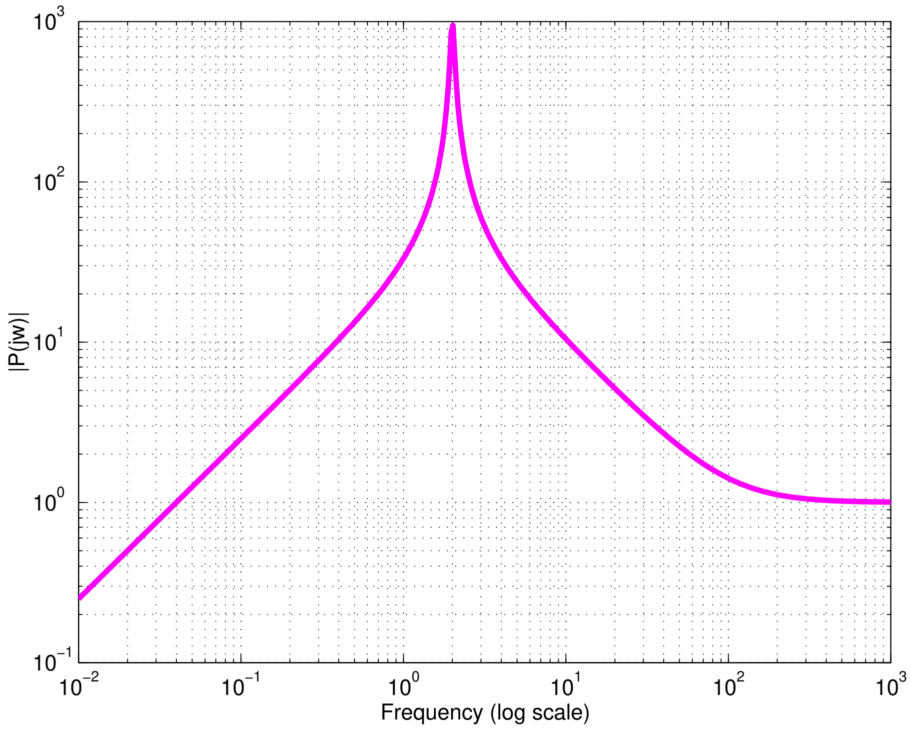


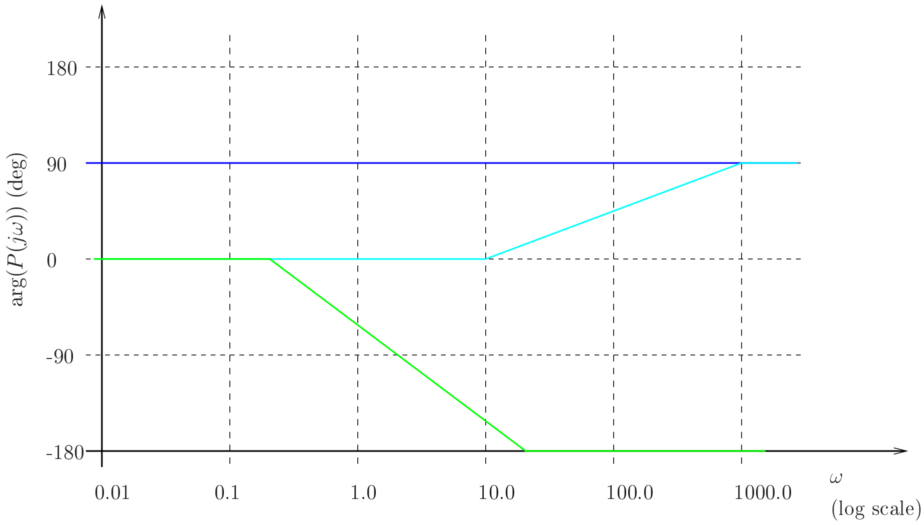
Figure 1: Magnitude and phase for $N(j\omega)$, as a function of $\delta = 0.01,\ 0.03,\ 0.1,\ 0.3,\ 0.7.$

From this plot, we see that $|N(j\omega)|$ has a peak magnitude of approximately 20dB above the DC gain for $\delta = 0.025$. Thus, we would expect the actual plot of $|P(j\omega)|$ to have a peak about 20dB above the asymptotic plot at $\omega = 2$. This is roughly the case, as can be seen on the actual plot is shown below, using MATLAB. Note that the y-axis is not plotted in dB here.

