OSNR game optimization with link capacity constraints in general topology WDM networks

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Abstract

This work studies games with coupled constraints in general topology optical wavelength-division multiplexed (WDM) networks towards optimizing channel optical signal-to-noise ratio (OSNR). We first develop a model to describe the network and an OSNR model for each link by investigating the interaction between the network and physical layers. The nonlinear threshold is considered as the link capacity constraint and we study the case in which channel powers are adjustable at optical switching nodes (e.g., optical cross-connects (OXCs)). An OSNR Nash game is formulated with coupled utilities and constraints, in which each player (channel) maximizes its own utility function related to minimizing its individual OSNR degradation. We exploit this OSNR Nash game in three types of network topologies: multi-link topology, quasi-ring topology and general topology. A hierarchical decomposition approach leads to a lower-level game for channels with no coupled constraints and a higher-level optimization problem for the network. Computation of equilibria based on this hierarchical algorithm is discussed and evaluated by simulation for each of the three network topologies.

1. Introduction

Optical WDM communication networks have emerged and have been evolving beyond statically designed point-to-point links to reconfigurable networks with arbitrary topologies. In WDM networks, channels travel starting from optical transmitters (Txs), ending at optical receivers (Rxs). Typically, at optical switching nodes, one or more channels will be dropped or added while preserving the integrity of other channels. Channel transmission performance and quality of service (QoS) need to be optimized and maintained after reconfiguration [1]. The impacts of various types of impairments to the signal at the physical layer are non-negligible in optical networks and some of these impairments depend on the network instantaneous traffic, e.g., node crosstalk [2,3]. OSNR is considered as the dominant performance parameter with a bound on nonlinear effects and dispersion by proper link design [2].

A simple network configuration is shown in Fig. 1. WDM optical switching nodes consist of optical line terminals (OLTs), optical add/drop multiplexers (OADMIs), and optical cross-connects (OXCs), etc. Optical amplifiers (OAs) are deployed along the fiber link to amplify the optical power. Optical switching nodes not only provide the advantage of transparency to protocols, bit-rates, etc., but also the flexibility of optical power adjustment [4]. There is a strong interaction between the network and physical layers such that the network design is required to consider the impairments and degradations.

All wavelength-multiplexed channels share the optical fiber in WDM networks. In order to limit the nonlinear effects, the total power launched into each fiber link has to be below the nonlinearity threshold [5]. This can be regarded as the link capacity constraint. We assume that...
channel input power can be adjusted not only at Tx, but also at optical switching nodes (OLTs, OADMs and OXCs). This is achievable since in WDM networks, the switching nodes will incorporate optical amplifiers to make up for power losses [4].

The problem we address here is to develop a cross-layer Nash game framework towards optimizing OSNR in the presence of link capacity constraints in general WDM networks. A short version of this work appeared in [6]. Previous related work in this area includes [7–9]. In [7,8], results were developed for a single point-to-point optical fiber link. The multi-link topology was studied in [9] by introducing a partitioned game with stages. Channel powers are assumed to be adjustable at bifurcation points where channels are dropped (exit).

This paper offers a complete framework for general network topologies. Besides two simple types of network topologies, multi-link topology and quasi-ring topology which were discussed in [6], we also address the general mesh topology. Furthermore, we discuss iterative computation of equilibria based on a three-level hierarchical algorithm and illustrate it with simulation examples.

The rest of this paper is organized as follows. In Section 2, we develop network and OSNR models for general WDM networks. We consider that channel powers are also adjustable at the input side of each optical link. In Section 3, we exploit the cross-layer Nash game and in Section 4 we implement it on three WDM network topologies and give simulation results in Matlab. Section 5 gives conclusions and directions for future work.

2. Network and OSNR models

Consider a generic WDM network configuration, with a set of optical links $\mathcal{L} = \{1, \ldots, L\}$ connecting optical switching nodes. We assume that each optical link is unidirectional. A link $l \in \mathcal{L}$ is composed of $N_l$ cascaded OAs. OAs are used to amplify the optical power of all channels in a link simultaneously, at the expense of introducing amplified spontaneous emission (ASE) noise. We denote the ASE power self-generated in the $k$th amplifier on the $l$th link, for the $i$th channel as $ASE_{i,k,l}$, which is wavelength-dependent. The following assumptions are used: all spans in a link have equal length and all the amplifiers have the same spectral shape. Therefore, on the $l$th link, each $i$th channel experiences a different wavelength-dependent gain, $G_{i,l}$. The OAs operate typically in automatic power control (APC) mode such that a specified target total power, $P_{0,l}$, is launched into each of the following spans in the link $l$, $l \in \mathcal{L}$. Moreover the target total power is selected to be below the threshold for nonlinear effects [10]. Since all spans have the same length, $P_{0,l}$ is the same for all spans in link $l$. For justification of the assumptions, see [11].

A total set of channels, $\mathcal{M} = \{1, \ldots, M\}$, corresponding to a set of wavelengths, is transmitted across the whole network. We denote by $\mathcal{M}_l$ the set of channels transmitted over link $l$, $l \in \mathcal{L}$. For a channel $i \in \mathcal{M}$, we denote by $R_i$ its optical path (or collection of optical links), from Tx to Rx.

2.1. Network model

We define two connection matrices for a general WDM network with $L$ links and $M$ channels: a channel transmission matrix and a system connection matrix. Moreover, we virtually construct a link from each Tx to the connected optical switching node as a virtual optical link (VOL), which is uni-directional. Obviously the total set of VOLs, $\mathcal{L}_v$, is $\mathcal{M}$. Therefore, the set of all optical links and virtual optical links is denoted by $L' = L \cup L_v$.

A channel transmission matrix (CTM) is defined as $A = [A_{ij}]_{i \in \mathcal{M}, j \in \mathcal{M}}$ with

$$A_{ij} = \begin{cases} 1 & \text{channel } i \text{ uses link } l, \text{virtual link } l; \\ 0 & \text{otherwise.} \end{cases}$$

We let $A = [A_1, \ldots, A_L]$ with $A_i = [A_{1,i}, \ldots, A_{M,i}]^T$.

