ECE 1648 Problem Set 1
January 18, 1999

Notation and Background: Let $\mathcal{C}^p(\mathcal{R}^n)$, or C^p for short, denote the vector space of functions that map $\mathcal{R}^n$ into $\mathcal{R}$ and are differentiable at least p times. A vector field in $\mathcal{R}^n$ can be thought of as an object that has a magnitude and a direction for each point of $\mathcal{R}^n$. Thus, we can denote a vector field $\boldsymbol{f}(\boldsymbol{x})$ as

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix}, \tag{1}$$

where $f_i(\boldsymbol{x}) \in \mathcal{C}^p$, for some positive integer p, $1 \leq i \leq n$. A vector field can also be defined as an *operator* on differentiable functions $\mathcal{C}^p$. Specifically, if $h \in \mathcal{C}^p$, then $\boldsymbol{f}(h)$ is defined as

$$\boldsymbol{f}(h) = f_1 \cdot \frac{\partial h}{\partial x_1} + \cdots + f_n \frac{\partial h}{\partial x_n},$$

which is the directional derivative of h in the $\boldsymbol{f}$ direction. Thus, we can *define* a vector field as a map $\boldsymbol{f} : \mathcal{C}^p \to \mathcal{C}^{p-1}$ that satisfies the following two properties

Linearity: $\boldsymbol{f}(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 \boldsymbol{f}(h_1) + \alpha_2 \boldsymbol{f}(h_2)$, where $\alpha_1,\, \alpha_2 \in \mathcal{R}$, and $h_1,\, h_2 \in \mathcal{C}^p$, $p \geq 1$; and

Derivation: $\boldsymbol{f}(h_1 \cdot h_2) = h_1 \cdot \boldsymbol{f}(h_2) + h_2 \cdot \boldsymbol{f}(h_1)$, where $h_1,\, h_2 \in \mathcal{C}^p$, $p \geq 1$.

(The second property is commonly called the product rule.) Thus, any operator on $\mathcal{C}^p$ that satisfies these properties is a vector field, and can be written in the operator form

$$\boldsymbol{f} = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}. \tag{2}$$

Sometimes the notation $\boldsymbol{f}(h)$ can be confusing. This is because $\boldsymbol{f}$ is also a function of the coordinates $\boldsymbol{x}$, so sometimes we write $\boldsymbol{f}(\boldsymbol{x})$. This is not the same thing as $\boldsymbol{f}(h)$! Thus, the Lie derivative notation is sometimes prefered. This is

$$\mathcal{L}_{\boldsymbol{f}} h = \boldsymbol{f}(h) = \sum_{i=1}^{n} f_i \frac{\partial h}{\partial x_i}.$$

Finally, the set of all vector fields on $\mathcal{R}^n$ is itself a vector space, and is denoted $\mathcal{T}(\mathcal{R}^n)$, and is often called the *tangent space* of $\mathcal{R}^n$.

Problem 1. Consider the third-order linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u. \tag{3}$$

Compute the coordinate transformation (similarity transformation) $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ and the control $u = \boldsymbol{\alpha}\boldsymbol{x} + \nu$, so that the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{\alpha}\boldsymbol{x} + \nu)$, when written in the $\boldsymbol{z}$-coordinates is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \nu. \tag{4}$$

Problem 2. Let $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{T}(\mathcal{R}^n)$, and let $h \in \mathcal{C}^\infty(\mathcal{R}^n)$. By definition, both $\boldsymbol{f} : \mathcal{C}^\infty(\mathcal{R}^n) \to \mathcal{C}^\infty(\mathcal{R}^n)$ and $\boldsymbol{g} : \mathcal{C}^\infty(\mathcal{R}^n) \to \mathcal{C}^\infty(\mathcal{R}^n)$. Therefore, $\mathcal{L}_f \left(\mathcal{L}_g h\right) \in \mathcal{C}^\infty(\mathcal{R}^n)$, and we can consider the composition $\mathcal{L}_f\mathcal{L}_g$ to be an operator that maps $\mathcal{C}^\infty(\mathcal{R}^n) \to \mathcal{C}^\infty(\mathcal{R}^n)$. Show that this operator is *not* a vector field. (Hint: Write $\boldsymbol{f}$ and $\boldsymbol{g}$ in operator form (2), then compute $\mathcal{L}_f\mathcal{L}_g h$, for some h, and show that this can't be written in the form $w_1 \cdot \frac{\partial h}{\partial x_1} + \cdots w_n \cdot \frac{\partial h}{\partial x_n}$ for any functions $w_i \in \mathcal{C}(\mathcal{R}^n)$.)

Problem 3. Continuing Problem 2, we know that $\mathcal{L}_f\mathcal{L}_g$ is not a vector field. However, we can define the Lie Bracket of the two vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$ as

$$[\boldsymbol{f}, \boldsymbol{g}] = \mathcal{L}_f\mathcal{L}_g - \mathcal{L}_g\mathcal{L}_f. \tag{5}$$

Obviously, $[\boldsymbol{f}, \boldsymbol{g}] : \mathcal{C}^\infty \to \mathcal{C}^\infty$. Show that this *is* a vector field, i.e., that the operator satisfies the two properties that define vector fields.

Problem 4. If we express vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$ as in (1), then show that

$$[\boldsymbol{f}, \boldsymbol{g}] = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \cdot \boldsymbol{f} \;-\; \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \cdot \boldsymbol{g}$$

where $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}$ is the Jacobian of the vector field $\boldsymbol{g}$,

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

Note: $[\boldsymbol{f}, \boldsymbol{g}] = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \cdot \boldsymbol{f} \;-\; \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \cdot \boldsymbol{g}$ does not mean that $\mathcal{L}_f\mathcal{L}_g = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}$!

Problem 5. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathcal{T}(\mathcal{R}^n)$. Show the Lie bracket satisfies the following properties:

- It is skew-commutative, meaning $[\boldsymbol{X}, \boldsymbol{Y}] = -[\boldsymbol{Y}, \boldsymbol{X}]$.
- It is bilinear over $\mathcal{R}$, meaning $[\alpha_1\boldsymbol{X} + \alpha_2\boldsymbol{Y}, \boldsymbol{Z}] = \alpha_1[\boldsymbol{X}, \boldsymbol{Z}] + \alpha_2[\boldsymbol{Y}, \boldsymbol{Z}]$ for $\alpha_1, \alpha_2 \in \mathcal{R}$.
- It satisfies the Jacobi identity, i.e., $[\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]] + [\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]] + [\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] = 0$.