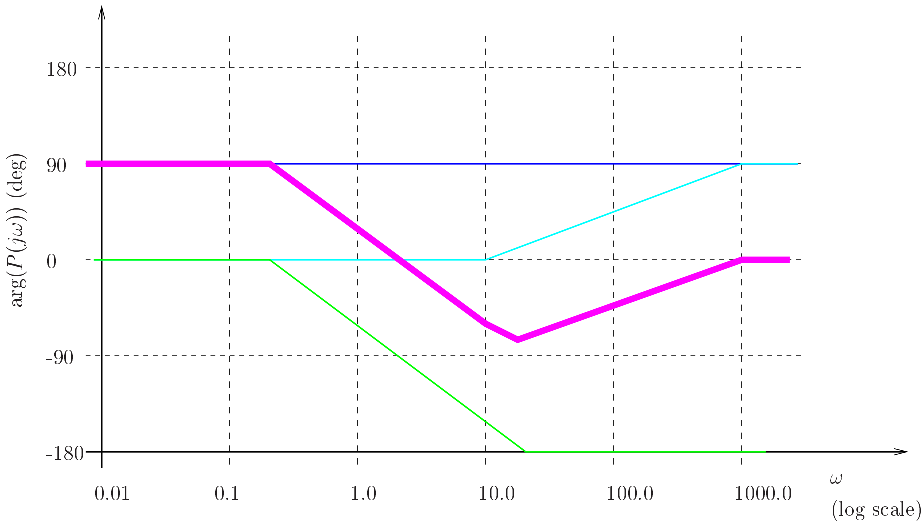
```
>> num=[1 100 0];
>> den=[1 0.1 4];
>> w=logspace(-2,3,500);
>> [mag,phase]=bode(num,den,w);
>> loglog(w,mag,'m','linewidth',3);
>> grid
>> xlabel('Frequency (log scale)')
>> ylabel('|P(jw)|')
```



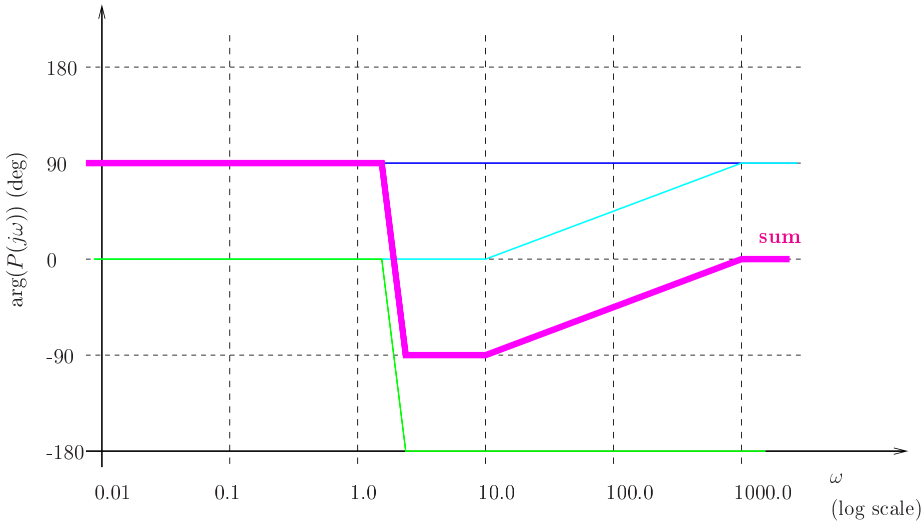
Step 4 Next, we plot the phase contribution from each term. There are three terms. The first is due to the zero at the origin, which contributes a constant $+90°$, and is drawn in blue below. The contribution due to the zero at -100 is shown in cyan. Finally, the contribution due to the double complex-conjugate pole is first sketched by assuming that the damping $\delta = 1$, i.e. the poles are both real and at -2, just as done for the magnitude plot.



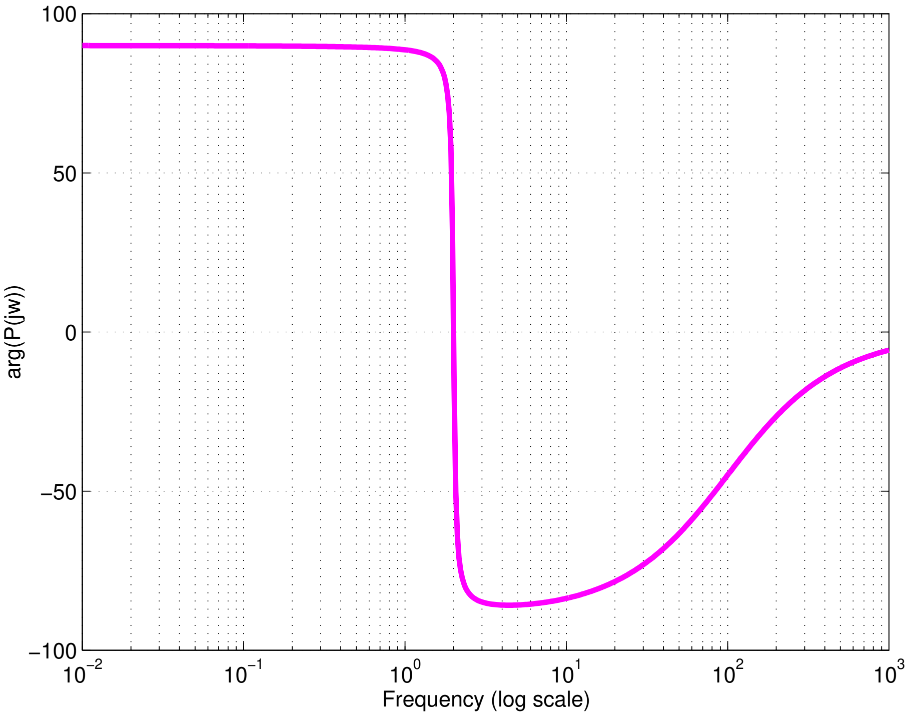
Step 5 Sum both terms to give the phase plot approximation.



Step 6 Finally, just as with the magnitude plot, the phase plot should be refined to take into account the damping is not one, but $\delta = 0.025$. To do this, we use the normalized phase plot shown in Figure 1. The small damping means the contribution due to the complex pole pair is "steeper," as shown in green below. The sum is also drawn.