A system connection matrix (SCM) is defined as $B = [B_{kl}]_{k \in \mathcal{M}, l \in \mathcal{L}}$ with

$$B_{kl} = \begin{cases} 1 & \text{link } k \text{ is connected at its output to link } l, \text{i.e., the signal transmission direction between } k \text{ and } l \text{ is } k \rightarrow l; \\ 0 & \text{otherwise.} \end{cases}$$

We also let $B = [B_1, \ldots, B_L]$ with $B_i = [B_{i,1}, \ldots, B_{i,L}]^T$.

The CTM and SCM sufficiently describe the signal routing situation and the system physical connection condition, as shown in the example below.

Consider a mesh topology example in Fig. 2a with channel routings shown in Fig. 2b and in Table 1. The channel transmission and system connection matrices are given as

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the $i$th channel, we denote by $u_i(t_0(i))$ and $p_i(t_0(i))$ the input and output signal (noise) power at Tx and Rx, respectively. We also let

$$\mathbf{u} = [u_1, \ldots, u_i, \ldots, u_M]^T$$
be the vector of channel powers at Tx and let \( u_{-i} \) denote the vector obtained from \( u \) by deleting the \( i \)th element, i.e., 
\[
u_{-i} = [u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_M]^T.
\]

We study the case where dynamic adjustment elements exist such that channel powers can be individually adjusted in the beginning of each optical link, see Fig. 3. For the \( i \)th channel, we denote \( p_{ki} \) and \( n_{ki}^n \) the output optical signal and noise power of the previous link/virtual link \( k, k \in \mathcal{R} \), respectively. We also denote \( u_i \) and \( n_{iL}^n \) the input channel signal and noise power, respectively, of the \( i \)th channel on the \( L \)th link. \( y_{li} \) is the adjustment parameter corresponding to the \( i \)th channel. Let \( u_i = [u_{i1}, \ldots, u_{iM}]^T \) and \( y_i = [y_{i1}, \ldots, y_{iM}]^T \), so that for each input channel power of the \( L \)th link, we have the following results.

**Lemma 1.** The optical signal power at the input of the \( L \)th link is given as 
\[
u_{ij} = A_{ij} + \sum_{k \in \mathcal{L}} (B_{ki}A_{ij}p_{ki}), \quad \forall i \in \mathcal{M}, j \in \mathcal{L},
\]
where 
\[
p_{kj} = \begin{cases} p_{kj}, & k \in \mathcal{L} \\ u_{ij}, & k \in \mathcal{L}_v. \end{cases}
\]

In a vector form, we have 
\[
u_i = \text{Diag}(y_i) \cdot \text{Diag}(V_i^T) \cdot p,
\]
where 
\[
y_i = \begin{bmatrix} y_{i1} \\ \vdots \\ y_{iM} \end{bmatrix}, \quad V_i = \begin{bmatrix} A_{i1}B_{ij} & A_{i2} & \cdots & A_{iM}B_{ij} \end{bmatrix}
\]

and
\[
p = \begin{bmatrix} p_1 \\ \vdots \\ p_M \\ \vdots \\ p_L \end{bmatrix},
\]
with 
\[
p_i = \begin{bmatrix} p_{i1} \\ \vdots \\ p_{iL} \end{bmatrix}.
\]

**Proof.** For channel \( i \), its optical signal power launched into link \( L \) is transmitted either from one of the previous links or one of the virtual links (Tx). Let \( k, k' \in \mathcal{L}' \), be this exact link such that \( B_{k,j} = 1, A_{k,j} = 1 \) and 
\[
u_{ij} = A_{ij} + \sum_{k \in \mathcal{L}', k \neq k'} B_{kj}A_{ij}p_{ki}.
\]
Recall that there is one and only one \( k' \) such that both \( B_{k,j} \) and \( A_{k,j} \) are non-zero. Therefore, the above equation can be rewritten as 
\[
u_{ij} = A_{ij} + \sum_{k \in \mathcal{L}', k \neq k'} B_{kj}A_{ij}p_{ki} = A_{ij} + \sum_{k \in \mathcal{L}', k \neq k'} B_{kj}A_{ij}p_{ki}.
\]
Recall that \( \mathcal{L}' = \{1, \ldots, L'\} \). Thus \( u_{ij} \) can be expressed as 
\[
u_{ij} = \gamma_i \begin{bmatrix} p_{i1} \\ \vdots \\ p_{iL} \end{bmatrix} = \gamma_i V_i^T p_i.
\]
In a vector form, this yields immediately (2). \( \square \)
Remark 1. We note that $\text{Diag} (V_l^T)$ is an augmented system configuration matrix, indicating not only the connections between links but also the channel transmission conditions. If we treat the whole network with channels traveling over it as a system, the inputs of this system are the signal optical power at the output side of each link and the outputs are the output signal power of each link.

The adjustment parameters affect both signal and noise components simultaneously. Similar to (1), the input noise $n_{li}^{\text{in}}$ is given as

$$n_{li}^{\text{in}} = A_{li} H_{li} \sum_{k \in L} (B_k A_k n_{ki}^{\text{out}}), \quad \forall i \in M, \ l \in L,$$

where

$$n_{ki}^{\text{out}} = \{ \begin{array}{ll} n_{ki}^{\text{out}}, & k \in L \setminus \{k_i\} \\ n_{0i}, & k \in L_v \end{array}$$

2.2. OSNR model

The OSNR of channel $i$ at RX is defined as

$$\text{OSNR}_i = \frac{p_i}{n_i}$$

At the output of link $l$ with $l \in R$, we denote the OSNR of channel $i$ after link $l$ by $\text{OSNR}_{li} = \frac{p_{li}}{n_{li}}$. The following lemma describes the signal, noise and OSNR at the output of link $l$, similar to Lemma 2 in [11].

