A Nash Game Approach for OSNR Optimization with Capacity Constraint in Optical Links

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***Abstract*—This paper develops a Nash game towards optimizing optical signal-to-noise ratio (OSNR) in the presence of link capacity constraint. In optical wavelength-division multiplexing (WDM) networks, all wavelength-multiplexed channels in a link share the optical fiber. In order to limit nonlinear effects, the total power launched into a fiber has to be below a nonlinearity threshold. This can be regarded as the optical link capacity constraint. We formulate an extended OSNR Nash game that incorporates this underlying system constraint. The status of an optical link is considered directly in the cost function. Sufficient conditions for existence and uniqueness of the Nash equilibrium (NE) solution are given. Two iterative algorithms for channel power control are proposed to compute the NE solution: a parallel update algorithm (PUA) and a relaxed PUA (r-PUA). Their convergence is studied under different conditions, both theoretically and numerically.**

***Index Terms*—OSNR, capacity constraint, Nash game, iterative control algorithms.**

I. INTRODUCTION

IN recent years, game theoretic models and approaches have attracted considerable interest in the communication, control and networking communities, [1]–[9], as an alternative to traditional system-wide optimization, [10]–[13]. The interest in game theory approaches is motivated by the departure from the assumption of cooperation among users and network utilities. In large-scale networks, decisions are often made independently by users according to their own performance objectives, [1]. This makes noncooperative game theory a suitable framework for networks.

In a noncooperative (Nash) game, players are self-interested: each player minimizes its own cost function (or maximizes its net utility) in the presence of all other players, [14]. The game settles at a Nash equilibrium (NE) if one exists. A beneficial feature of a game formulation is that it can lead itself to iterative decentralized algorithms, which are useful in networking environments. There exists a rich literature on game theoretic approaches for network flow optimization (congestion control), [2]–[4], [15], or power allocation in wireless networks, [5]–[8]. Fewer results exist for game theoretic approaches in optical networks. Very recently a first such result appeared in [9], where a noncooperative game was formulated and efficiently solved towards optimizing channel optical signal-to-noise ratio (OSNR). Studies of noncooperative behavior and game theory in optical networks represent a promising new area, relevant in the context of evolution from statically designed links towards reconfigurable WDM networks. The dominant impairment affecting OSNR is spectrally dependent noise accumulation in chains of optical amplifiers, [16]. By adjusting channel launched optical power at the transmitter (Tx), channel OSNR at the receiver (Rx) can be optimized.

In optical networks cascaded amplified spans are present, as well as accumulation and self-generation of optical noise, cross-talk, possible coupling and saturation. Considering these differences and specific realistic physical features, the work in [9] concentrated on OSNR modelling and basic Nash game formulation, while [17] extended it to a more general optical link with dynamic gain filters. In order for game methods to be practical, they must incorporate realistic constraints of the underlying network systems. In optical networks one such important constraint is the power capacity constraint. Such a constraint arises because when all wavelength-multiplexed channels share the optical fiber, the total power launched into the fiber needs to be restricted below the nonlinearity threshold, [18], [19]. This constraint is satisfied at intermediary amplifier sites, which are operated in automatic power control, [16], but not at the transmitter sites in a link. Each channel aims to utilize an optimal Tx power such that its own performance is optimized (e.g., OSNR). Since total power of all channels has to be kept below the nonlinearity threshold, a network game with coupled constraints has to be considered. The initial work in [9], [17] did not consider this underlying total power constraint.

This is the problem we address in this paper: continuing the work in [9], [17], we formulate an extended OSNR game that considers the total capacity constraint in optical networks. We indirectly impose the link capacity constraint at the transmitter site by adding an extra term to the channel cost function, that reflects the link status as a regulatory function. As a first step we study a single point-to-point link. Part of this work appeared in [20]. We characterize the NE solution for the extended OSNR game. We develop some iterative algorithms towards finding numerically the NE solution. As needed in networks, the iterative algorithms must be scalable, predictable, with provable convergence properties and fast response in reaching an optimum Nash equilibrium state. We show that the physical features of optical networks and the conditions for NE uniqueness and convergence of game-based algorithms are intricately related.

The organization of the paper is as follows. In Section II, we review the OSNR model for a single link. In Section III, we

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formulate an extended Nash game model for channel OSNR optimization with capacity constraint. We prove the existence of a unique NE solution. In Section IV we discuss decentralized pricing strategies and characterize properties of the NE solution. We present two distributed iterative algorithms to compute the unique NE solution in Section V. We also study their local and global convergence properties towards the NE solution. Numerical simulations and experimental results are shown in Section VI, followed by conclusions.

II. Background: Link OSNR Model

In the following we review the network OSNR model, [9]. Consider a single optical link composed of N cascaded optical amplifiers (OAs) and optical fiber spans. A set $\mathcal{M} = \{1, \dots, m\}$ of channels, corresponding to a set of wavelengths, are transmitted across the same optical fiber by wavelength-multiplexing. For the i^{th} channel, let p_i, s_i, $n_{0,i}$ and n_i denote the optical power of the input signal (at Tx), output signal (at Rx), input noise (at Tx) and output noise (at Rx), respectively. Let $\mathbf{p} = [p_1, \dots, p_m]^T$ be the vector of input channel powers and let $\mathbf{p}_{-i}$ be the vector obtained by deleting the i^{th} element from $\mathbf{p}$. In a link, each channel's input signal power can be adjusted independently at Tx.

OAs are used to amplify the optical power of all channels simultaneously, at the expense of introducing amplified spontaneous emission (ASE) noise. The OA's gain is wavelength-dependent so that each channel experiences a different gain and so does the ASE noise. We denote for the i^{th} channel the ASE noise self-generated in the k^{th} optical amplifier as $ASE_{k,i}$, which is wavelength-dependent. Typically, OAs are operated in automatic power control (APC) mode such that a target total power is maintained, [18]. In order to make the problem simple but practical, the following assumptions are used: all OAs in a link have the same spectral shape, G_i, and all spans have equal length, such that the same target total power, P_0, is launched into each span. Keeping a constant total optical power, or a constant total launching power after each optically amplified span, compensates variations in fiber-span loss across a link, [16]. Moreover, the target total power, P_0, is selected to be below the threshold for nonlinear effects, such that an optimal gain distribution is achieved across the whole link, [19]. Then at the input of each intermediary optical span, the following condition holds,

$$\sum_{i \in \mathcal{M}} p_{k,i} = P_0, \ \forall k = 1, \dots, N \tag{1}$$

where $p_{k,i}$ is the i^{th} channel power at the output of the k^{th} span, or input of the $(k+1)^{th}$ span. Thus the inherent scaling and propagation across a link due to the total power target, as in (1), translates into the coupling between all channels' power, which leads to a complex mathematical OSNR model, [9], [12], [16].

The OSNR for the i^{th} channel at the output of a link, denoted by γ_i, defined as $\gamma_i = \frac{s_i}{n_i}$, was given as Lemma 2 in [9], which we restate here.

Lemma 1: In a single optical link, the optical powers of the output signal and noise for the i^{th} channel are given as

$$\begin{aligned} s_i &= H_{N,i} p_i \\ n_i &= H_{N,i} n_i^0 + \sum_{v=1}^{N} ASE_{v,i} \frac{H_{N,i}}{H_{v,i}} \end{aligned}$$

where

$$H_{v,i} = \frac{P_0 G_i^v}{\sum_{j \in \mathcal{M}} G_j^v p_j}, \ \forall v = 1, \dots, N$$

The OSNR of channel i at the output of a link is given by

$$\gamma_i = \frac{p_i}{n_i^0 + \sum_{j \in \mathcal{M}} \Gamma_{i,j} p_j} \tag{2}$$

where $\mathbf{\Gamma} = [\Gamma_{i,j}]$ is the system-related matrix with

$$\Gamma_{i,j} = \sum_{r=1}^{N} \frac{G_j^r}{G_i^r} \frac{ASE_{r,i}}{P_0}, \ \forall i, j \in \mathcal{M}$$

While the OSNR model is mathematically similar to the wireless signal to interference ratio (SIR) model, [7], it has a richer system structure: $\mathbf{\Gamma}$ is a full matrix, with cross-coupling terms, non-zero diagonal elements and with all elements dependent on specific network parameters. We can rewrite γ_i, (2), as

$$\gamma_i = \frac{p_i}{X_{-i} + \Gamma_{ii} p_i}, \tag{3}$$

where $X_{-i} = n_{0,i} + \sum_{j \neq i} \Gamma_{ij} p_j$, so that unlike SIR, channel OSNR is no longer a linear function of individual channel optical power.