The MATLAB plot of phase for $P(s)$ is shown below.



Example 4. Draw the magnitude and phase for the system

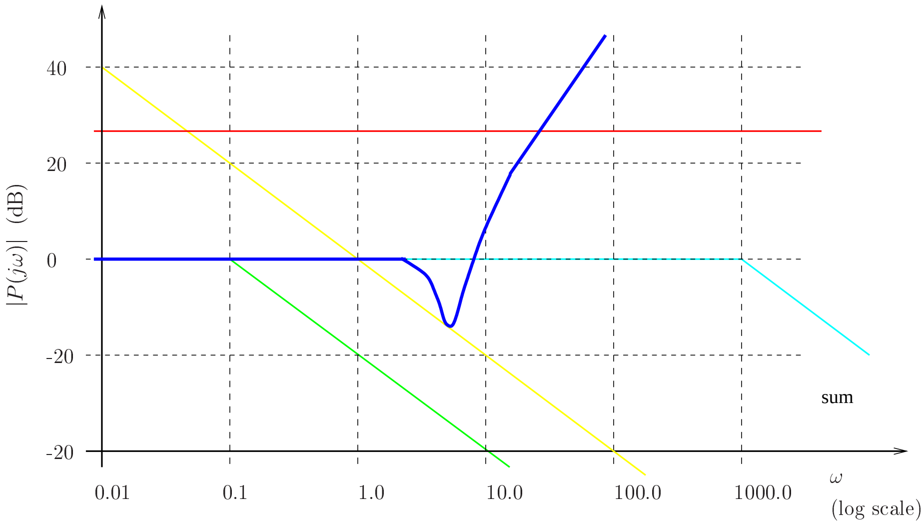
$$P(s) = \frac{100(s^2 + s + 49)}{s(s + 0.1)(s + 1000)} \tag{4}$$

Step 0 Again, first write $P(s)$ in Bode form:

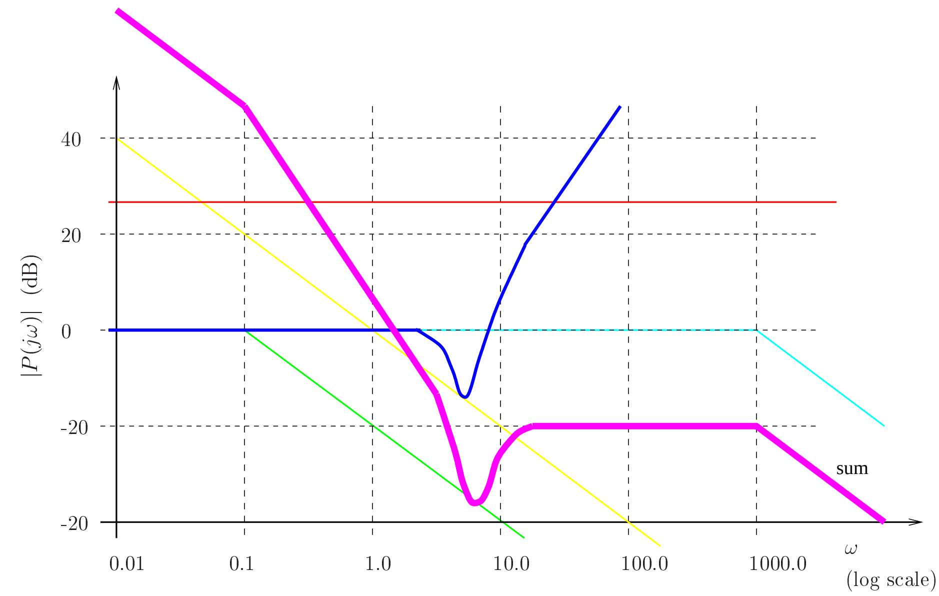
$$P(s) = \frac{4900}{100} \frac{(1 + (s/49) + (s/7)^2)}{s(1 + s/0.1)(1 + s/1000)}$$

Thus, the DC gain is $49 \approx 33$dB.