Lemma 2. The optical signal and noise power of channel $i$ at the output of link $l$ are given as

$$p_{li} = H_{li} H_{li}^T u_{li}$$

$$n_{li}^{\text{out}} = H_{li} n_{li}^{\text{in}} + \sum_{v = 1}^{N_l} A_{li} H_{li}^T H_{li} u_{li}$$

where

$$H_{li} = \frac{p_{li} C_{li}^T}{\sum_{j \in R_l} C_{lj}^T u_{lj}}, \quad \forall v = 1, \ldots, N_l$$

The OSNR of channel $i$ at the output of link $l$ is

$$\text{OSNR}_{li} = \frac{u_{li}}{n_{li}^{\text{in}} + \sum_{j \in R_l} A_{li} H_{li}^T H_{li} u_{li}}, \quad \forall i \in M_l$$

where $R_l = \{r_{li}\}$ is the system matrix of link $l$ with

$$R_l = \sum_{v = 1}^{N_l} C_{li}^T A_{li} H_{li}^T H_{li} C_{li}^T, \quad \forall i, j \in M_l$$

Remark 2. A similar approach to this one in [11] can be used to extend the OSNR model to include crosstalk terms due to WDM components at the optical nodes (e.g., OADM or OXC), such as optical filters, demultiplexers, add/drop modules, routers or switches. The only difference is the structure of $R_l$, which in the extended OSNR model will have an extra term given by the specific contribution of crosstalk accumulation.

Based on Lemma 2, especially from (6), a recursive OSNR model for the whole network can be obtained directly.

Lemma 3. Consider link $l$ (Fig. 3), the OSNR of channel $i$ at the output of link $l$ is given recursively as

$$\frac{1}{\text{OSNR}_{li}} = \frac{1}{\text{OSNR}_{R_l}} + \sum_{j \in M_l} A_{li} F_{lj} \frac{u_{lj}}{u_{li}}, \quad i \in M_l,$$

where link $l'$ is the link precedent to link $l$ for channel $i$.

Using Lemma 3, we define

$$\frac{1}{\delta Q_{li}} = \frac{1}{\text{OSNR}_{li}} - \frac{1}{\text{OSNR}_{R_l}},$$

i.e.,

$$\frac{1}{\delta Q_{li}} = \sum_{j \in M_l} A_{li} F_{lj} \frac{u_{lj}}{u_{li}}$$

Usually the low noise at Tx, $n_{0i}$, is negligible, such that at Tx, we let

$$\frac{1}{\text{OSNR}_{0i}} \approx 0$$

Using Lemma 3 and (8) recursively, and neglecting the low noise, $n_{0i}$, we have for the OSNR of channel $i$ at RX:

$$\frac{1}{\text{OSNR}_i} = \sum_{i \in R_i} \left( \frac{1}{\delta Q_{li}} \right).$$

It follows that $1/\delta Q_{li}$ is working as a measure of OSNR degradation for channel $i$, from link $l'$ to link $l$. Recall that we consider that channel powers are also adjustable at optical switching nodes, thus instead of maximization of OSNR$_i$ from Tx to Rx, we consider minimization of each $1/\delta Q_{li}$ between links, i.e., minimization of individual OSNR degradation.

3. OSNR Nash game

3.1. Game formulation

Recall that in an OSNR noncooperative (Nash) game, each channel maximizes its own utility function, $U_i$, or minimizes its individual cost function, $J_i$. The utility function reflects the channel’s preference for optimal OSNR at RX, OSNR$_i$, and typically is selected to be monotonically increasing in OSNR$_i$ [7,8]. Therefore, by minimizing its cost function, in response to the other channels’ actions, each channel minimizes its overall OSNR degradation, $1/\text{OSNR}_i$, from Tx to Rx. This can be denoted as

$$J_i \sim \frac{1}{\text{OSNR}_i}.$$  (10)

Using (9), the foregoing relation can be rewritten as

$$J_i \sim \sum_{l \in R_i} \left( \frac{1}{\delta Q_{li}} \right).$$

Since for each channel $i \in M$, links $l \in R_i$ are ordered sequentially, such that for channel $i$, maximizing OSNR$_i$ can be equivalently formulated by minimizing each of $1/\delta Q_{li}$, i.e., minimizing the OSNR degradation from one link to another.

From this point of view, on each link $l$, we specify a channel cost function for channel $i$, $J_{li}$. First we introduce
a channel utility function, $U_{l,i}$ on each link $l$:

$$U_{l,i} = \ln \left(1 + \frac{A_{l,i}a_{l,i}}{\delta Q_{l,i}}\right), \quad \forall i \in \mathcal{M},$$

(11)

where $a_{l,i} > 0$ is for scalability. The utility function $U_{l,i}$ is monotonically decreasing in $1/\delta Q_{l,i}$ and maximizing utility is related to minimizing the OSNR degradation. Then on each link $l$, we define the channel cost function $J_{l,i}$ as

$$J_{l,i} = -\beta_{l,i}U_{l,i} = -\beta_{l,i} \ln \left(1 + \frac{A_{l,i}a_{l,i}}{\delta Q_{l,i}}\right), \quad \forall i \in \mathcal{M},$$

(12)

where $\beta_{l,i} > 0$ is the channel-defined parameter, indicating the strength of the channel’s desire to minimize its OSNR degradation. We also note that when $A_{l,i} = 0$, i.e., channel $i$ does not travel over link $l$, $J_{l,i} = 0$. Thus we have $J_{l,i} \sim 1/\delta Q_{l,i}$ and $J_{l} \sim \sum_{i \in \mathcal{M}} J_{l,i}$.

Using (8), we can express the cost function, (12), as

$$J_{l,i}(u_{l,-i}, u_{l,i}) = -\beta_{l,i} \ln \left(1 + \frac{A_{l,i}a_{l,i}}{\sum_{l \neq i, j \in \mathcal{M}} A_{l,j} f_{l,i} u_{l,i}}\right),$$

(13)

or

$$J_{l,i}(u_{l,-i}, u_{l,i}) = -\beta_{l,i} \ln \left(1 + \frac{A_{l,i}u_{l,i}}{\sum_{l \neq i, j \in \mathcal{M}} A_{l,j} f_{l,i} u_{l,i}}\right),$$

(14)

where $X_{l,-i} = \sum_{l \neq i, j \in \mathcal{M}} A_{l,j} f_{l,i} u_{l,i}$.

The OSNR Nash game is played under constraints. Note that in order to limit nonlinear effects, the link capacity constraint has to be imposed at the beginning of each optical link, i.e.,

$$\sum_{l \in \mathcal{L}} u_{l,i} \leq P_{0,i}, \quad \forall l \in \mathcal{L},$$

(15)

or

$$\sum_{i \in \mathcal{M}} A_{l,i} u_{l,i} \leq P_{0,i}, \quad \forall l \in \mathcal{L}.$$

(16)

We rewrite (16) as $g_i(u_i) \leq 0$ with

$$g_i(u_i) = \sum_{i \in \mathcal{M}} A_{l,i} u_{l,i} - P_{0,i}.$$  

Thus for each link $l \in \mathcal{L}$, $g_i(u_i) \leq 0$ is convex with respect to $u_i$. Furthermore, the coupled action set defined as

$$\Omega_l = \{u_i \geq 0 | g_i(u_i) \leq 0\}$$

(18)

is compact and convex.