III. Extended OSNR Nash Game

A. Game Formulation with Capacity Constraint

Based on the OSNR model, a noncooperative Nash game can be formulated towards the OSNR optimization with link capacity constraint. A Nash game [14] is defined between channels, where each channel (player) minimizes an individual cost function J_i, by adjusting its own optical power p_i, in response to the other channels' actions.

The cost function $J_i(\mathbf{p}_{-i}, p_i)$ is defined as the difference between a pricing and a utility function, $J_i(\mathbf{p}_{-i}, p_i) = P_i(\mathbf{p}_{-i}, p_i) - U_i(\mathbf{p}_{-i}, p_i)$, while satisfying the constraint,

$$\sum_{j=1}^{m} p_j \leq P_0, \text{with } p_j \geq 0. \tag{4}$$

Note that (4) is the link capacity constraint. Although condition, (1), holds at the input of each intermediary optical span, the total launched power at Tx has to be constrained also, such that nonlinearity is limited, [18]. This was not considered in [9], [17].

Typically, the power per channel of the transmitter is limited to be below a certain threshold, p_{max}, i.e., $p_i \in [0, p_{max}]$, $\forall i \in \mathcal{M}$. In the multi-channel case, especially with a large number of channels, the total power constraint can be stronger than the individual channel power constraint.

Unless otherwise stated, we use $\bar{p}$, $\bar{p}_{-i}$ to denote $\sum_{j=1}^{m} p_j$ and $\sum_{j \neq i} p_j$. We denote a set by $\bar{\mathbb{P}}$, where power vector $\mathbf{p}$ belongs in view of (4), which is compact and convex, [20].

The utility function, U_i, is related to channel's OSNR level, indicating a preference for higher OSNR and can be defined as in [9],

$$U_i = \beta_i \ln \frac{1 + (\lambda_i - \Gamma_{ii})\gamma_i}{1 - \Gamma_{ii}\gamma_i}, \quad (5)$$

where $\beta_i > 0$ is the channel specific parameter and $\lambda_i > 0$ is introduced for flexibility. U_i in (5) is monotonic in γ_i, hence maximizing utility is related to maximizing channel's OSNR level. In this sense, β_i indicates the strength of the channel's desire to maximize its OSNR level, γ_i. Equivalently using (3) yields

$$U_i(\mathbf{p}_{-i}, p_i) = \beta_i \ln(1 + \lambda_i \frac{p_i}{X_{-i}}), \forall i \in \mathcal{M}, \quad (6)$$

which is twice continuously differentiable, monotone increasing and strictly concave in p_i.

Now for the extended game formulation with power capacity constraint, (4), we consider a modified pricing function, P_i, that indicates the current state of the network, or link, and is defined as:

$$P_i(\mathbf{p}_{-i}, p_i) = \alpha_i p_i + \frac{1}{P_0 - \bar{p}}, \forall i \in \mathcal{M}, \quad (7)$$

where $\alpha_i > 0$ is a pricing parameter, determined by the network. Thus the link capacity constraint, (4), is considered indirectly. The pricing function (7) not only sets the actual price for each channel, but also considers the link status via the extra term $\frac{1}{P_0 - \bar{p}}$. When the sum of optical power of channels approaches the target power P_0, the price increases without bound. Hence the power resources are preserved by forcing all channels to decrease their input power, and indirectly satisfies the link capacity constraint, (4).

Note that if we consider the constraint as a certain threshold on the power per channel, p_{max}, the extra term can be defined as $\frac{1}{p_{max} - p_i}$, such that when the individual channel power approaches the threshold, p_{max}, the corresponding price increases without bound. This could be treated as a special case of the general total power case, (7).

Therefore the cost function that the i^{th} channel seeks to minimize is

$$J_i(\mathbf{p}_{-i}, p_i) = \alpha_i p_i + \frac{1}{P_0 - \bar{p}} - \beta_i \ln(1 + \lambda_i \frac{p_i}{X_{-i}}), \forall i \in \mathcal{M}. \quad (8)$$

Hence the underlying extended m-player Nash game is defined here in terms of the cost function $J_i(\mathbf{p}_{-i}, p_i)$, (8), which satisfies the constraint, (4).

Remark 1: Compared to the case when no extra term exists in the cost function in [9], intuitively our price is higher with the same α_i. In other words, α_i should be smaller in order to be a good approximation for the intended game problem. We will see that there is lots of flexibility in selecting the pair of α_i and β_i and this extra term does ensure the constraint, (4).

Remark 2: The formulation is similar to the wireless power control problem, [5], [7], [8], but it's more challenging in optical networks due to specific features: coupled gain, cascaded amplified spans, accumulation and self-generation of ASE noise. Moreover, the total power capacity is a novel feature, essentially needed here to limit the nonlinear effects in optical links.

B. Existence and Uniqueness of NE Solution

The solution to the Nash game is the Nash equilibrium (NE), [14], [21]. Recall that the term $\frac{1}{P_0 - \bar{p}}$ in J_i, (8), ensures that $p_i = P_0 - \bar{p}_{-i}$ is not a solution to the minimization of J_i, (8), i.e., $J_i(\mathbf{p} : p_i = P_0 - \bar{p}_{-i}) > J_i(\mathbf{p})$, $\forall \mathbf{p}, p_i \neq P_0 - \bar{p}_{-i}$. This is a necessary condition for an inner NE solution. We make another assumption to guarantee the NE solution is inner.

(A.i.1) $p_i = 0$ is not a solution to the minimization of the cost function, J_i, (8), i.e., there always exists some $p_i' \neq 0$ such that $J_i(\mathbf{p} : p_i = 0) > J_i(\mathbf{p_{-i}}, p_i' \neq 0)$, $\forall \mathbf{p_{-i}}$.

We can rewrite the above inequality as

$$\beta_i \ln(1 + \lambda_i \frac{p_i'}{X_{-i}}) + \frac{1}{P_0 - \bar{p}_{-i}} > \alpha_i p_i' + \frac{1}{P_0 - \bar{p}_{-i} - p_i'}$$

Thus, if β_i and α_i in (8) are selected properly (for example, by letting $\beta_i \gg \alpha_i$), (A.i.1) will be satisfied. Since β_i and α_i are determined by the channel and the network, respectively, this selection can always be made. The following result characterizes the NE solution.

Theorem 1: Assume (A.i.1) holds. Consider the extended m-player OSNR Nash game problem with individual cost function defined as in (8) and capacity constraint, (4). Such a game admits a unique inner NE solution, $\mathbf{p}^*$, if α_i, β_i and λ_i are selected such that, $\forall i \in \mathcal{M}$:

$$\lambda_i > (m-1)\Gamma_{ij},\ j \neq i, \quad (9)$$

$$\beta_{min} \leq \beta_i < \beta_{min} / \textstyle\sum_{j \neq i} (\Gamma_{ji}/\lambda_j), \quad (10)$$

$$\alpha_{max} \sqrt{\beta_i \textstyle\sum_{j \neq i} (\Gamma_{ji}/(\lambda_j \beta_j))} < \alpha_i \leq \alpha_{max}, \quad (11)$$

where $\beta_{min} = \min_j \beta_j$ and $\alpha_{max} = \max_j \alpha_j$.

Proof: The proof is presented in Appendix.

Remark 3: Some discussion on pricing is presented in the next section. When $\alpha_i = \alpha$ and $\beta_i = \beta$, the same price is set for channels and each channel has the same preference for OSNR. Therefore the cost function is reduced to

$$J_i(\mathbf{p}_{-i}, p_i) = \alpha p_i + \frac{1}{P_0 - \bar{p}} - \beta \ln(1 + \lambda_i \frac{p_i}{X_{-i}}), \forall i \in \mathcal{M}.$$

The conditions in Theorem 1 are reduced to (9) only, which is a system condition.

Remark 4: In a Nash game, each channel is assumed to act noncooperatively. Intuitively, the NE might not be optimal from a centralized cost prospective. However, a game formulation has the advantage that it leads directly to decentralized algorithms.