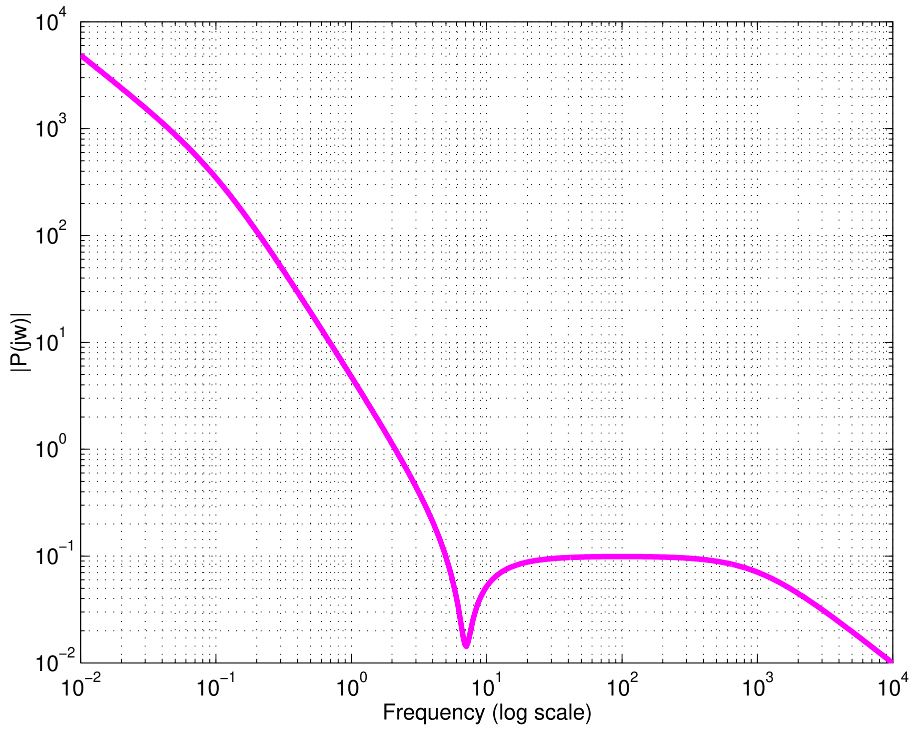
Step 1 Sketch the magnitude contribution for each term. The damping of the complex-conjugate zero pair is $\delta = 1/(2 \cdot 7) = 0.07$. This will attenuate the magnitude of the complex-conjugate term by about 15 dB at $\omega = 7$ (or increase the gain by -15 dB), which can be determined by the normalized plot of $|N(j\omega)|$ in Figure 1. The contribution to the magnitude plot due to the complex zero-pair is drawn in blue, the contribution due to the pole at the origin is drawn in yellow, that due to the pole at -0.1 is drawn in green, and that due to the pole at -100 is drawn in cyan.



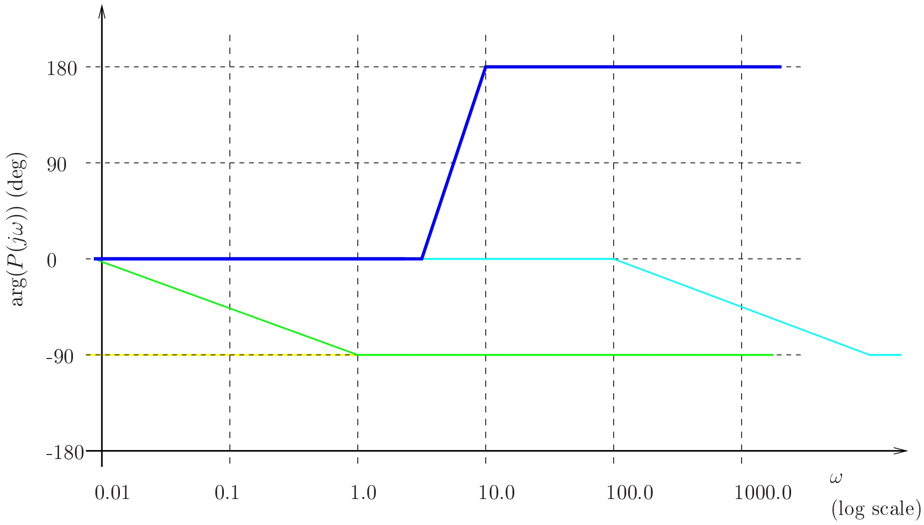
Step 2 Sum each term. The sum is shown in magenta below.



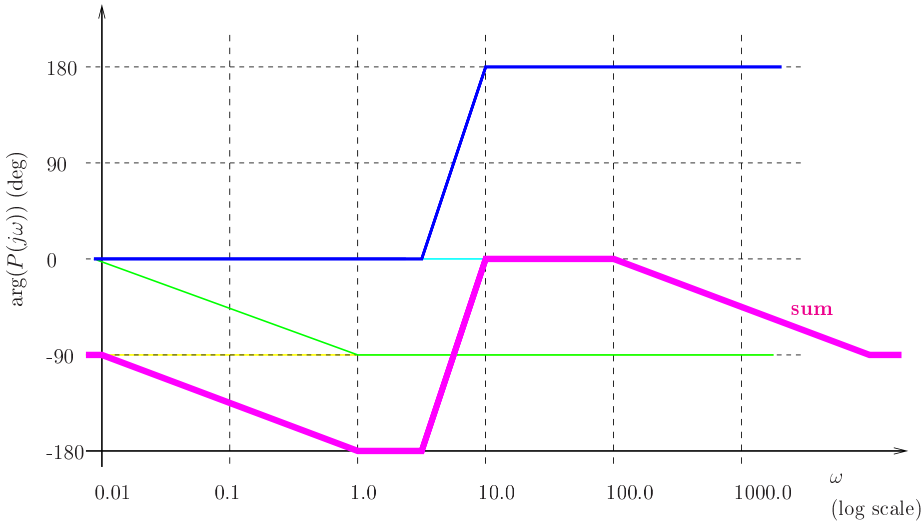
The MATLAB plot is shown below.



Step 3 Next, we plot the phase contribution from each term. There are four terms. The first is due to the pole at the origin, which contributes a constant $-90°$, and is drawn in yellow below. The contribution due to the pole at -0.1 is shown in green. The contribution due to the pole at -1000 is drawn in cyan. Finally, the contribution due to the double complex-conjugate zero is drawn in blue. Note that the effect of $\delta = 1/14$ is taken into account by referring to the plot of $\arg(N(j\omega))$ in Figure 1, for $\delta = 1/14 \approx 0.07$.