Therefore, the OSNR Nash game is defined as each player (channel) minimizing its own cost function, (13), which is subject to the coupled constraint, (17), in the presence of all other players. A vector $u^*_l = (u^*_{l,-i}, u^*_{l,i}) \in \Omega_l$ is called an NE solution if

$$J_{l,i}(u^*_{l,-i}, u^*_{l,i}) \leq J_{l,i}(u_{l,-i}, u_{l,i}), \quad \forall X_{l,i} \in \Omega_{l,i}$$

$$= \{x_{l,i} \geq 0 | (u^*_{l,-i}, u_{l,i}) \in \Omega_l\}, \quad \forall i \in \mathcal{M} \quad (19)$$

Since each of the channel cost functions, $J_{l,i}$, is strictly convex over a compact and convex set, $\Omega_l$, the existence of an NE solution, $u^*_l$, is guaranteed by Theorem 4.4 in [12].

However, due to the coupled cost function and the coupled constraints, solving directly for an NE solution of this game requires coordination among channels, which is not practical. A natural way is to apply the duality results developed in [8] for Nash games with coupled constraints, to obtain a hierarchical decomposition.

3.2. Hierarchical decomposition

For Nash games with convex coupled constraints, duality based on Lagrangian extension enables a hierarchical decomposition into a lower-level Nash game with no coupled constraints and a higher-level optimization problem for link pricing.

First of all, we use the system-link cost interpretation [12], such that the game on each link $l$ can be equivalently defined as a two-argument function:

$$\bar{J}_l(u_l; x_l) = \sum_{i \in \mathcal{M}} J_{l,i}(u_{l,i}, x_{l,i})$$

where $J_{l,i}$ is defined in (13). Similarly, the two-argument constraints (vector) are given as

$$\bar{g}_l(u_l; x_l) = \sum_{i \in \mathcal{M}} g_i(u_{l,i}, x_{l,i})$$

Both $\bar{J}_l(u_l; x_l)$ and $\bar{g}_l(u_l; x_l)$ are separable in the second argument, $x_l$. The associated two-argument Lagrangian function $\bar{L}_l$ is defined by

$$\bar{L}_l(u_l; x_l; \mu_l) = \bar{J}_l(u_l; x_l) + \mu_l \bar{g}_l(u_l; x_l)$$

(20)

Assume that $u^*_l$ minimizes $\bar{L}_l$, (20), over $x_l \in \Omega_l$ as a fixed point, i.e.,

$$u^*_l = \arg \min_{x_l \in \Omega_l} \bar{L}_l(u_l; x_l; \mu_l)|_{x_l = u_l}$$

(21)

then it has been proved in [8] that $(u_l; x_l) = (u^*_l; u^*_l)$ is an associated NE solution, i.e.,

$$\bar{J}_l(u^*_l; u^*_l) \leq \bar{J}_l(u_l; x_l) \quad \forall x_l \in \Omega_l, \quad \bar{g}_l(u^*_l; x_l) \leq 0,$$

and hence $u^*_l$ is an NE solution in the sense of (19), which is a function of $\mu_l$, i.e., $u^*_l = u^*_l(\mu_l)$.

The associated dual cost function, $D_l(\mu_l)$, is defined as

$$D_l(\mu_l) = \bar{L}_l(u_l^*_l; u^*_l; \mu_l).$$

(22)

The dual problem is defined as $\max_{\mu_l \geq 0} D_l(\mu_l)$ with an optimal dual solution, $\mu^*_l = (u^*_l; \mu^*_l)$ is an optimal NE solution-Lagrange multiplier pair if and only if $u^*_l \in \Omega_l$, $g_l(u^*_l; x^*_l) \leq 0$, $u^*_l \geq 0$, $\mu^*_l \geq 0$, $\bar{g}_l(u^*_l; x^*_l) = 0$ and (21) is satisfied. The dual cost function $D_l(\mu_l)$ can be decomposed such that it can be found by solving a modified Nash game with no coupled constraints. The hierarchical decomposition of $D_l(\mu_l)$ is similar to the result in [8] (Corollary 1), which we restate here.

**Corollary 1.** Consider the coupled OSNR Nash game with cost functions, $J_{l,i}(u_{l,i}, u_{l,-i})$, (14), subject to the linear constraint (18). Then the associated dual cost function $D_l(\mu_l)$, (22), can be decomposed as

$$D_l(\mu_l) = \sum_{i \in \mathcal{M}} \bar{L}_{l,i}(u^*_{l,-i}, u^*_l, \mu_l) + \sum_{i \in \mathcal{M}} \mu_l \bar{e}_{l,i}^T u^*_{l,-i} - P_{0,l}$$

(23)

where

$$\bar{L}_{l,i}(u_{l,-i}, x_{l,i}, \mu_l) = J_{l,i}(u_{l,-i}, x_{l,i}) + \mu_l A_{l,i} x_{l,i}, \quad i \in \mathcal{M}$$

(24)
\[ \mathbf{e}_l = [A_{i1}, \ldots, A_{i(j-1), A_{i(j+1)}, \ldots, A_{iM}}]^T \] and \( \mathbf{u}^i(\mu_i) \) is an NE solution to the lower-level OSNR Nash game with the foregoing modified cost function \( T_{i,j}, (24) \), and no coupled constraints.

For a given price \( \mu_i \), results in [13] have shown that if \( \alpha_i \) selected such that a diagonal dominance condition is satisfied, such a lower-level OSNR Nash game with no coupled constraints admits a closed-form explicit solution.