IV. Discussion

We formulated an extended OSNR Nash game, that considers the link capacity constraint implicitly by using the cost function, (8). The NE solution of the extended game is analytically intractable and highly nonlinear. Thus developing iterative algorithms is not immediate. First we study some pricing strategies and characterize properties of this NE solution. In the next section we consider iterative algorithms towards finding the NE solution.

A. Decentralized Pricing Strategy

Note that the network (the link) sets fixed channel prices, and channels decide their willingness (β_i) to obtain higher

OSNR levels. From (3) and the first-order necessary condition (FONC), $\frac{\partial J_i}{\partial p_i} = 0$, at the NE point we have

$$\beta_i(\gamma_i) = \frac{\alpha_i}{\lambda_i}(X_{-i} + \lambda_i p_i(\gamma_i)) + \frac{X_{-i} + \lambda_i p_i(\gamma_i)}{\lambda_i(P_0' - p_i(\gamma_i))^2}, \quad (12)$$

where, $P_0' = P_0 - \bar{p}_{-i}$ and $p_i(\gamma_i) = \frac{X_{-i}}{1/\gamma_i - \Gamma_{ii}}$. For a given lower OSNR bound, γ_i^*, we can show that if β_i is adjusted to satisfy the lower bound

$$\beta_i > \Big(\frac{\alpha_i}{\lambda_i}\frac{1+(\lambda_i - \Gamma_{ii})\gamma_i^*}{1-\Gamma_{ii}\gamma_i^*} + \frac{(1-\Gamma_{ii}\gamma_i^*)(1+(\lambda_i-\Gamma_{ii})\gamma_i^*)}{\lambda_i(P_0' - (P_0'\Gamma_{ii} + X_{-i})\gamma_i^*)^2}\Big)X_{-i}, \quad (13)$$

each channel will achieve at least the γ_i^* level, i.e., $\gamma_i > \gamma_i^*$.

B. Properties of the NE Solution

Note that the cost function, (8), can be rewritten as:

$$J_i(p_i, \gamma_i) = \alpha_i p_i + \frac{1}{P_0 - \bar{p}} - \beta_i \ln \frac{1 + (\lambda_i - \Gamma_{ii})\gamma_i}{1 - \Gamma_{ii}\gamma_i}. \quad (14)$$

The FONC used for finding the unique inner NE solution is

$$0 = \left(\frac{\partial J_i}{\partial p_i}\right) = \frac{\partial J_i(p_i,\gamma_i)}{\partial p_i} + \frac{\partial J_i(p_i,\gamma_i)}{\partial \gamma_i} \cdot \frac{\partial \gamma_i}{\partial p_i}, \quad (15)$$

where

$$\frac{\partial J_i(p_i,\gamma_i)}{\partial p_i} = \alpha_i + \frac{1}{(P_0 - \bar{p})^2}$$
$$\frac{\partial J_i(p_i,\gamma_i)}{\partial \gamma_i} \cdot \frac{\partial \gamma_i}{\partial p_i} = -\frac{\beta_i \lambda_i}{X_{-i} + \lambda_i p_i}$$

Equivalently, (15) can be rewritten as

$$\alpha_i + \frac{1}{(P_0 - \bar{p})^2} = \frac{\beta_i \lambda_i}{X_{-i} + \lambda_i p_i}, \forall i \in \mathcal{M}. \quad (16)$$

By Theorem 1, there exists a unique and inner NE solution, which is a vector solution of the set of (15) or (16). When $\mathbf{p} \in \bar{\mathbb{P}}$, the left hand side (LHS) of (16) is a monotonic (increasing) function with respect to p_i, while the right hand side (RHS) of (16) is monotonically decreasing. Therefore, for all i, there exists a unique intersection between the LHS and RHS of (16), which is the NE solution point, $\mathbf{p}^*$. Given $\mathbf{p}^*_{-i}$ and (16), p_i^* can be solved for each i by

$$p_i^* = \arg\min_{p_i \in [0, P_0 - \bar{p}^*_{-i}]} J_i(\mathbf{p}^*_{-i}, p_i). \quad (17)$$

We define an individual nonlinear *best response* [14] function here,

$$I_i(\mathbf{p}_{-i}) := \arg\min_{p_i \in [0, P_0 - \bar{p}_{-i}]} J_i(\mathbf{p}_{-i}, p_i), \forall i \in \mathcal{M}. \quad (18)$$

Therefore,

$$p_i = I_i(\mathbf{p}_{-i}), \forall i \in \mathcal{M}. \quad (19)$$

Equivalently, in vector form, (19) can be written as

$$\mathbf{p} = \mathbf{I}(\mathbf{p}). \quad (20)$$

Lemma 2: The nonlinear best response function $\mathbf{I}(\mathbf{p})$, (20), satisfies the following properties:

- non-negativity: $\mathbf{I}(\mathbf{p}) \geq 0$;
- monotonicity: If $\mathbf{p} > \mathbf{p}'$, then $\mathbf{I}(\mathbf{p}) < \mathbf{I}(\mathbf{p}')$.

Proof: See Appendix for the details.

V. Iterative Algorithms and Convergence

In this section we develop iterative algorithms and study their convergence to the extended NE solution. We will use the nonlinear best response function, (20), and the FONC, (16), in developing these algorithm. From Lemma 2, the best response function is similar to a *standard function* in [22], but without the property of scalability, which means that convergence can not be proved by direct application of results in [22].

First, using (3), we rewrite (16) as

$$\alpha_i + \frac{1}{(P_0 - \bar{p})^2} = \frac{\beta_i \lambda_i}{p_i/\gamma_i + (\lambda_i - \Gamma_{ii})p_i}, \ \forall i. \quad (21)$$

Thus from (21) we see that individual channels need only the sum of powers of all channels, rather than the specific powers of other channels.

Based on (19) and (21), we first investigate a "best-response" algorithm, which we call *parallel update algorithm* (PUA). As a relaxation of PUA, r-PUA used an identical relaxation parameter, μ, to determine the step size that each channel takes towards the NE solution at each iteration. The main advantage of r-PUA compared with PUA is that wide fluctuations in channels' updates can be largely avoided. r-PUA will be discussed in the following part of this section. All proofs for Lemmas are presented in Appendix.

Algorithm 1 (PUA): 1). Set the initial power at iteration $k = 0$: $\mathbf{p}(0) = \mathbf{0}$;

2). At each iteration $k+1$, given $\mathbf{p}_{-i}(k)$, set

$$p_i(k+1) = I_i(\mathbf{p}_{-i}(k)), \quad (22)$$

i.e., solve the FONC, (16). Equivalently, in vector form, (22) is

$$\mathbf{p}(k+1) = \mathbf{I}(\mathbf{p}(k)). \quad (23)$$

From Lemma 2, it follows that the nonlinear best response function $\mathbf{I}(\mathbf{p})$ is monotonically decreasing, which is not *standard*, [22]. This kind of monotonic property makes the sequence of each channel's power non-monotone. However, $\mathbf{I}(\mathbf{p})$ has interesting properties, which are useful in studying global convergence of Algorithm 1 (PUA) for two channels ($m = 2$).

Let $\mathbf{p}(k)$ denote the power vector at iteration k, as in (22) or (23), with the initial power vector given as $\mathbf{p}(0) = \mathbf{0}$. Let $\{\mathbf{p}(k)\}$ denote the sequence of generated power vectors. Two subsequences, $\{\mathbf{p}(2k)\}$ and $\{\mathbf{p}(2k+1)\}$, are defined as $\{\mathbf{p}(2k)\} := \{\mathbf{p}(0), \mathbf{p}(2), \ldots, \mathbf{p}(2k), \ldots\}$ and $\{\mathbf{p}(2k+1)\} := \{\mathbf{p}(1), \mathbf{p}(3), \ldots, \mathbf{p}(2k+1), \ldots\}$, respectively. The following two lemmas hold only for a link with two channels.

Lemma 3: For two channels ($m = 2$) with $\mathbf{p}(0) = \mathbf{0}$, the sequence $\{\mathbf{p}(2k)\}$ is strictly monotonically increasing and $\{\mathbf{p}(2k+1)\}$ is strictly monotonically decreasing. Each of the sequences $\{\mathbf{p}(2k)\}$ and $\{\mathbf{p}(2k+1)\}$ will converge to a fixed point $\mathbf{p}^1$ and $\mathbf{p}^2$, respectively.

