1. This is a standard LQR problem, with \( Q = I_{2\times2} \) and \( R = \epsilon > 0 \). The pair \((A,B)\) is controllable, hence stabilizable, which guarantees this problem to be solvable. The optimal control is \( K^* = -R^{-1}B^TP \), where \( P \) is the positive definite solution of (ARE):

\[
A^TP + PA - PBR^{-1}B^TP + Q = 0,
\]

i.e.

\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix} P + P \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} - \frac{1}{\epsilon} P \begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix} P + \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = 0.
\]

Solving the above equation, we obtain

\[
P = \begin{bmatrix}
\sqrt{1 + 2\sqrt{\epsilon}} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & \sqrt{(1 + 2\sqrt{\epsilon})\epsilon}
\end{bmatrix}.
\]

Therefore, the associated control law is

\[
K^* = \begin{bmatrix}
K^*x \\
-\frac{1}{\sqrt{\epsilon}} & -\sqrt{\frac{1 + 2\sqrt{\epsilon}}{\epsilon}}
\end{bmatrix}.
\]

The magnitude of \( u \) at \( t = 0 \), \( \|u(0)\| \), is

\[
u(0) = \|K^*x(0)\| = \frac{1}{\sqrt{\epsilon}} + \sqrt{\frac{1 + 2\sqrt{\epsilon}}{\epsilon}}.
\]

In Figure 1, we observe that as \( \epsilon \to 0 \), the control effort at the initial time goes to \( \infty \), as expected.

2. The state-space model of the system is

\[
\begin{align*}
\dot{x} &= \lambda x + u \\
y &= x.
\end{align*}
\]

Stabilizability and detectability are easily verified so the problem is solvable. We solve

\[
A^TP + PA - PBR^{-1}B^TP + C^TC = 0
\]

to obtain \( 2\lambda P - \frac{\lambda^2}{\epsilon} + 1 = 0 \). Solving for \( P \) and keeping in mind that \( P > 0 \), we get

\[
P = \epsilon \lambda + \sqrt{\lambda^2 \epsilon^2 + \epsilon}.
\]
and

\[ u^* = \left(-\lambda - \sqrt{\lambda^2 + \frac{1}{\epsilon}}\right)x. \]

Next we consider the closed-loop poles of the system

\[ \text{eig}(A + BK^*) = -\sqrt{\lambda^2 + \frac{1}{\epsilon}}. \]

We see that regardless of whether \( \lambda < 0 \) or \( \lambda > 0 \), if \( \epsilon \to 0 \) then the closed-loop poles approach \(-\infty\), whereas if \( \epsilon \to \infty \) then the poles approach \(-|\lambda|\).

3. Let \( A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \). It is easily verified that \((C, A, G)\) is stabilizable and detectable. Therefore, the Kalman filter can be computed as

\[ \dot{x} = (A - KC)x + Ky + Bu. \]

The optimal Kalman gain \( K \) is obtained by

\[ K = PC^T V^{-1}, \]

where \( P \) is the symmetric, positive definite solution of

\[ 0 = AP + PA^T + GWG^T - PC^T V^{-1} CP. \]

Solving the above equation, we obtain

\[ P = V \begin{bmatrix} -1 + \sqrt{1 + 2\beta} & 1 + \beta - \sqrt{1 + 2\beta} \\ 1 + \beta - \sqrt{1 + 2\beta} & -1 - 2\beta + (1 + \beta)\sqrt{1 + 2\beta} \end{bmatrix}, \]

and the optimal Kalman gain is

\[ K = P \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{V} & -1 + \sqrt{1 + 2\beta} \\ 1 + \beta - \sqrt{1 + 2\beta} \end{bmatrix}. \]
The poles of the Kalman filter are

\[ \text{eig}(A - KC) = \frac{1}{2} \left( -\sqrt{1 + 2\beta} \pm \sqrt{1 - 2\beta} \right). \]

When \( W = 1 \) and \( V \to 0 \), then \( \beta \to \infty \), and

\[ \frac{1}{2}(-\sqrt{1 + 2\beta} \pm \sqrt{1 - 2\beta}) \to \frac{\beta}{2} (-1 \pm j). \]

In other words, as \( \beta \to \infty \) the filter acts faster.