**Corollary 2.** For a given price \( \mu_i \), the Nash game with cost functions, \( T_{i,j}, (24) \), admits a unique NE solution \( \mathbf{u}^i(\mu_i) \), if \( a_{ij} \) are selected such that

\[ \sum_{j \neq l, j \in M} A_{ij} \Gamma_{ij} < a_{ij}, \quad \forall i \in M_l \]  

(25)

Thus we can obtain the optimal NE solution-Lagrange multiplier pair \( (\mathbf{u}^i, \mu^i) \) in the following way. For each given \( \mu_i \), we see that \( \mathbf{u}^i(\mu_i) \) can be found as a unique NE solution to the modified cost function, \( T_{i,j}, (24) \), with no coupled constraints. Furthermore, \( u^n_i(\mu_i) \) is a linear function of \( 1/\mu_i \) [8], such that the value of \( \sum_{j \in M} u^n_i(\mu_i) \) decreases with \( \mu_i \). Hence the optimal \( \mu^i \) can be obtained by letting the following complementary slackness condition hold:

\[ \mu^i \left( \sum_{i \in M} A_{ij} u^n_{ij} - P_{0,i} \right) = 0 \]  

(26)

### 3.3. Recursive hierarchical algorithm

Next we propose a recursive hierarchical algorithm. The algorithm is developed based on the following coordination mechanism: the link acts as the coordinator and sets the price \( \mu_i \) according to the slackness condition. For each given price \( \mu_i \), each channel responds by adjusting its power \( u_{ij} \) to minimize the individual cost as a lower-level Nash game.

**Channel algorithm:** Based on a given price \( \mu_i(t) \), the optimal channel power is computed by solving

\[ \frac{\partial T_{i,j}}{\partial u_{ij}} (\mathbf{u}_{-i}, u_{ij}, \mu_i(t)) = 0, \quad \forall i \in M_l \]

i.e.,

\[ u_{ij}(n+1) = \frac{\beta_{ij}}{\mu_i(t)} - \frac{X_{\gamma_{ij}}(n)}{a_{ij}}, \quad \forall i \in M_l \]  

(27)

which requires global information. By substituting (8) into (27), the following decentralized channel update algorithm is obtained:

\[ u_{ij}(n+1) = \frac{\beta_{ij}}{\mu_i(t)} \left( \frac{1}{\text{OSNR}_{ij}(n)} - \Gamma_{ij} \right) u_{ij}(n) / a_{ij}, \quad \forall i \in M_l \]

(28)

where link \( l \) is the precedent of link \( i \) and \( \text{OSNR}_{ij,l} \) is invariant during the channel iteration on link \( l \). Here the channel algorithm is developed similar to the one in [8].

The link price, \( \mu_i(t) \), is constant during the channel iteration process.

**Theorem 1.** If \( a_{ij} \) is selected such that (25) is satisfied, then given \( \mu^i \), channel algorithm, (28), converges to the unique NE solution, \( \mathbf{u}^i(\mu^i) \).

**Theorem 1** provides a diagonally dominant condition for the convergence of the channel algorithm, similar to [8].

**Link algorithm:** According the slackness condition, (26), the new link price \( \mu_i \) is computed based on the total received power. Practically, on each link \( l \), after every \( N \) iterations of the channel algorithm such that each channel power has arrived its unique solution, the new link price, \( \mu_i \), is generated at each iteration time \( t \), according to the following link algorithm:

\[ \mu_i(t+1) = \mu_i(t) + \eta \left( \sum_{j \in M} u_{ij}(\mu_i(t)) - P_{0,i} \right) \]

or

\[ \mu_i(t+1) = \left[ \mu_i(t) + \eta \left( \sum_{j \in M} A_{ij} u_{ij}(\mu_i(t)) - P_{0,i} \right) \right]^{+} \]

where \( \eta > 0 \) is a step-size and \( |z|^+ = \max(z, 0) \). This algorithm is a typical gradient projection algorithm [14,15]. Thus given the total power of all channels on the link \( l \), the link (29) is completely distributed and can be implemented by individual links using only local information. For notation simplicity, we denote \( \sum_{j \in M} u_{ij}(\mu_i(t)) \) by \( s(\mu_i(t)) \) or sometimes just \( s(\mu) \) if without causing confusion. In what follows, we further omit all the subscript \( l \) in the equation. Then it becomes

\[ \mu_i(t+1) = [\mu(t) + \eta s(\mu_i(t) - P_0)]^+. \]  

(30)

Before proving the convergence of the link update algorithm, (30), we note the following facts.

**Fact 1:** The function \( s(\mu) \) is a decreasing function with respect to \( \mu \). That is, \( s'(\mu) < 0 \).

**Fact 2:** The function \( s(\mu) \) is a strictly convex function.

**Fact 3:** The system (30) is a time-invariant system.

By Fact 1, we know there is a unique solution \( \mu^* > 0 \) for \( s(\mu) = \Delta P_0 = 0 \), i.e., \( s(\mu^*) - P_0 = 0 \). Also from the above fact, we know \( s(\mu) > P_0 \) when \( \mu < \mu^* \), and \( s(\mu) < P_0 \) when \( \mu > \mu^* \).

Let

\[ \theta(\mu) = \begin{cases} \frac{\mu - \mu^*}{P_0 - s(\mu)} & \text{when } \mu \neq \mu^* \\ \frac{1}{-s'(\mu^*)} & \text{when } \mu = \mu^* \end{cases} \]

**Lemma 4.** The function \( \theta(\mu) \) is positive, continuous everywhere and increasing with \( \lim_{\mu \to 0} \theta(\mu) = 0 \) and \( \lim_{\mu \to \infty} \theta(\mu) = \infty \).

**Proof.** The proof is presented in Appendix A.

Define

\[ \sigma_0 = \inf_{x \in \mathbb{R}^+} x - s^{-1}[2P_0 - s(x)] \frac{P_0 - s(x)}{P_0 - s(\mu)} \]

For \( \mu > \mu^* \), we define

\[ \sigma(\mu) = \inf_{x \in \mathbb{R}^+} x - s^{-1}[2P_0 - s(x)] \frac{P_0 - s(x)}{P_0 - s(\mu)} \]

\[ \inf_{x \in \mathbb{R}^+} x - s^{-1}[2P_0 - s(x)] \frac{P_0 - s(x)}{P_0 - s(\mu)} \]

\[ \inf_{x \in \mathbb{R}^+} x - s^{-1}[2P_0 - s(x)] \frac{P_0 - s(x)}{P_0 - s(\mu)} \]
It can be easily shown that $\sigma(\mu)$ exists and is positive and with $\lim_{\mu \to \infty} \sigma(\mu) = \sigma_0$. Moreover, it is obvious that the function is continuous and non-increasing. Hence, if $\sigma_0 > \theta(\mu_0)$, then there is a unique $\mu^* > \mu_0$ such that $\sigma(\mu^*) = \theta(\mu)$.