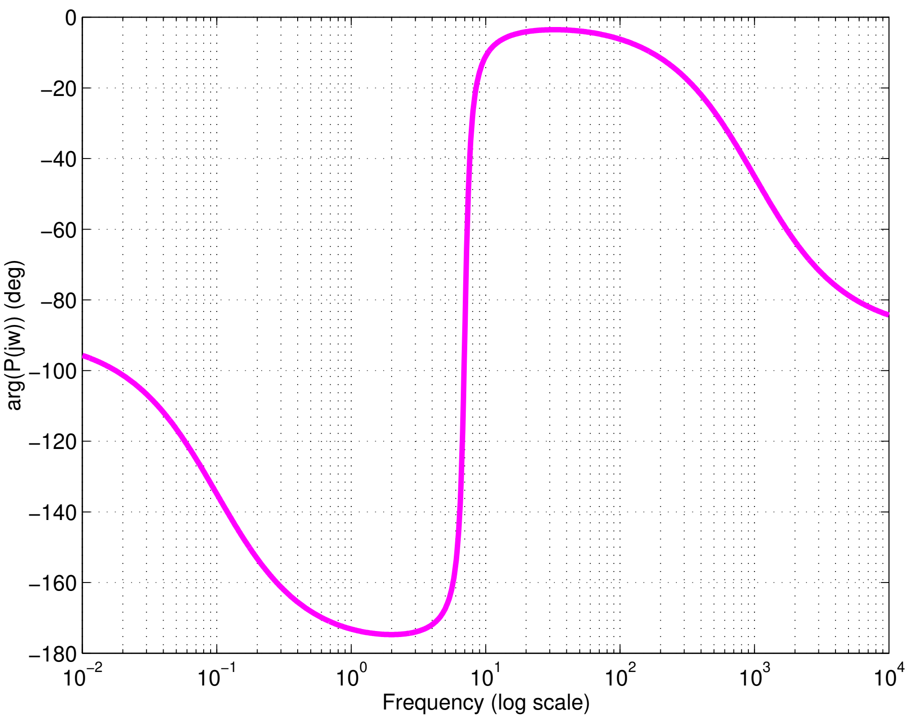


Step 4 Sum all terms to give the phase plot approximation.



The MATLAB plot of phase for $P(s)$ is shown below. Note the steep slope near $\omega = 7$, due to the double zero near the $j\omega$-axis. (Think about the phasor from one of these zeros, to the $j\omega$-axis, as the frequency is swept past the zero. Its phase will change very quickly as a function of ω from $-90°$ to $+90°$.)

```
>> num=100*[1 1 49];
>> den=[1 0.1 0];
>> den=conv(den,[1 1000]);
>> w=logspace(-2,4,1000);
>> [mag,phase]=bode(num,den,w);
>> semilogx(w,phase,'m','linewidth',3);
>> grid;
>> xlabel('Frequency (log scale)');
>> ylabel('arg(P(jw)) (deg)');
```



3 Non-Minimum Phase and/or Unstable Systems

If $P(s)$ has a pole and / or a zero in the right-half plane, then the magnitude and phase plots may be sketched as follows.

1. Transform $P(s)$ into a related minimum-phase, stable system, denoted by $\widehat{P}(s)$.
2. Sketch $|\widehat{P}(j\omega)|$, just as in Examples 1-4. We shall show that $|\widehat{P}(j\omega)| = |P(j\omega)|$.
3. Transform $P(s)$ into a different, stable, minimum-phase, stable system, denoted by $\widetilde{P}(s)$.
4. Sketch the plot of $\arg(\widetilde{P}(j\omega))$. We shall show that $\arg(P(j\omega)) = \arg(\widetilde{P}(j\omega)) + k\cdot\pi$, i.e., the phase of P differs from $\widehat{P}$ by a fixed constant. Thus, *shifting* the plot of $\arg(\widetilde{P}(j\omega))$ by $k\cdot\pi$ radians gives the plot of $\arg(P(j\omega))$.

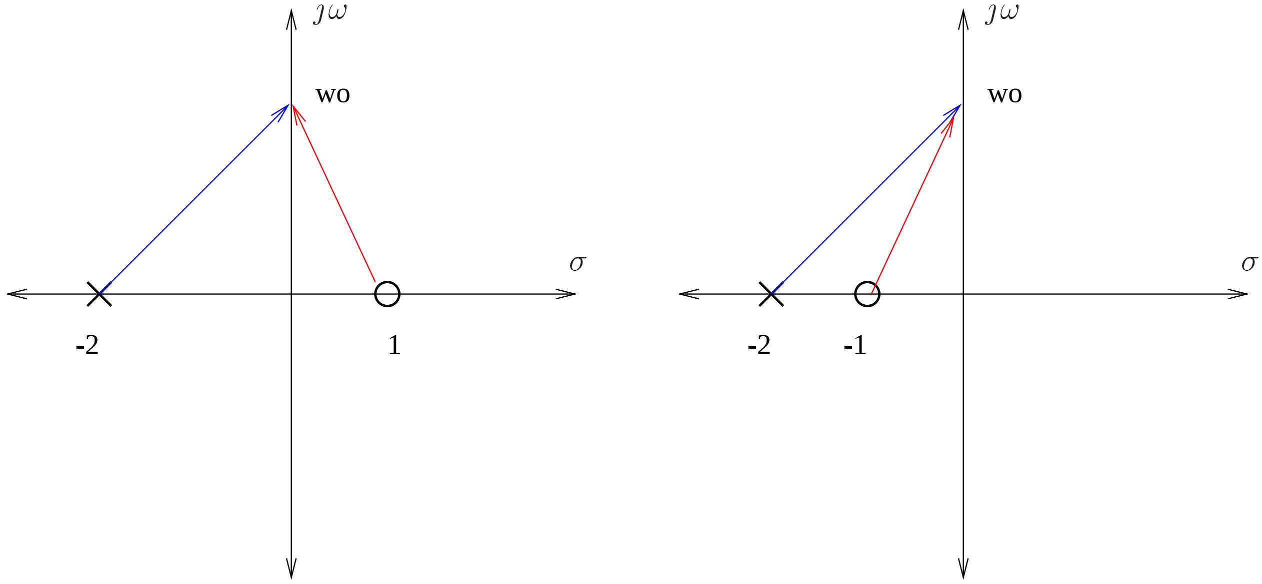
The technique will be illustrated by four examples.

