Instructions: Duration 1 hour
Please answer all questions.
All questions have equal value.
A one-page aid sheet is allowed.

1. (a) The state equations of a multivariable 2 input, 2 output system are given by:

\[ \begin{align*}
    x_1 &= x_2 + u_1 \\
    x_2 &= -x_1 - 6x_2 + u_2 \\
    y_1 &= x_1 \\
    y_2 &= x_2 + u_1
\end{align*} \]

The initial state is \( x(0) = [3, 2] \), and a unit step function input of \( (u_1, u_2) = (1, 1) \) is applied at \( t = 0 \). Determine the steady-state output response of the system as \( t \to \infty \).

(Hint: there is a "fast" way of doing this.)

(b) Consider the following systems:

\[ \begin{align*}
    \hat{x} &= \begin{bmatrix} \cos \omega & \sin \omega \\
                        -\sin \omega & \cos \omega \end{bmatrix} x + \begin{bmatrix} 0 \\
                                                 1 \end{bmatrix} u \\
    y &= \begin{bmatrix} 0 & 0 \\
                        1 & 0 \end{bmatrix} x
\end{align*} \]

where the voltage \( e(t) \) can be considered as the input to the system. Let \( x = \begin{bmatrix} x_1 \\
                                 x_2 \end{bmatrix} \) be the state of the system. Find conditions on \( R, L, C \) so that the system is controllable.

2. (a) Given a system described by:

\[ \begin{align*}
    \dot{x} &= \begin{bmatrix} -a_1 + \sin x_1 & \cos x_1 \\
                             \cos x_1 & -a_2 + \sin x_2 \end{bmatrix} x + \begin{bmatrix} 0 \\
                                                 1 \end{bmatrix} \begin{bmatrix} 0 \\
                                                 0 \end{bmatrix} u \\
    y &= \begin{bmatrix} 0 & 0 \\
                        1 & 0 \end{bmatrix} x
\end{align*} \]

for what values of \( a_1, a_2 \) is the system locally asymptotically stable about the origin?

(b) Consider the following system:

\[ \begin{align*}
    \dot{x} &= \begin{bmatrix} a_1 & a_2 & a_3 \\
                             a_4 & a_5 & a_6 \\
                             a_7 & a_8 & a_9 \end{bmatrix} x + \begin{bmatrix} 0 \\
                                                 0 \\
                                                 0 \end{bmatrix} u \\
    y &= \begin{bmatrix} 0 & 0 & 0 \\
                        1 & 0 & 0 \end{bmatrix} x
\end{align*} \]

For what values of \( a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \) is this system minimum phase?

3. Consider the following system:

\[ \begin{align*}
    \begin{align*}
    u_1 &
    \end{align*}
    \begin{align*}
    u_2 &
    \end{align*}
    \begin{align*}
    + &
    \end{align*}
    \begin{align*}
    y_1 &
    \end{align*}
    \begin{align*}
    y_2 &
    \end{align*}
\end{align*} \]

(a) Find a state space model of the system.
(b) Assume that \( \theta \neq -1, 1, 2 \). Find the minimal realization of the system obtained in (a).
(c) The following unstable system

\[ \begin{align*}
    \dot{x} &= \begin{bmatrix} 2 & 1 \\
                             0 & \theta \end{bmatrix} x + \begin{bmatrix} 2 \\
                                                       0 \end{bmatrix} u \\
    y &= \begin{bmatrix} 12 & 20 \end{bmatrix} x
\end{align*} \]

where \( \theta \) is unknown, can be stabilized by using the controller \( u = (-12, -13)x \) to produce closed loop eigenvalues \(-1, -2\). Determine if the resultant closed loop system obtained by applying \( u = (-12, -13)x \) to the system is observable.