The following result proves convergence of link, (30), for sufficient small step-size $\eta$.

**Theorem 2.** If $\sigma_0 \leq \theta(\mu_*)$, let $\eta < \theta(\mu_*)$, otherwise, $\eta < \theta(\mu)$. Then algorithm (30) converges to $\mu^*$.

**Proof.**

**Case 1:** Suppose $\eta \leq \theta(\mu_*)$. Then by Lemma 4 we know that $\theta^{-1}(\eta) < \mu^*$ and that for $\mu \geq \theta^{-1}(\eta)$, $\eta < \theta(\mu)$, which implies $0 < \eta + \eta(\theta(\mu) - P_0) \leq \mu^*$ if $\mu \in (\theta^{-1}(\eta), \mu^*)$ and $\mu + \eta(\theta(\mu) - P_0) > \mu^*$ if $\mu > \mu^*$. That means for initial condition $\mu(0) \in (\theta^{-1}(\eta), \mu^*)$, it is monotonically increasing with upper bound $\mu^*$ and for initial condition $\mu(0) > \mu^*$, it is monotonically decreasing with lower bound $\mu^*$. So they converge. For initial condition $\mu(0) < \theta^{-1}(\eta)$, we obtain $\eta > \theta(\mu(0)) = (\mu(0) - \mu^*)/(P_0 - s(\mu(0)))$ and therefore $\mu + \eta(s(\mu(0)) - P_0) > \mu^*$ (i.e., $\mu(1) > \mu^*$). Then from the above, it converges. In other words, for this initial condition, it jumps one step and then monotonically converges. Case 1 is illustrated in Fig. 4a.

**Case 2:** Suppose $\theta(\mu_*) < \eta < \theta(\mu)$ when $\theta(\mu) < \sigma_0$.

1. Since $\theta(\mu_*) < \eta$, by Lemma 4 one obtains that for any $\mu < \mu^*$, $\eta > \theta(\mu) = (\mu - \mu^*)/(P_0 - s(\mu))$ and therefore $\mu + \eta(s(\mu) - P_0) < \mu^*$ meaning with one step it jumps to above $\mu^*$.

2. For $\mu \in (\mu^*, \theta^{-1}(\eta))$, we have $\eta > \theta(\mu) = (\mu - \mu^*)/(P_0 - s(\mu))$ and therefore $\mu + \eta(s(\mu) - P_0) < \mu^*$ meaning with one step it jumps to below $\mu^*$.

3. For $\mu \in \theta^{-1}(\eta), \infty$, we have $\eta < \theta(\mu) = (\mu - \mu^*)/(P_0 - s(\mu))$ and therefore $\mu + \eta(s(\mu) - P_0) < \mu^*$ meaning with one step it remains above $\mu^*$.

4. Suppose in this part that $\mu \in (\mu^*, \theta^{-1}(\eta))$. First, we have $\eta > \theta(\mu) = (\mu - \mu^*)/(P_0 - s(\mu))$. Second, we have $\eta < \theta(\mu) = (\mu - \mu^*)/(P_0 - s(\mu))$, which implies by the definition that for any $\mu \in (\mu^*, \theta^{-1}(\eta))$, $\eta < \mu - s^\eta(2P_0 - s(\mu))$.

Hence, one obtains

- $\eta(P_0 - s(\mu)) < \mu - s^{-\eta}(2P_0 - s(\mu))$

Combining (1), (2) and the last inequality above, we conclude that for $\mu(0) \in (\mu^*, \theta^{-1}(\eta))$, $\mu(2k)$ is always above $\mu^*$ and is strictly decreasing. Now look at the subsequence $\mu(2k+1)$. From (2), the subsequence $\mu(2k+1)$ is always below $\mu^*$.

5. For any initial condition $\mu(0) \geq \theta^{-1}(\eta)$, we show that it will go into the zone $(\mu^*, \theta^{-1}(\eta))$ at a finite step. (To see this, suppose not. That is, it remains greater than $\theta^{-1}(\eta)$ forever. So it is strictly decreasing. And still by Fact 3, it cannot converge to some other value other that the unique equilibrium $\mu^*$. As long as it goes into the zone $(\mu^*, \theta^{-1}(\eta))$, then by (4) the sequence converges.

6. For any initial condition $\mu(0) < \mu^*$, by (1), it jumps to above $\mu^*$ with one step. So by (4) and/or (5) it converges. Case 2 is illustrated in Fig. 4b.

**4. Implementation cases and simulation results**

In this section, we study the OSNR Nash game with link capacity constraints for WDM network topologies. Note that the implementation of single point-to-point optical link case was studied in [8]. First we study two typical network topology cases: the multi-link topology and the quasi-ring topology. Both of them are representative graphs.

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**Fig. 4.** Illustration for Algorithm (30), Illustration for (a) Case 1 and (b) Case 2.
for selected paths extracted from a mesh configuration. Specifically, the quasi-ring topology is a ring-type topology, with partially closed-loops being formed between channel optical paths [5]. After that, we present and discuss a mesh network topology case. Simulation examples are given after each topology case.

Recall that at each optical switching node (OXC), dynamic adjustment elements, $\gamma_{li}$, $i \in M_l$, exist such that each channel power $u_{li}$ can be individually adjusted at the beginning of each optical link $l$. The dynamic adjustment parameter $\gamma_{li}$ is typically bounded within $[\gamma_{min}, \gamma_{max}]$, say, $[0.1, 10]$. We assume that each channel $i$ is assigned an individual cost function, $J_{li}$, (13), on each link $l$. The diagonal dominant conditions, (25), are satisfied for each game on each link. In the examples simulating the iterative hierarchical algorithm for all cases, the channel update algorithm is repeated with

$$
\rho_{li} = 1 + 0.1 \times i \\
\alpha_{li} = 10 \times I_{li} \\
\alpha_{li} = 50 \times I \times I_{li},
$$

and the initial price for each link is $\mu_{li} = 2$.