An illustration of the convergence of sequence $\{p_i(k)\}$ for channel i with $p_i(0) = 0$ is shown in Fig.1.

Lemma 4: For two channels ($m = 2$), the sequence $\mathbf{p}(k)$ converges to a limit point $\mathbf{p}^0$, i.e., $\lim_{k\to\infty} \mathbf{p}(k) = \mathbf{p}^0$.

Based on Lemma 3 and Lemma 4, the following result shows global convergence.

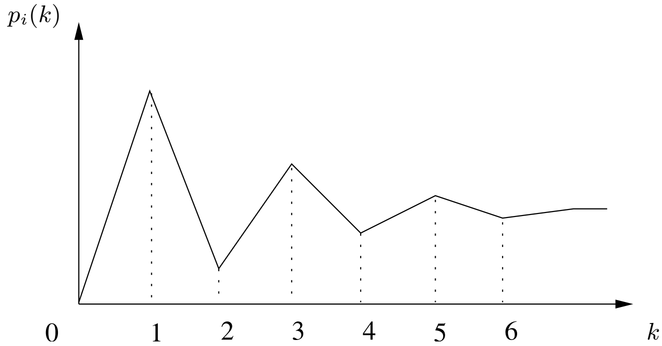


Fig. 1. Iteration for channel i.

Theorem 2: For two channels ($m = 2$), Algorithm 1 (PUA) converges to the NE solution, $\mathbf{p}^*$,

$$\lim_{k\to\infty} \mathbf{p}(k) = \mathbf{p}^*. \tag{24}$$

Proof: We take the limits of both sides of (23),

$$\lim_{k\to\infty} \mathbf{p}(k+1) = \lim_{k\to\infty} \mathbf{I}(\mathbf{p}(k)). \tag{25}$$

From Lemma 4, the LHS of (25) equals $\mathbf{p}^0$. From Lemma 2, $\mathbf{I}(\mathbf{p})$ is a monotonic function. It is easy to show that $\mathbf{I}(\mathbf{p})$ is also continuous for two channels. Therefore,

$$\lim_{k\to\infty} \mathbf{I}(\mathbf{p}(k)) = \mathbf{I}\left(\lim_{k\to\infty} \mathbf{p}(k)\right) = \mathbf{I}(\mathbf{p}^0). \tag{26}$$

Using (26), (25) can be rewritten as $\mathbf{p}^0 = \mathbf{I}(\mathbf{p}^0)$. As the NE solution $\mathbf{p}^*$ is unique and satisfies $\mathbf{p}^* = \mathbf{I}(\mathbf{p}^*)$, we have $\mathbf{p}^0 = \mathbf{p}^*$. Thus $\{\mathbf{p}(k)\}$ converges to the unique NE solution $\mathbf{p}^*$. ■

Remark 5: PUA may not converge globally when more than two channels exist, the intuitive reason being that each channel updates its optical power based on instant costs and parameters, ignoring future implications of its action. Therefore at some iterations, the total optical power of all other channels will exceed the target power P_0 when $m > 2$, such that the nonlinear best response function, (22), is no longer monotonic.

However, Algorithm 1 (PUA) converges locally when $m > 2$, as shown in the next result.

Theorem 3: There exists some open neighborhood $\mathcal{V}$ of $\mathbf{p}^*$, such that if $\mathbf{p}(0) \in \mathcal{V}$, then $\mathbf{p}(k)$, (23), converges to the unique NE solution $\mathbf{p}^*$, if the following sufficient condition is satisfied:

$$\sum_{j\neq i} \frac{\Gamma_{ij}}{\lambda_i} < 1 - \frac{2(m-2)B_i}{A(\alpha_i A^2 + 1)}, \tag{27}$$

where $A = P_0 - \bar{p}^*$ and $B_i = \frac{n_{0i}}{\lambda_i} + \sum_{j\neq i} \frac{\Gamma_{ij}}{\lambda_i} p_j^* + p_i^*$.

Proof: The proof is presented in Appendix.

Algorithm 2 (r-PUA): 1). Set the initial power at iteration $k = 0$: $\mathbf{p}(0) = \mathbf{0}$;

2). At each iteration $k+1$, given $\mathbf{p}_{-i}(k)$, set

$$p_i(k+1) = (1-\mu)p_i(k) + \mu I_i(\mathbf{p}_{-i}(k)), \tag{28}$$

where $0 < \mu < 1$ and typically selected as $\mu = \frac{1}{m}$.

Local convergence for r-PUA can be proved similarly as in Theorem 3, under the same sufficient condition.

Theorem 4: There exists some open neighborhood $\mathcal{V}$ of $\mathbf{p}^*$, such that if $\mathbf{p}(0) \in \mathcal{V}$, then $\mathbf{p}(k)$, (28), converges to the unique NE solution $\mathbf{p}^*$, if the sufficient condition, (27), is satisfied.

VI. Numerical Simulations and Experimental Results

In this section, we present both MATLAB simulations and experimental results. Even though both of the iterative algorithms use the best response function, (20), depending on the powers of the other channels, the algorithms can be implemented decentralized using only the corresponding OSNR level and the total power. This is shown in a discretized version of the FONC, (21), $\forall i \in \mathcal{M}$:

$$\alpha_i + \frac{1}{(P_0 - \bar{p}_{-i}(k) - p_i(k+1))^2} = \frac{\beta_i\lambda_i}{(\frac{1}{\gamma_i(k)} - \Gamma_{ii} + \lambda_i)p_i(k+1)} \tag{29}$$

In other words, the iterative algorithms use decentralized information, rather than centralized one which would appear from the individual best response function, (19).

The two iterative algorithms (PUA and r-PUA) are simulated for a single point-to-point link in MATLAB, as well as implemented experimentally on a testbed with 2 channels ($m = 2$). The optical link has a total power capacity constraint $P_0 = 1.5$ mW/1.76 dBm. The system matrix, Γ, is obtained as

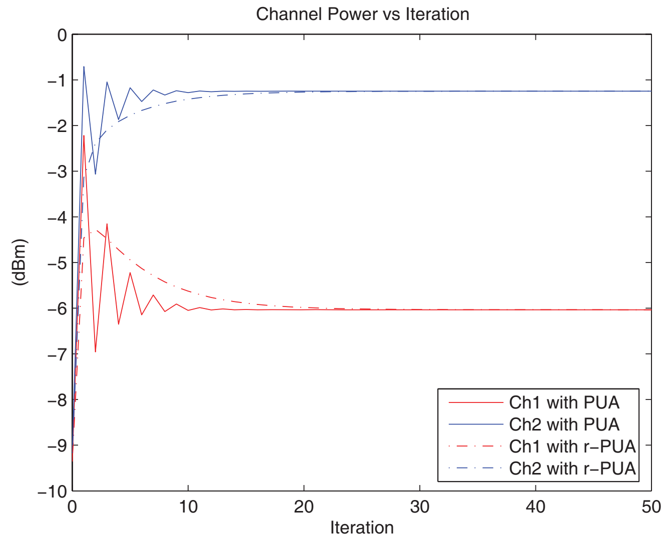
$$\Gamma = \begin{bmatrix} 1.2438 & 1.2296 \\ 1.2418 & 1.2276 \end{bmatrix} \times 10^{-4}.$$

The channel parameters are selected in accordance with the assumption, (A.i.3). They also satisfy the sufficient conditions for existence of a unique NE solution. Simulations are repeated with the following selected parameters: $\alpha_i = [0.01, 0.01]$, $\beta = [1, 3]$, $\lambda = [1, 1]$ and $\mu = 1/m$.