3.1 Magnitude Plots for Non-Minimum Phase / Unstable Systems

Example 5 Sketch the magnitude of the system

$$P(s) = \frac{(s-1)}{(s+2)}. \tag{5}$$

The pole-zero diagram for this system is drawn below, to the left. Recall that $|P(j\omega_0)|$ is the ratio of the length of the phasor drawn from the zero at $+1$ to $j\omega_0$, to the length of the phasor drawn from the pole at -2 to $j\omega_0$. These phasors are also drawn below, to the left.



Now, the *length* of the phasor from $+1$ to $j\omega_0$ is identical to the *length* from -1 to $j\omega_0$. Thus, if we define a new *stable, minimum-phase* transfer function

$$\widehat{P}(s) = \frac{s+1}{s+2}$$

by moving the zero at $+1$ to a zero at -1, we must have $|P(j\omega)| = |\widehat{P}(j\omega)|$. Plotting $|P(j\omega)|$ is similar to Example 1, and is left to the reader.

In general, to plot the magnitude of an unstable and/or non-minimum phase plant $P(s)$, define a new transfer function $\widehat{P}(s)$ by **reflecting all unstable poles and/or zeros across the $j\omega$ axis**. Plot $|\widehat{P}(j\omega)|$, which is stable and minimum phase.

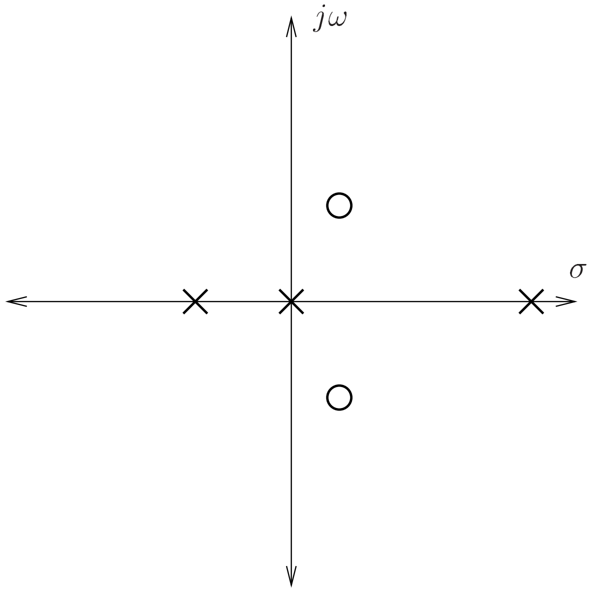
Example 6 Plot the magnitude of the plant

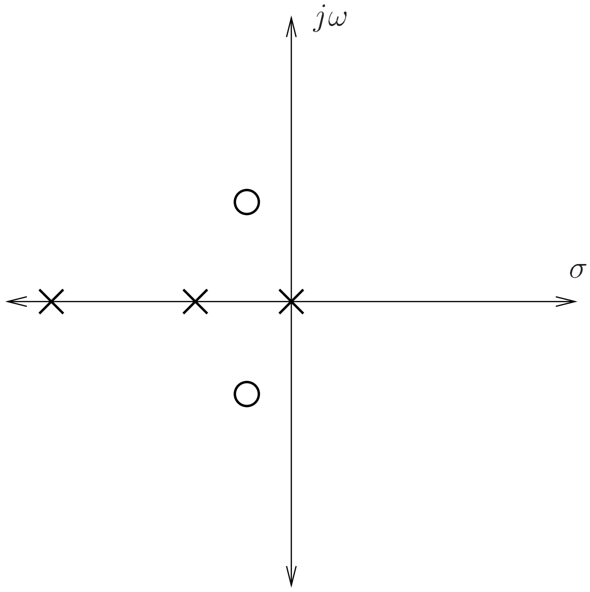
$$P(s) = \frac{(s^2 - s + 4)}{s(s+1)(s-10)}. \tag{6}$$

For this plant,

$$\widehat{P}(s) = \frac{(s^2 + s + 4)}{s(s+1)(s+10)}$$

The pole-zero plots for $P(s)$ and $\widehat{P}(s)$ are shown below at the left and right, respectively.