### 4.1. Multi-link topologies

We study a simple multi-link topology (Fig. 5) with three links and three channels. The multi-link case has been extensively studied in [9] by introducing a partitioned game with stages. Channel powers are assumed to be adjustable at the bifurcation points by taking advantage of this flexibility in optical networks [4]. Therefore the multi-link OSNR Nash game can be partitioned into stages depending on the bifurcation points, where channels are dropped (exit). Each stage has a single sink multi-link structure and stages are partitioned into stages depending on the bifurcation points. After that each link is a stage, for example in Fig. 5, $K = L = 3$. This case simplifies the partition structure, making it regular and more robust or scalable.

Thus the convexity condition is satisfied with respect to $u_i$ based on (17), i.e., $g_i(u_i)$, $l = 1, 2, 3$ with

$$
g_1(u_1) = u_{1,1} + u_{1,2} - P_{0,1} \\
g_2(u_2) = u_{2,1} + u_{2,2} + u_{2,3} - P_{0,2} \\
g_3(u_3) = u_{3,1} + u_{3,2} - P_{0,3},
$$

are all convex with respect to $u_i$. Therefore on each link $l$, the coupled OSNR Nash game with channel cost functions $J_{li}$, (13), subject to the constraints, (18), can be hierarchically decomposed into a lower-level Nash game for channels with no coupled constraints and a higher-level optimization problem for link price.

We exploit the set of $L$ partitioned games. Each game $l$, $l = 1, \ldots, L$, is played such that $1/\partial Q_{li}$ is minimized for each channel $i$ travelling over link $l$ in a game sense. The $L$ link games are played sequentially (in a precedence or parallel order) with the interpretation that across all $L$ links, $\sum_{l} Q_{li}$ is related to overall OSNR degradation for channel $i$, and with solutions inter-connected in a nested manner.

Channel powers are adjustable at the switching nodes, such that (1) is satisfied, or $u_{li} = \gamma_{li} p_{li}$, where $\gamma_{li}$ is the adjustable parameter for channel $i$ on link $l$ and link $l'$ is the link precedent to link $l$ for channel $i$. The parameters $\alpha_{li}$ are selected such that the diagonal dominant conditions, (25), are satisfied. Then on link $l$, for the given link price $\mu_{li}$, the OSNR Nash game between channels with the cost function $J_{li}$, (13), admits a unique inner NE solution, $\mathbf{u}^* = [u_{li}^* \beta_{li}^*]$. Given the precedent actions for channel $i$, i.e., $u_{li}^*$, $\beta_{li}^*$, $p_{li}^*$ and $\gamma_{li}^*$, the unique solution for channel $i$ on link $l$ is obtained as $\gamma_{li}^* = u_{li}^*/p_{li}^*/(\mu_{li}^*)$. A threshold, $\eta = 0.8$. The game on link $l_1$ is played first. Fig. 6a shows the evolution in time of channel OSNR on link $l_1$. For every $N = 20$ iteration, the link adjusts its price via the link update algorithm and then channels readjust their powers as shown in Fig. 6b. After link $l_1$ settles down, the games on link $l_2$ is played (see Fig. 7) and then link $l_3$ (see Fig. 8).

Each of these games settle down at the NE solution points. After that each link $l$ determines its $\gamma_{li}^*$. In this case, the final value of adjustable parameters is achieved as

$$
\gamma^* = \begin{bmatrix}
0 & 0 & 0 \\
3.6429 & 0.6240 & 0 \\
5.1592 & 1.7449 & 0
\end{bmatrix}
$$

For channel $i$ which is added on link $l$ directly from the transmitter, we set $\gamma_{li} = 0$, since channel power is adjusted at the transmitter. Thus $\gamma^*$ is feasible with respect to the predefined range $[\gamma_{min}, \gamma_{max}]$.

### 4.2. Quasi-ring topologies

In multi-link topologies, the $L$ partitioned games can be automatically played sequentially (in a precedence or parallel order). However in quasi-ring topologies, we need...
to decide the order of links first, such that games among links will be played sequentially. In other words, this requires to choose the starting point (link) for each quasi-ring.

We study a simple quasi-ring topology with three links shown in Fig. 9a, with three channels whose optical paths are shown in Fig. 9b. Similar to the multi-link case, the convexity of link capacity constraints is automatically satisfied.

In this case we select link $l_3$ as the starting link, as shown in Fig. 10. The overall recursive process is such that games on links are played sequentially: $l_3 \rightarrow l_1 \rightarrow l_2$. Specifically the game on each link is hierarchically decomposed by applying the results in Section 3.2.

On link $l_3$, the adjustable parameters for channel 1 and channel 2 are initially set as $\gamma_{1,3}^0$ and $\gamma_{2,3}^0$, respectively, where the superscript 0 indicates the number of iteration of the game among links. The game on link $l_3$ settles down at $u_{3}^0(\mu_{3}^0)$ with the corresponding output channel power $p_{3}^0(\mu_{3}^0)$. Sequentially, the game on link $l_1$ is played with an NE solution, $u_{1}^0(\mu_{1}^0)$. The output channel power is $p_{1}^0(\mu_{1}^0)$. Given $p_{1}^0(\mu_{1}^0)$, the adjustable parameter on link $l_1$ is determined, $\gamma_{1,1}^* = u_{1}^0(\mu_{1}^0)/p_{1}^0(\mu_{1}^0)$. The game on link $l_2$ is played after that. The NE solution of this game is $u_{2}^0(\mu_{2}^0)$ and the output channel power is $p_{2}^0(\mu_{2}^0)$. Then the adjustable parameters on link $l_2$ are determined, $\gamma_{2,2}^* = u_{2}^0(\mu_{2}^0)/p_{2}^0(\mu_{2}^0)$, $i = 1, 2$. With the given $p_{2}^0(\mu_{2}^0)$, link $l_1$ determines its adjustable parameters by $\gamma_{1,1}^* = u_{1}^0(\mu_{1}^0)/p_{1}^0(\mu_{1}^0)$, $i = 1, 2$.

The following is a numerical example. The user-defined step-size in this case is $\eta = 0.1$.

The evolution in time of channel OSNR, total power and link price on each link is shown in Figs. 11–13, respectively.