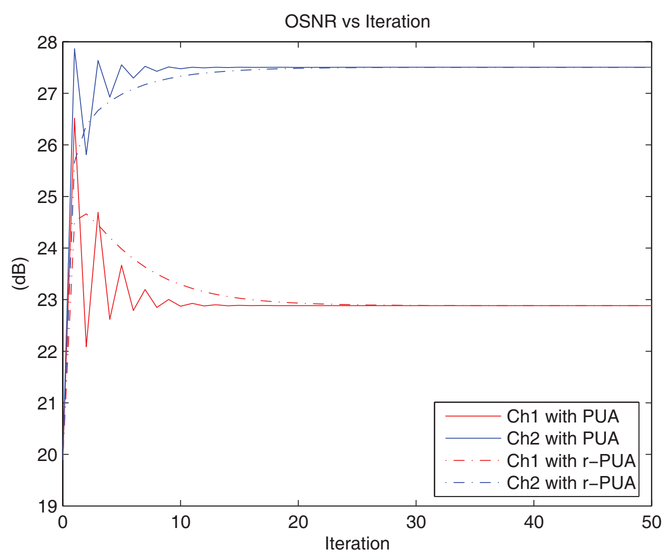
The input power, OSNR and total power vs iteration time are plotted in Fig.2 and Fig.3. In Fig.2, two channels compete for the power resources and settle down at the NE solution via PUA and r-PUA. Numerical simulations show r-PUA has better convergence properties than PUA. It can be seen that the wide fluctuations in PUA are largely avoided via r-PUA. Moreover, during the iterative process, the total power constraint is violated in PUA, while it is not in r-PUA. Fig.3 shows the experimental results. The difference between the simulation results and experiment results is due to the difference in the input noise. Both simulation and experimental results showed above prove the performance of the algorithms and validate the analytic results. For multi-channel case, Fig.4 shows the results for six channels ($m = 6$) iteratively adjusted via r-PUA. The parameters are selected as $\alpha_i = 0.01$, $\lambda_i = 1$, $\forall i = 1, \ldots, 6$, $\beta = [1.8\,2\,3\,5\,6\,6.5]$, and $\mu = 1/m$. The input power and the total power are plotted in the same figure. The simulation result also indicates a good performance of r-PUA.

VII. Conclusions

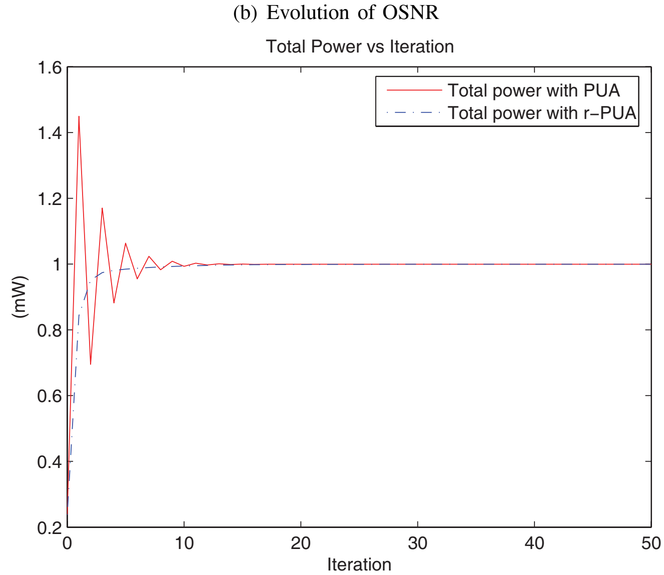
We presented a framework of a noncooperative game, towards OSNR optimization with capacity constraint in optical links. This link capacity threshold was not imposed on the total launched power at Tx in the basic game, recently developed in [9]. We formulated an extended Nash game by modifying



(a) Evolution of channel power

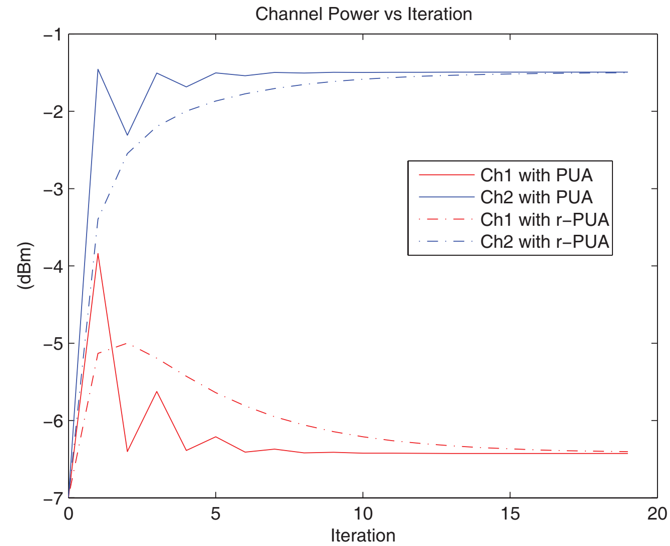


(b) Evolution of OSNR

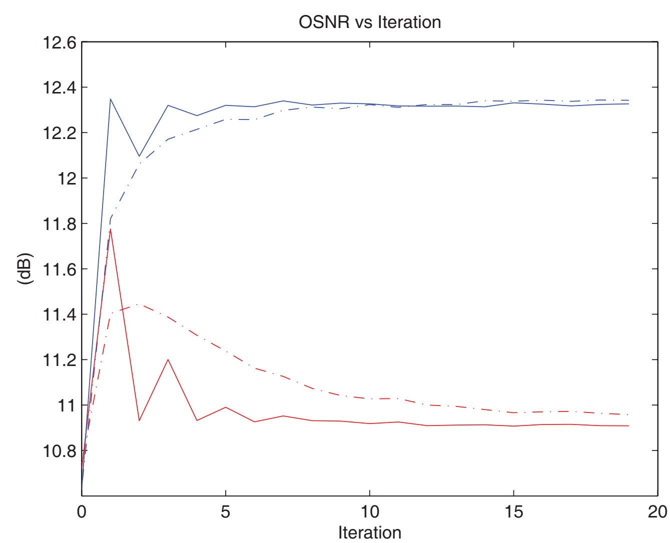


(c) Evolution of total power

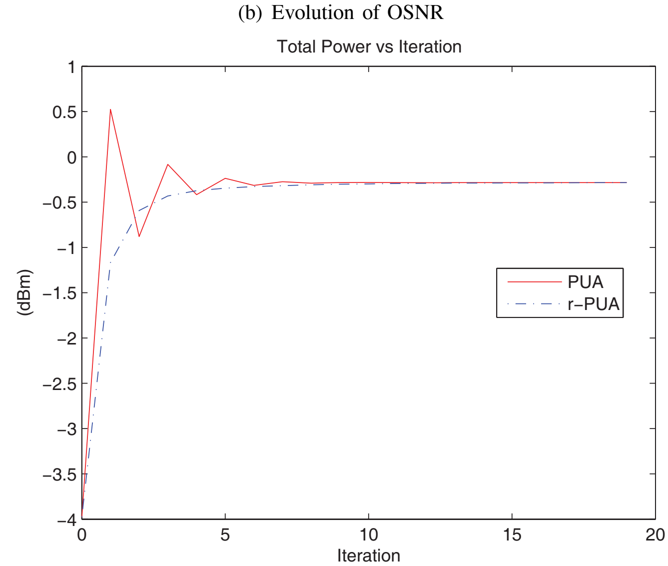
Fig. 2. PUA and r-PUA for m = 2.



