ECE 1648 Problem Set 2
January 25, 1999

Problem 6. Suppose we are given a system

$$\Sigma: \quad \dot{x} = f(x) + u \cdot g(x),$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}$. Now, Σ is feedback linearizble, i.e., there exist a change of coordinates $z = T(x)$ and a nonlinear input $u = \alpha(x) + \beta(x)\nu$ such that $\dot{x} = f(x) + \alpha(x) \cdot g(x) + \beta(x)\nu$ has the form

$$\dot{z} = Az + b\nu$$

in the z-coordinates, if and only if

1. $\{g,\ \mathrm{ad}_f g,\ \mathrm{ad}_f^n g, \ldots,\ \mathrm{ad}_f^{n-1} g\}$ is full-rank (n), and
2. $\{g,\ \mathrm{ad}_f g,\ \mathrm{ad}_f^n g, \ldots,\ \mathrm{ad}_f^{n-2} g\}$ is involutive.

Suppose Σ itself is linear, i.e., $f(x) = Fx$ and $g(x) = G$, for some $n \times n$ matrix F and n-vector G. Write 1 and 2 in terms of F and G, and show 2 is always satisfied.

Problem 7. Consider the system

$$\Sigma: \quad \begin{array}{rcl} \dot{x}_1 &=& x_2 - 3x_1x_2^2 + 3x_2^5 + 3x_2^2 u \\ \dot{x}_2 &=& -x_1 + x_2^3 + u. \end{array}$$

Is Σ feedback linearizable? If so, determine a control $u = u_1(x_1, x_2)$ that makes 0 globally exponentially stable. Next, design a linear control $u = u_2 = k_1x_1 + k_2x_2$, and compare the two via simulation. Note that to make a fair comparison, the two linear models should have the same eigenvalues.

Problem 8. Consider the second-order nonlinear system

$$\Sigma: \quad \dot{x} = f(x) + u \cdot g(x),$$

where $x \in \mathcal{R}^2$, $u \in \mathcal{R}$, $f(0) = 0$. Show that Σ is feedback linearizable if and only if its Jacobian linearization at the origin is completely controllable. (The Jacobian linearization is $\dot{z} = Fz + Gu$, where $F = \frac{\partial f}{\partial x}|_{x=0}$ and $G = g(0)$.)

Problem 9. Show that the statement from Problem 8 is false if Σ is third-order.

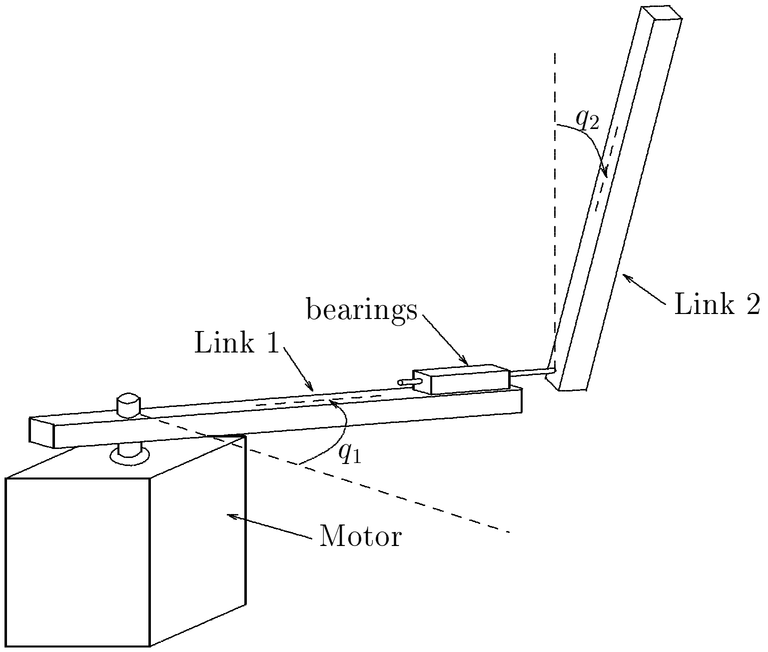

Figure 1: The rotating inverted pendulum.

Problem 10. The rotating inverted pendulum is similar to the classical inverted pendulum problem, i.e., the cart and stick problem, except the cart is not linear, but rotates. See Figure 1. This makes the practical construction easier and consumes less laboratory space when compared with the linear version. However, the control problem, which usually involves stabilizing the system in its inverted position, is more challenging. If we define the states and parameters as in Table 1, then the dynamic equations can be expressed

$$\begin{bmatrix} J_1 + m_2 l_1^2 + J_2\left(1 - \cos\left(2q_2\right)\right) & -m_2 l_1 l_{c2} \\ -m_2 l_1 l_{c2} & J_2 \end{bmatrix} \cdot \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} m_2 l_1 l_{c2} \sin\left(q_2\right) \dot{q}_2^2 + J_2 \sin\left(2q_2\right) \dot{q}_1 \dot{q}_2 \\ -0.5 J_2 \sin\left(2q_2\right) \dot{q}_1^2 - m_2 g l_{c2} \sin\left(q_2\right) \end{bmatrix} = \begin{bmatrix} K_7 \\ 0 \end{bmatrix} \cdot u. \tag{1}$$

In order to simplify the analysis and design, we define the seven parameters $K_1, \ldots, K_7$ as in Table 2. Note that all quantities are in the appropriate MKS units, and the control input u is unitless. That is, K_7 has units of nt·m. With these definitions, we rewrite the equations of motion as

$$\begin{bmatrix} K_1 + K_2\left(1 - \cos\left(2q_2\right)\right) & -K_3 \cos\left(q_2\right) \\ -K_3 \cos\left(q_2\right) & K_2 \end{bmatrix} \cdot \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} +$$

Table 1: Rotating inverted pendulum parameter descriptions.

Parameter	**Description**
q_1	Angle of link 1 (motor)
q_2	Angle of link 2 (pendulum)
J_1	Inertia of link 1 around axis 1
J_2	Inertia of link 2 around axis 2
m_2	Mass of link 2
l_1	Distance from axis 1 to link 2
l_{c2}	Distance from axis 2 to CM, link 2
g	Acceleration due to gravity (9.81)

Table 2: Rotating inverted pendulum parameter definitions.

Parameter	**Definition**	**Value**	**Type**
K_1	$J_1 + m_2 l_1^2$	0.096	Inertial
K_2	J_2	0.017	Inertial
K_3	$m_2 l_1 l_{c2}$	0.0192	Inertial
K_4	$m_2 g l_{c2}$	0.539	Gravity on link 2
K_5	b_1	0.056	Damping, link 1
K_6	b_2	0.006	Damping, link 2
K_7	K_7	0.44	Normalizing gain

$$\begin{bmatrix} K_3 \sin(q_2)\, \dot{q}_2^2 + K_2 \sin(2q_2)\, \dot{q}_1 \dot{q}_2 + K_5 \dot{q}_1 \\ -0.5 K_2 \sin(2q_2)\, \dot{q}_1^2 - K_4 \sin(q_2) + K_6 \dot{q}_2 \end{bmatrix} = \begin{bmatrix} K_7 \\ 0 \end{bmatrix} \cdot u. \tag{2}$$

The first step in our design is to express (2) in state space. Define the state as

$$\boldsymbol{x} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T = [q_1 \;\; q_2 \;\; \dot{q}_1 \;\; \dot{q}_2]^T,$$

and write $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + u \cdot \boldsymbol{g}(\boldsymbol{x})$. Is this system feedback linearizable? The use of *Mathematica* is encouraged in this problem. Also, we will return to this example throughout the semester.