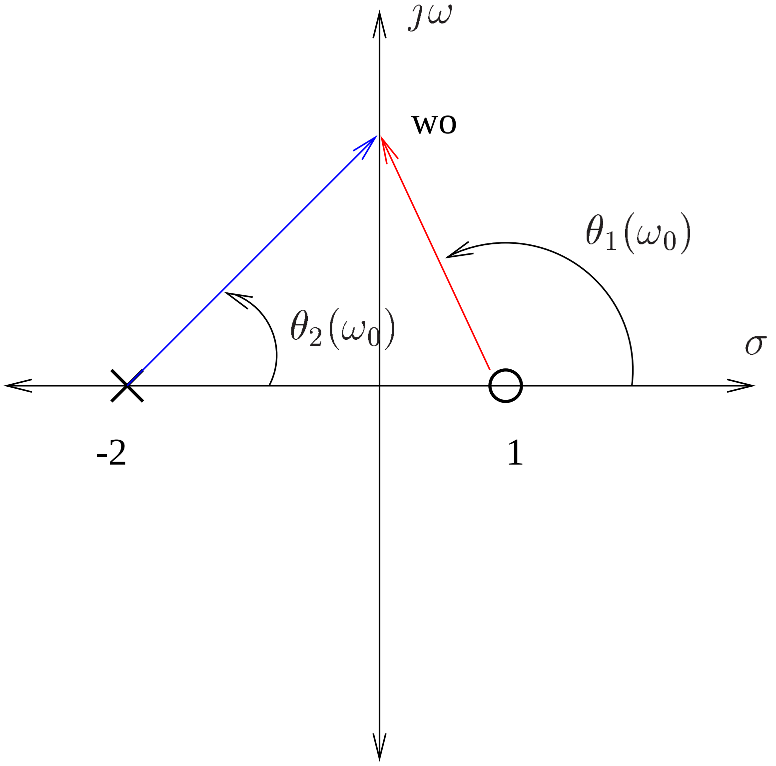


3.2 Phase Plots for Non-Minimum Phase / Unstable Systems

Example 7 Plot the phase for the system

$$P(s) = \frac{(s-1)}{(s+2)}. \tag{7}$$

The pole-zero diagram for $P(s)$ is sketched below. Recall that, in general, **the phase of** $P(j\omega_0)$



is the sum of the phases of the zeros, minus the sum of the phases of the poles. (The phrase "phase of a pole" means the phase of the phasor from the pole to a point on the $j\omega$-axis. Ditto for "phase of a zero.") In this example, $\arg(P(j\omega)) = \theta_1(\omega_0) - \theta_2(\omega_0)$, where θ_1 and θ_2 are defined in the figure.

Now, consider a new plant, denoted $\widetilde{P}(s)$ defined from $P(s)$ as follows:

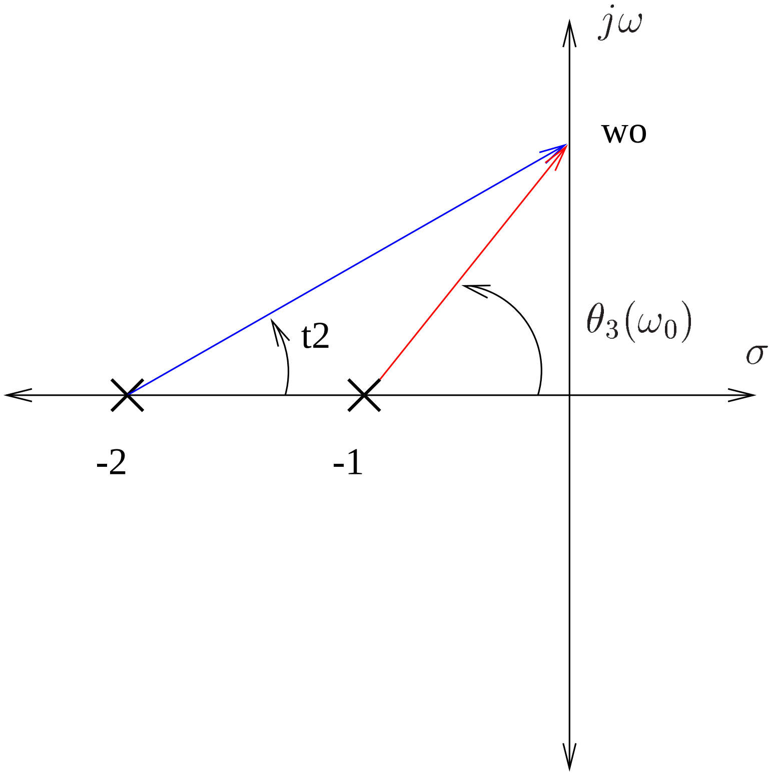
- Take any **pole** of $P(s)$ in the right-half plane, at $s = \sigma_1 \pm \omega_1$, where $\sigma_1 > 0$, and make it a **zero** of $\widetilde{P}(s)$ at $s = -\sigma_1 \pm \omega_1$.
- Take any **zero** of $P(s)$ in the right-half plane, at $s = \sigma_2 \pm \omega_2$, where $\sigma_2 > 0$, and make it a **pole** of $\widetilde{P}(s)$ at $s = -\sigma_2 \pm \omega_2$.
- The remainder of the poles and zeros of $P(s)$, i.e., those in the closed left-half plane, are also poles and zeros of $\widetilde{P}(s)$.