(a) Evolution of channel power

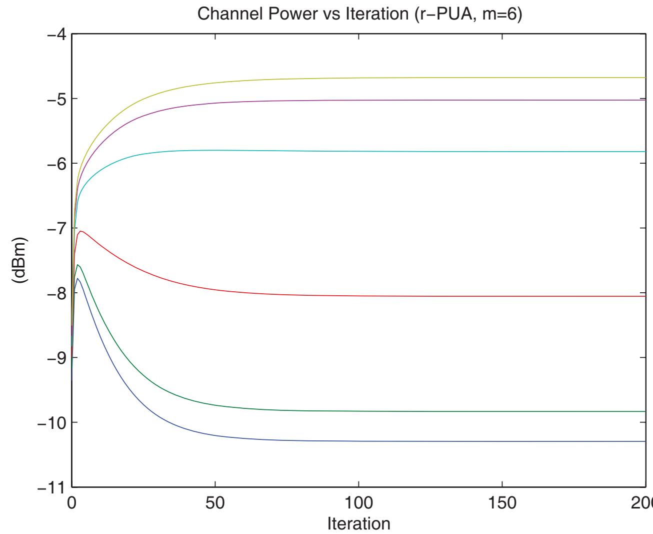


(b) Evolution of OSNR

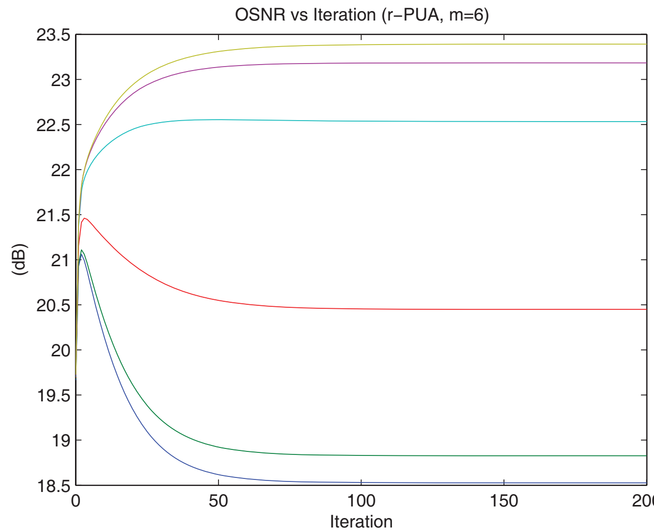


(c) Evolution of total power

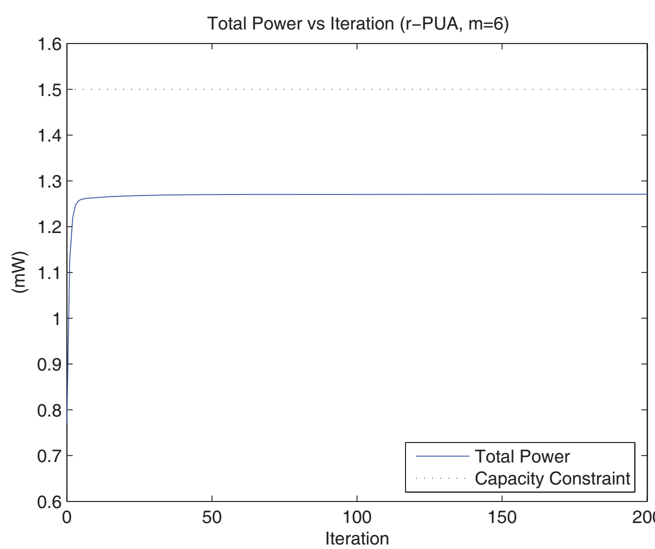
Fig. 3. Experiment: PUA and r-PUA for m = 2.



(a) Evolution of channel power



(b) Evolution of OSNR



(c) Evolution of total power

Fig. 4. r-PUA for m = 6.

the cost function to consider the status of the link. We studied the single link case and obtained conditions for the existence of a unique NE solution to this extended Nash game. The NE solution is analytically intractable and highly nonlinear, and developing iterative algorithms is not immediate. We studied the properties of the NE solution and presented two update algorithms, PUA and a relaxation of PUA (r-PUA), towards finding the unique NE solution. We also studied their convergence properties both theoretically and numerically. The cost function considered in this paper only indirectly reflects the OSNR. The tuning of the parameters is important in order to meet a rough OSNR target. This is one of the directions for future research. Another direction is the extension to general multi-link configurations, where the length of links and the number of optical components in different links are changing. In general WDM networks, different channels can travel via different optical paths such that the number of channels on each link may be different. Moreover, different links may be designed for a different number of maximum channels and thus have different total power targets.

Appendix

1) Proof of Theorem 1: For more details of the proof see [20]. We note that $\bar{\mathbb{P}}$ is compact and convex. Each $J_i(\mathbf{p}_{-i}, p_i)$ is continuous and bounded except on the hyperplane defined by (4). Hence for a given sufficiently small $\epsilon > 0$, if we replace (4) with $\bar{p} \le P_0 - \epsilon$ and denote the corresponding constraint set by $\bar{\mathbb{P}}_\epsilon$. This set is compact, convex and non-rectangular. On $\bar{\mathbb{P}}_\epsilon$, differentiating (8) with respect to p_i, we have $f_i(\mathbf{p}) := \frac{\partial J_i}{\partial p_i} = \alpha_i + \frac{1}{(P_0-\bar{p})^2} - \frac{\beta_i \lambda_i}{X_{-i}+\lambda_i p_i}$. Differentiating $f_i(\mathbf{p})$ with respect to p_i and p_j, $j \neq i$, yields

$$A_{ii}(\mathbf{p}) := \frac{\partial^2 J_i}{\partial p_i^2} = \frac{2}{(P_0-\bar{p})^3} + \frac{\beta_i \lambda_i^2}{(X_{-i}+\lambda_i p_i)^2}$$
$$A_{ij}(\mathbf{p}) := \frac{\partial^2 J_i}{\partial p_i \partial p_j} = \frac{2}{(P_0-\bar{p})^3} + \frac{\beta_i \lambda_i \Gamma_{ij}}{(X_{-i}+\lambda_i p_i)^2} \quad \text{(A.1)}$$

From (A.1), it follows directly that $\frac{\partial^2 J_i}{\partial p_i^2}$ is positive on $\bar{\mathbb{P}}_\epsilon$. By a standard theorem in game theory [14], this game admits a NE solution. Furthermore, the NE solution is independent of ϵ for $\epsilon > 0$ sufficiently small. Thus it provides also an NE solution to the original game on $\bar{\mathbb{P}}$.

We next show the uniqueness of the NE solution. The inner solution satisfies the FONCs, i.e., the set of equations: $f_i(\mathbf{p}) = 0$, $i = 1, \ldots, m$. Suppose that there are two Nash equilibria, represented by two optical power vectors $\mathbf{p}^1$ and $\mathbf{p}^0$, respectively. Let $\triangle \mathbf{p} = \mathbf{p}^0 - \mathbf{p}^1$. Define the pseudo-gradient vector:

$$g(\mathbf{p}) := [\nabla_{p_1} J_1(\mathbf{p}), \cdots, \nabla_{p_m} J_m(\mathbf{p})]^T$$
$$= [f_1(\mathbf{p}), \cdots, f_m(\mathbf{p})]^T \quad \text{(A.2)}$$

From the FONC, it follows that $g(\mathbf{p}^1) = 0$ and $g(\mathbf{p}^0) = 0$. Define a vector $\mathbf{p}(\theta)$ as a convex combination of the two equilibrium points $\mathbf{p}^1$ and $\mathbf{p}^0$:

$$\mathbf{p}(\theta) = \theta \mathbf{p}^0 + (1-\theta)\mathbf{p}^1, \; 0 < \theta < 1. \quad \text{(A.3)}$$

Differentiating $g(\mathbf{p}(\theta))$ with respect to θ, we get

$$\frac{dg(\mathbf{p}(\theta))}{d\theta} = G(\mathbf{p}(\theta))\frac{d\mathbf{p}(\theta)}{d\theta} = G(\mathbf{p}(\theta))\triangle\mathbf{p}, \quad (A.4)$$

where $G(\mathbf{p})$ is the Jacobian of $g(\mathbf{p})$. Using (A.1) and (A.2) yields $G(\mathbf{p}) = [A_{ij}]_{m\times m}$. Integrating (A.4) over θ, we obtain

$$0 = g(\mathbf{p}^1) - g(\mathbf{p}^0) = \left[\int_0^1 G(\mathbf{p}(\theta))d\theta\right]\triangle\mathbf{p}. \quad (A.5)$$

For simplicity, we define $\mathcal{G}(\mathbf{p}^1,\mathbf{p}^0) = \int_0^1 G(\mathbf{p}(\theta))d\theta = [\overline{A}_{ij}]$ with

$$\overline{A}_{ij} := \overline{A}_{ij}(\mathbf{p}^1,\mathbf{p}^0) = \int_0^1 A_{ij}(\mathbf{p}(\theta))d\theta. \quad (A.6)$$

Therefore, we can rewrite (A.5) as

$$0 = \mathcal{G}(\mathbf{p}^1,\mathbf{p}^0)\triangle\mathbf{p}. \quad (A.7)$$

We compute next $\overline{A}_{ii}$ and $\overline{A}_{ij}$ in (A.6). Using (3), (A.1) and (A.3) into (A.6) yields

$$\overline{A}_{ii} = \int_0^1 [\frac{2}{(P_0 - \bar{p}^1 - \theta\sum_j \triangle p_j)^3} + \frac{\beta_i\lambda_i^2}{(n_{0,i} + \sum_{j\neq i}\Gamma_{ij}(\theta\triangle p_j + p_j^1) + \lambda_i p_i(\theta))^2}]d\theta$$

It can be shown that $\overline{A}_{ii}$ can be written as $\overline{A}_{ii} = W_1 + W_2\cdot\frac{1}{\beta_i} + W_3\cdot\frac{\alpha_i}{\beta_i} + \frac{\alpha_i^2}{\beta_i}$, with $W_1 = \frac{(P_0-\bar{p}^0)+(P_0-\bar{p}^1)}{(P_0-\bar{p}^0)^2(P_0-\bar{p}^1)^2}$, $W_2 = \frac{1}{(P_0-\bar{p}^0)^2(P_0-\bar{p}^1)^2}$ and $W_3 = \frac{(P_0-\bar{p}^0)^2+(P_0-\bar{p}^1)^2}{(P_0-\bar{p}^0)^2(P_0-\bar{p}^1)^2}$.