The adjustable parameters for three links and three channels are obtained as in the following $3 \times 3$ matrix:

$$
\gamma^* = \begin{bmatrix}
0 & 0 & 0.7208 \\
7.5983 & 1.4868 & 0 \\
1.5277 & 0.6042 & 0
\end{bmatrix}
$$
where row indicates the link index and column is the channel index.

The overall game settles down since $\gamma^*$ is feasible. Note that a different starting point can be selected, say, link $l_1$, such that games on links are played sequentially: $l_1 \rightarrow l_2 \rightarrow l_3$. Typically we select the starting point where channels are added directly from transmitters.

4.3. Mesh network topologies

We study a mesh network topology as shown in Fig. 2a, where there are 8 channels transmitted over 6 links. The channel routes are shown in Fig. 2b and Table 1. It can be seen from Fig. 2b that there is a quasi-closed loop among links $l_1$, $l_4$ and $l_6$, being formed between the optical paths of channels 3, 4, and the optical paths of channels 5, 6. We select link $l_1$ as the starting link and unfold the closed-loop, which results in Fig. 14.

Based on Fig. 14, the overall recursive play process is in a ladder-nested form. The games on links $l_1$, $l_4$ and $l_6$ are played in a precedence order: $l_1 \rightarrow l_6 \rightarrow l_4$. Since the quasi-closed loop among links $l_1$, $l_4$ and $l_6$ is unfolded, on link $l_1$ the adjustable parameters for channels 5 and 6 are initially set as $\gamma_{15}^*$ and $\gamma_{16}^*$, respectively. The games on links $l_1$, $l_2$ and $l_5$ can be played in parallel order, while the game on link $l_3$ is played after both the games on links $l_2$ and $l_5$. The last game to be played is the one on link $l_4$. After all the games settle down on $u_i(\mu_f)$ with corresponding output channel powers $p_i(\mu_f)$ and adjustable parameters $\gamma_{i1}^*$, link $l_1$ re-determines its adjustable parameter $\gamma_{11}^*$ according to $p_3(\mu_f)$ because of the quasi-closed loop among links $l_1$, $l_4$ and $l_6$.

We note that link $l_1$ is not the only choice for the starting point. Typically we select the link where channels are added directly from transmitters as the starting point.

The simulation is executed with the user-defined step-size $\eta = 0.6$. The evolution in time of channel OSNR, total power and link price on each link $l$ is shown in Figs. 15–20.

Fig. 8. Multi-link case: Link 3. (a) Channel OSNR, (b) total power and price.

Fig. 9. Quasi-ring topology. (a) Three-link quasi-ring topology, (b) channel optical paths (quasi-ring routes).

Fig. 10. Unfolded with the starting point/link, $l_3$. 

Fig. 8. Multi-link case: Link 3. (a) Channel OSNR, (b) total power and price.
Fig. 11. Quasi-ring case: Link 3. (a) Channel OSNR, (b) total power and price.

Fig. 12. Quasi-ring case: Link 1. (a) Channel OSNR, (b) total power and price.

Fig. 13. Quasi-ring case: Link 2. (a) Channel OSNR, (b) total power and price.
The final adjustable parameter values for 6 links and 8 channels are obtained as in the $6 \times 8$ matrix below:

$$\gamma^* = \begin{bmatrix}
0 & 0 & 0 & 0 & 0.7067 & 1.3548 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.6893 & 0.4308 & 0 & 0 & 0 & 0 & 0 & 0.6535 \\
1.1210 & 0.7271 & 0.8486 & 0.4335 & 0 & 0 & 0 & 1.0415 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2.5839 & 1.1778 & 0 & 0 & 0 & 0
\end{bmatrix}$$

with the same interpretation as before.

5. Conclusions

We have considered the OSNR optimization problem with link capacity constraints in general topology WDM networks. Earlier work in [8] theoretically studied OSNR Nash game with capacity constraints in single point-to-point optical links. We have extended the previous results to games in generic WDM networks. The nonlinear threshold is considered as the link capacity constraint and we assume that channel powers are adjustable at each switching node (OXC). The game was formulated based on the model for the network and the OSNR model for each link.

By selecting a starting point (link), the game on each link can be played sequentially and the hierarchical decomposition led to a lower-level game for channels with no coupled constraints and a higher-level optimization problem for the link pricing. We exploited this OSNR Nash game in three network topologies: multi-link topology, quasi-ring topology and mesh topology. Computation of equilibria based on this hierarchical algorithm is discussed and simulation examples are provided.

Appendix A

Proof of Lemma 4. First, the function $\theta(\mu)$ is obviously continuous everywhere except at $\mu^*$. Note that

$$\lim_{\mu \to \mu^*} \frac{\mu - \mu^*}{\bar{P}_0 - s(\mu) - s(\mu^*)} = 0.$$  

Hence, it is continuous everywhere.

Second, it can be easily checked that the function is positive when $\mu > 0$. Moreover, since $s(\mu) \to \infty$ as $\mu \to 0$ and $s(\mu) \to 0$ as $\mu \to \infty$, it follows that

$$\lim_{\mu \to 0} \frac{\mu - \mu^*}{\bar{P}_0 - s(\mu)} = 0.$$

Fig. 14. Unfolded general topology with the starting point/link, $l_1$.

Fig. 15. Mesh case: Link 1. (a) Channel OSNR, (b) total power and price.
Fig. 16. Mesh case: Link 2. (a) Channel OSNR, (b) total power and price.

Fig. 17. Mesh case: Link 5. (a) Channel OSNR, (b) total power and price.

Fig. 18. Mesh case: Link 6. (a) Channel OSNR, (b) total power and price.
and
\[
\lim_{\mu \to \infty} \frac{\mu - \mu^*}{P_0 - s(\mu)} = \infty.
\]

Third, taking the derivative of \( \theta \) with respect to \( \mu \), we obtain
\[
\theta'(\mu) = \frac{(P_0 - s(\mu)) + (\mu - \mu^*)s'(\mu)}{(P_0 - s(\mu))^2}.
\]

Note that \( s(\mu) \) is a strictly convex function, so \((P_0 - s(\mu)) + (\mu - \mu^*)s'(\mu) > 0\) and therefore \( \theta'(\mu) > 0 \), which implies \( \theta(\mu) \) is increasing. Hence, the conclusion follows. □

References