Note: $\widetilde{P}(s)$ may not be proper, but that does not matter. You may still plot $\arg(\widetilde{P}(j\omega))$.

For the example at hand, we have

$$\widetilde{P}(s) = \frac{1}{(s+1)(s+2)},$$

which has the pole-zero plot drawn below.



For this system, $\arg(\tilde{P}(j\omega_0)) = -\theta_2(\omega_0) - \theta_3(\omega_0)$. But, referring to the pole-zero diagrams of both P and $\tilde{P}$, $\theta_1(\omega_0) + \theta_3(\omega_0) = \pi$. Thus, $\arg(P(j\omega_0)) = \arg(\tilde{P}(j\omega_0)) + \pi$. **To plot** $\arg(P(j\omega_0))$**, first plot** $\arg(\tilde{P}(j\omega_0))$**, which is stable and minimum-phase, and then add** π**.** This is similar to Example 1, and is omitted.

Example 8 Plot the phase for the system

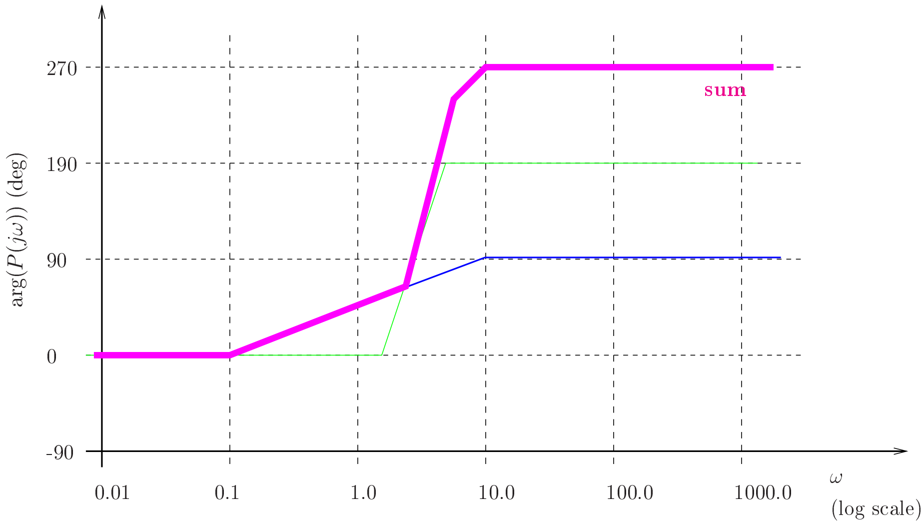
$$P(s) = \frac{(s+1)}{(s^2 - s + 10)}. \tag{8}$$

Step 0 For this system,

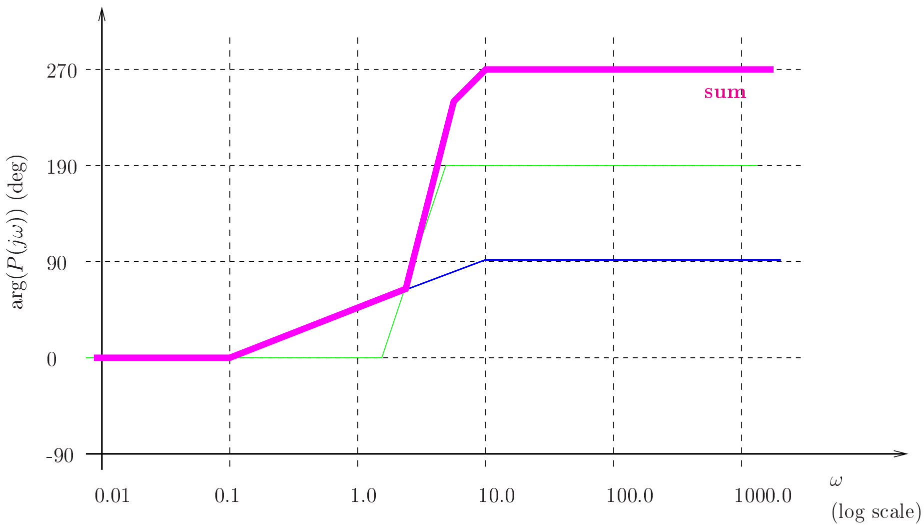
$$\tilde{P}(s) = \frac{(s+1)(s^2 + s + 10)}{1}.$$

Although this is not proper, its phase can still be sketched as in Examples 1-4.

Step 1 Sketch $\arg(\widetilde{P}(j\omega))$. This is shown below, following the same steps as in Examples 1-4.



Step 2 Shift the plot of $\arg(\widetilde{P}(j\omega))$ by 0 radians to get the plot of $\arg((j\omega))$ Why zero? Draw a pole-zero plot of P. Determine the phase at low frequency. It is 0, which is also the phase of $\widetilde{P}$ at low frequency.



The MATLAB phase plot for P is shown below.

```
>> num=[1 1];
>> den=[1 -1 10];
>> w=logspace(-2,4,100);
>> [mag,phase]=bode(num,den,w);
>> semilogx(w,phase,'m');
>> xlabel('Frequency (log scale)');
>> ylabel('arg(P(jw))');
>> grid;
```