Similarly for $\overline{A}_{ij}$, $j\neq i$, we can obtain $\overline{A}_{ij} = W_1 + W_2\cdot\frac{\Gamma_{ij}}{\beta_i\lambda_i} + W_3\cdot\frac{\alpha_i\Gamma_{ij}}{\beta_i\lambda_i} + \frac{\alpha_i^2\Gamma_{ij}}{\beta_i\lambda_i}$. Let $B_{ii} = 1$, $C_{ii} = \frac{1}{\beta_i}$, $D_{ii} = \frac{\alpha_i}{\beta_i}$ $E_{ii} = \frac{\alpha_i^2}{\beta_i}$ and $B_{ij} = 1$, $C_{ij} = \frac{\Gamma_{ij}}{\beta_i\lambda_i}$, $D_{ij} = \frac{\alpha_i\Gamma_{ij}}{\beta_i\lambda_i}$, $E_{ij} = \frac{\alpha_i^2\Gamma_{ij}}{\beta_i\lambda_i}$, for $j\neq i$. Therefore, for any i,j, $\overline{A}_{ij}$ above can be written as $\overline{A}_{ij} = W_1\cdot B_{ij} + W_2\cdot C_{ij} + W_3\cdot D_{ij} + E_{ij}$, and matrix $\mathcal{G}(\mathbf{p}^1,\mathbf{p}^0)$ can be expressed as $\mathcal{G}(\mathbf{p}^1,\mathbf{p}^0) = W_1\cdot B + W_2\cdot C + W_3\cdot D + E$. It is obvious that matrix B is positive semidefinite. If (9) holds, it follows $\lambda_i > \sum_{j\neq i}\Gamma_{ij}$. Hence C, D and E are all strictly diagonally dominant. If (10) holds, i.e., $\frac{1}{\beta_i} > \sum_{j\neq i}\frac{\Gamma_{ji}}{\beta_{min}\lambda_j}$, we can obtain $\frac{1}{\beta_i} > \sum_{j\neq i}\frac{\Gamma_{ji}}{\beta_j\lambda_i}$, so that C^T is strictly diagonally dominant. It also follows $\beta_i\sum_{j\neq i}\frac{\Gamma_{ji}}{\beta_j\lambda_j} < 1$, such that $\sqrt{\beta_i\sum_{j\neq i}\frac{\Gamma_{ji}}{\beta_j\lambda_j}} > \beta_i\sum_{j\neq i}\frac{\Gamma_{ji}}{\beta_j\lambda_j}$. From (11) and the foregoing, it follows that $\alpha_{max}\beta_i\sum_{j\neq i}\frac{\Gamma_{ji}}{\lambda_j\beta_j} < \alpha_i \leq \alpha_{max}$, from which we get $\frac{\alpha_i}{\beta_i} > \sum_{j\neq i}\frac{\alpha_j\Gamma_{ji}}{\beta_j\lambda_j}$. Thus D^T is strictly diagonally dominant. Following similar arguments, it can be shown that $\frac{\alpha_i^2}{\beta_i} > \sum_{j\neq i}\frac{\alpha_j^2\Gamma_{ji}}{\beta_j\lambda_j}$, so that E^T is also strictly diagonally dominant. Thus matrix C, D and E are all positive definite and B is positive semidefinite. Thus the matrix $\mathcal{G}(\mathbf{p}^1,\mathbf{p}^0)$ is full rank. From (A.7), it readily follows $\triangle\mathbf{p} = 0$, i.e., $\mathbf{p}^0 = \mathbf{p}^1$. Thus the NE solution is unique. ■

2) Proof of Lemma 2.: Non-negativity is obvious from the definition of $\mathbf{I}(\mathbf{p})$. We prove the monotonicity of $\mathbf{I}(\mathbf{p})$. Suppose $\mathbf{p}$ and $\mathbf{p}'$ are two distinct power vectors and satisfy $\mathbf{p} > \mathbf{p}'$, i.e., $\mathbf{p}_{-i} > \mathbf{p}'_{-i}$. We denote $\widetilde{\mathbf{p}} = \mathbf{I}(\mathbf{p})$ and $\widetilde{\mathbf{p}}' = \mathbf{I}(\mathbf{p}')$. For the i^{th} channel, using the FONC, (16), we have $\partial J_i(p_{-i},p_i)/\partial p_i|_{p_i=\widetilde{p}_i} = \partial J_i(p'_{-i},p_i)/\partial p_i|_{p_i=\widetilde{p}'_i} = 0$, i.e.,

$$\alpha_i + \frac{1}{(P_0 - \bar{p}_{-i} - \widetilde{p}_i)^2} = \frac{\beta_i}{\widetilde{p}_i + (\frac{n_{0,i}}{\lambda_i} + \sum_{j\neq i}\frac{\Gamma_{ij}}{\lambda_i}p_j)} \quad (A.8)$$

$$\alpha_i + \frac{1}{(P_0 - \bar{p}'_{-i} - \widetilde{p}'_i)^2} = \frac{\beta_i}{\widetilde{p}'_i + (\frac{n_{0,i}}{\lambda_i} + \sum_{j\neq i}\frac{\Gamma_{ij}}{\lambda_i}p'_j)} \quad (A.9)$$

Note that $0 < P_0 - \bar{p}_{-i} < P_0 - \bar{p}'_{-i}$ and $-\left(\frac{n_{0,i}}{\lambda_i} + \sum_{j\neq i}\frac{\Gamma_{ij}}{\lambda_i}p_j\right) < -\left(\frac{n_{0,i}}{\lambda_i} + \sum_{j\neq i}\frac{\Gamma_{ij}}{\lambda_i}p'_j\right) < 0$, such that $\widetilde{p}_i < \widetilde{p}'_i$. Therefore, $\widetilde{\mathbf{p}} < \widetilde{\mathbf{p}}'$, and hence $\mathbf{I}(\mathbf{p}) < \mathbf{I}(\mathbf{p})'$ form their definition. ■

3) Proof of Lemma 3: We prove by induction. It is obvious $\mathbf{p}(1) > 0 = \mathbf{p}(0)$ and $\mathbf{p}(2) > 0 = \mathbf{p}(0)$, and we assume $\mathbf{p}(0) < \mathbf{p}(2) < \cdots < \mathbf{p}(2k)$. Recall (23) such that $\mathbf{p}(2k+1) = \mathbf{I}(\mathbf{p}(2k))$, then from the monotonicity property of $\mathbf{I}(\mathbf{p})$ in Lemma 2 we have

$$\mathbf{p}(1) > \mathbf{p}(3) > \cdots > \mathbf{p}(2k-1) > \mathbf{p}(2k+1). \quad (A.10)$$

Also from (23), we have $\mathbf{p}(2(k+1)) = \mathbf{I}(\mathbf{p}(2k+1))$ and $\mathbf{p}(2k) = \mathbf{I}(\mathbf{p}(2k-1))$. From monotonicity of $\mathbf{I}(\mathbf{p})$ and (A.10), we have $\mathbf{p}(2(k+1)) = \mathbf{I}(\mathbf{p}(2k+1)) > \mathbf{I}(\mathbf{p}(2k-1)) = \mathbf{p}(2k)$. Hence $\{\mathbf{p}(2k)\}$ is strictly monotonically increasing. Similarly it follows $\mathbf{p}(2k+3) < \mathbf{p}(2k+1)$, therefore $\{\mathbf{p}(2k+1)\}$ is strictly monotonically decreasing. Note that $\mathbf{P}$ is a compact set. Since a non-decreasing sequence (or a non-increasing sequence) has a limit point in a compact set [23], we have $\lim_{k\to\infty}\mathbf{p}(2k) = \mathbf{p}^1$ and $\lim_{k\to\infty}\mathbf{p}(2k+1) = \mathbf{p}^2$. ■

