ECE 1649 Project Number 1: System Identification
Monday January 13, 1997

Problem 1. (25 points) The least squares identification method minimizes the cost $J_{LS} = \int_0^t e_1(\tau) d\tau$ with respect to the parameter estimate $\theta(t)$. Using standard notation introduced in class, the solution is given as

$$\theta_{LS}(t) = \left[\int_0^t w(\tau) w^T(\tau) d\tau\right]^{-1} \cdot \left[\int_0^t w(\tau) y_p(\tau) d\tau\right], \tag{1}$$

where $y(t) = w(t)^T \theta^*$. Because we are interested in recursive formulations, we derived the following set of ordinary differential equations, whose solution is given by (1)

$$\begin{aligned} \dot{\theta}_{LS}(t) &= -P(t) w(t) e_1(t) && (2) \\ \dot{P}(t) &= -P(t) w(t) w^T(t) P(t), && (3) \end{aligned}$$

where $P(t) = \left[\int_0^t w(\tau) w^T(\tau) d\tau\right]^{-1}$ and where the initial condition $P(0)$ must exist. In practice, the differential equations (2)-(3) are *simulated* (in real time) with arbitrarily chosen initial conditions $P(0) = P_0$ and $\theta(0) = \theta_0$. In this case, the solution to (1)-(2) is *not* given by (1).

1. (10 points) Show that if P_0 and θ_0 are arbitrary, then the solution to (2) - (3) is given by

$$\theta_{RLS}(t) = \left[P_0^{-1} + \int_0^t w(\tau) w^T(\tau) d\tau\right]^{-1} \cdot \left[P_0^{-1}\theta_0 + \int_0^t w(\tau) y_p(\tau) d\tau\right]. \tag{4}$$

   Hint: Follow closely the derivation of (2)-(3) from class, and figure out what $P(t)$ is in this case. You do not require the matrix inversion lemma for this problem.

2. (5 points). Show that the parameter error is given by

$$\theta_{RLS}(t) - \theta^* = \left[P_0^{-1} + \int_0^t w(\tau) w^T(\tau) d\tau\right]^{-1} \cdot \left[P_0^{-1}(\theta_0 - \theta^*)\right]. \tag{5}$$

3. (10 points). Show that $\lim_{t\to\infty} \theta(t) = \theta^*$ if the *smallest* singular value of the matrix

$$\int_0^t w(\tau) w^T(\tau)\, d\tau,$$

   denoted $\sigma_{\min}(t)$, satisfies $\lim_{t\to\infty} \sigma_{\min}(t) = \infty$. In this case, the matrix can be considered an operator that is unbounded "in every direction."

Problem 2. (25 points). This problem considers the gradient identification algorithm with projection. Suppose it is known that the true parameters (θ^*) of a plant lie within a convex set Ω. Define the gradient algorithm with projection as

$$\dot{\theta}(t) = \begin{cases} -g e_1(t) w(t) & \text{if} \quad \theta(t) \in \text{ interior } \Omega \\ \text{proj}_{\partial\Omega}[-g e_1(t) w(t)] & \text{if} \quad \theta(t) \notin \Omega \end{cases}, \tag{6}$$

where $\partial\Omega$ is the boundry of the set Ω, and $\text{proj}_{\partial\Omega}[-g e_1(t) w(t)]$ means the projection of the vector $-g e_1(t) w(t)$ onto $\partial\Omega$. Using the "Liapunov" function $V = \tilde{\theta}^T \tilde{\theta}$, where $\tilde{\theta} = \theta(t) - \theta^*$, show $\dot{V}(t) \le 0$. What does this conclusion imply about the signals $\theta(t)$ and $e_1(t)$ as $t \to \infty$? HINT: look at the

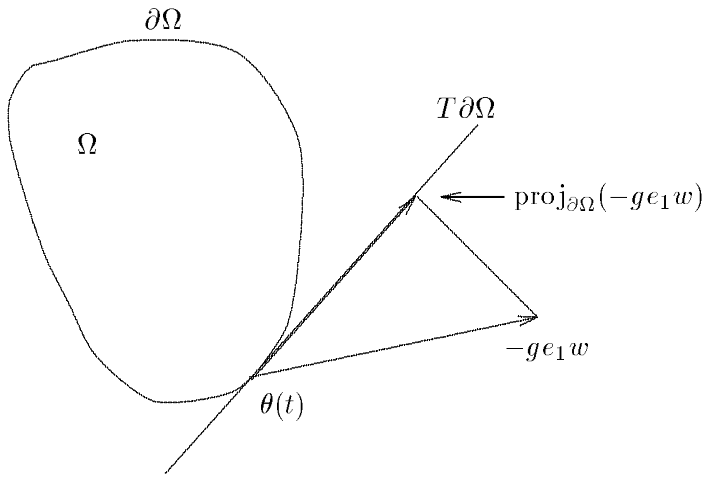

Figure 1: Gradient Projection for Problem 2.

theorems presented in the second class, and use the fact that the set $\boldsymbol{\Theta}$ is convex. Also, decompose the gradient vector into a normal and tangent component.

Problem 3. (50 points). Consider the second-order linear system

$$\hat{P}(s) = \frac{\hat{y}_p(s)}{\hat{r}(s)} = \frac{s-1}{s^2+s+10}. \tag{7}$$

This problem is a simulation study of the gradient and least squares methods of parameter identification.

1. (20 points). Simulate the system with the gradient identification algorithm, for the following three inputs $r(t)$:

   - $r(t) = 1$ (constant);
   - $r(t) = \sin(t)$;
   - $r(t) =$ square wave, of period 2π seconds.

   Set the gain $g = 10$, and use the same set of initial conditions for each of the three simulation studies. Choose the initial parameter estimates $\theta(0)$ to be zero. Set all other initial states to be zero. For which inputs do the parameter estimates converge to their actual values? How long does this take?

2. (20 points). Simulate the system for the same three inputs using the normalized recursive least squares identification algorithm. Choose the gain $g = 10$, and $\gamma = 1.0$. Choose the initial condition $P_0 =$ Identity matrix. For which inputs do the parameters converge? How long does this take? Compare with the gradient algorithm.

3. (10 points). Simulate the system for the same three inputs using the normalized recursive least squares identification algorithm with covariance reset. Reset $P(t)$ to the identity when $\lambda_{\min}(P(t)) = 0.01$. Compare the rate of convergence with your results for the non-resetting case.