4) Proof of Lemma 4: Based on Lemma 3, if we can show that $\mathbf{p}^1 = \mathbf{p}^2$, then all subsequences of the sequence $\{\mathbf{p}(k)\}$ converge to the same limit point. Therefore $\{\mathbf{p}(k)\}$ converges to this limit point, $\mathbf{p}^0 = \mathbf{p}^1 = \mathbf{p}^2$. Next we will use discrete-time Lyapunov stability theory, [24], to prove the result. We construct a new system with state vector defined as

$$\mathbf{q}(k) = \mathbf{p}(2k+1) - \mathbf{p}(2k). \quad (A.11)$$

Since from (23), $\mathbf{p}(k+1) = \mathbf{I}(\mathbf{p}(k))$ and from (A.11), we make the following notation:

$$\mathbf{q}(k) = \mathbf{G}_1(\mathbf{p}(2k)), \quad (A.12)$$

where $\mathbf{G}_1(\mathbf{p}(2k)) = \mathbf{I}(\mathbf{p}(2k)) - \mathbf{p}(2k)$. Note that $\mathbf{G}_1(\mathbf{x})$ here is a strictly decreasing function with respect to $\mathbf{x}$ (from Lemma 2). Therefore, $\mathbf{q}(k+1) = \mathbf{G}_1(\mathbf{p}(2k+2)) = \mathbf{G}_1\circ\mathbf{I}\circ\mathbf{I}(\mathbf{p}(2k))$, where (23) was used twice. We define $\mathbf{G}_2 := \mathbf{G}_1\circ\mathbf{I}\circ\mathbf{I}$, thus

$$\mathbf{q}(k+1) = \mathbf{G}_2(\mathbf{p}(2k)). \quad (A.13)$$

From (A.12), (A.13), we can write

$$\mathbf{q}(k+1) = \mathbf{G}_2\circ\mathbf{G}_1^{-1}(\mathbf{q}(k)). \quad (A.14)$$

Since $\mathbf{G}_1(\mathbf{x})$ is strictly decreasing, $\mathbf{G}_1^{-1}$ exists. We define $\mathbf{F} := \mathbf{G}_2\circ\mathbf{G}_1^{-1}$, therefore

$$\mathbf{q}(k+1) = \mathbf{F}(\mathbf{q}(k)). \quad (A.15)$$

When $\mathbf{q} = 0$, $\mathbf{p}$ is a constant vector. Hence, $\mathbf{F}(0) = 0$.

Therefore we obtain a nonlinear discrete-time state space system (A.15), with

$$\mathbf{q}(0) = \mathbf{p}(1) - \mathbf{p}(0),\ \mathbf{F}(0) = 0. \quad (A.16)$$

Now we turn to construct a Lyapunov function $V(\mathbf{q}(k)) : \mathbb{R}^M \mapsto \mathbb{R}$, for the new system (A.15)-(A.16). We select $V(\mathbf{q}(k)) = \mathbf{q}(k)^T \cdot \mathbf{q}(k)$, which is positive definite.

Notice that using (A.11) yields $\mathbf{q}(k+1) - \mathbf{q}(k) = \mathbf{p}(2k+3) - \mathbf{p}(2k+2) - \mathbf{p}(2k+1) + \mathbf{p}(2k)$. From Lemma 3, it follows that $\mathbf{p}(2k+3) - \mathbf{p}(2k+1) < 0$ and $\mathbf{p}(2k) - \mathbf{p}(2k+2) < 0$. Therefore, $\mathbf{q}(k+1) - \mathbf{q}(k) < 0$, or component-wise, $q_i(k+1) < q_i(k)$. Thus,

$$V(\mathbf{q}(k+1)) - V(\mathbf{q}(k)) = \mathbf{q}(k+1)^T\mathbf{q}(k+1) - \mathbf{q}(k)^T\mathbf{q}(k) = \sum_{i=1}^{M}[q_i^2(k+1) - q_i^2(k)] < 0$$

Since $V(\mathbf{q})$ is a continuous positive function and $\Delta V(\mathbf{q})$ is negative definite, from discrete-time Lyapunov stability theory (pp193, [25]), $V(\mathbf{q})$ is a strong Lyapunov function for system (A.15, A.16), and the origin of (A.15) is asymptotically stable.Hence $\mathbf{q}(k)$ converges to zero. From (A.11), this implies two subsequences $\{\mathbf{p}(2k)\}$ and $\{\mathbf{p}(2k+1)\}$ will converge to the same limit point, i.e., $\mathbf{p}^1 = \mathbf{p}^2$. Therefore, $\{\mathbf{p}(k)\}$ converges to the limit point $\mathbf{p}^0 = \mathbf{p}^1 = \mathbf{p}^2$. ■

5) Proof of Theorem 3: The proof follows the approach in [26]. Recall the FONC, (16), satisfied by $\mathbf{p}^*$, i.e.,

$$\alpha_i + \frac{1}{(P_0 - \bar{p}^*)^2} = \frac{\beta_i}{\frac{n_{0i}}{\lambda_i} + \sum_{j\neq i}\frac{\Gamma_{ij}}{\lambda_i}p_j^* + p_i^*}. \tag{A.17}$$

We define $A = P_0 - \bar{p}^*$ and $B_i = \frac{n_{0i}}{\lambda_i} + \sum_{j\neq i}\frac{\Gamma_{ij}}{\lambda_i}p_j^* + p_i^*$, so that (A.17) can be rewritten as

$$\alpha_i + \frac{1}{A^2} = \frac{\beta_i}{B_i}. \tag{A.18}$$

Now for all k and for all channels i, let $\Delta p_i(k) := p_i(k) - p_i^*$. Then at $(k+1)$ when channel i updates its power $p_i(k+1) = p_i^* + \Delta p_i(k+1)$, the power satisfies the following equation:

$$\alpha_i + \frac{1}{(A - \sum_{j\neq i}\Delta p_j(k) - \Delta p_i(k+1))^2} = \frac{\beta_i}{B_i + \sum_{j\neq i}\frac{\Gamma_{ij}}{\lambda_i}\Delta p_j(k) + \Delta p_i(k+1)} \tag{A.19}$$

We rewrite (A.19) and omit those high-order terms, $o(\cdot)$, that satisfy $\lim_{x\to\infty} o(x)/|x| = 0$. We get

$$\Delta p_i(k+1) = \sum_{j\neq i}\frac{\frac{\Gamma_{ij}}{\lambda_i}(\alpha_i A^2+1) + 2\beta_i A - 2\alpha_i A B_i}{2\alpha_i A B_i - \alpha_i A^2 - 2\beta_i A - 1}\Delta p_j(k), \tag{A.20}$$

which using (A.18) yields

$$\Delta p_i(k+1) = -\sum_{j\neq i}\frac{\frac{\Gamma_{ij}}{\lambda_i}(\alpha_i A^2+1) + \frac{2B_i}{A}}{\frac{2B_i}{A} + \alpha_i A^2 + 1}\Delta p_j(k). \tag{A.21}$$

Rewriting it in matrix form yields $\Delta\mathbf{p}(k+1) = \mathbf{C}\Delta\mathbf{p}(k)$, where $\mathbf{C} = [C_{ij}]$ with $C_{ii} = 0$ and

$$C_{ij} = -\frac{\frac{\Gamma_{ij}}{\lambda_i}(\alpha_i A^2+1) + \frac{2B_i}{A}}{\frac{2B_i}{A} + \alpha_i A^2 + 1},\ j \neq i,$$

From Gersgorin's Theorem [27], it is seen if the sufficient condition (27) holds, then

$$\sum_{j\neq i}\frac{\frac{\Gamma_{ij}}{\lambda_i}(\alpha_i A^2+1) + \frac{2B_i}{A}}{\frac{2B_i}{A} + \alpha_i A^2 + 1} < 1,$$

and therefore, all eigenvalues of $\mathbf{C}$ are in the interior of the unit disk. Hence the sequence $\{\mathbf{p}(k)\}$ converges locally to the NE solution $\mathbf{p}^*$. ■

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