A Nested Noncooperative OSNR Game in Distributed WDM Optical Links

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***Abstract*— This paper develops a Nash game formulation for optical signal-to-noise ratio (OSNR) in distributed optical links. The starting point is a recent network OSNR model developed for optically amplified links whereby channel powers are adjusted independently only at transmitter sites. A more general case is considered here where channel powers are also adjustable at intermediary dynamic sites, specific to optical networks. For this inherent distributed configuration a nested Nash game is formulated towards minimizing channel OSNR degradation along the link. Existence and uniqueness of the Nash equilibrium solution is shown and a recursive procedure for constructing it is given. Based on this, an iterative algorithm that is distributed with respect to both channels and γ-spans is proposed.**

***Index Terms*— Game theory, noncooperative games, optical networks, optical signal-to-noise ratio, optimization.**

I. INTRODUCTION

THERE has been recent interest in optical wavelength-division multiplexed (WDM) communication networks and their dynamic performance aspects [1]-[2]. An important question is how to realize reconfigurable networks that maintain stability [3] and optimal channel performance after reconfiguration. Channel performance depends on optical signal-to-noise ratio (OSNR), dispersion and nonlinear effects, [4]. Typically, in link optimization OSNR is considered the performance parameter, with dispersion and nonlinearity being kept low by proper link design, [5], [6]. The dominant impairment affecting OSNR is spectrally dependent noise accumulation in chains of optical amplifiers, [6]. Channel OSNR at receiver (Rx) can be equalized by adjusting channel input power at transmitter (Tx). Some static equalization approaches have been developed for single optical links, [6]-[8]. In reconfigurable optical networks different channels can travel via different optical paths. On-line decentralized algorithms are needed, and this problem was recently addressed via either a central optimization or a game theoretic approach, [9]-[11].

Game theoretic models have started to be used recently in networks, [12] -[16], as an alternative to traditional system-wide optimization, [17]-[21]. In large-scale networks decisions are often made by users independently [13], each according to its own performance objective. This is also appropriate for large-scale optical networks, where it is difficult to maintain a centralized system for transmitting real-time information between all channels, and cooperation among channels is impractical. This makes noncooperative game theory a suitable framework, [22], [23]. In a noncooperative (Nash) game players are self-interested; each player minimizes its own cost function (or maximizes its net utility) in the presence of all other players. The game settles at a Nash equilibrium (NE) if one exists. A beneficial feature of a Nash game formulation is that it lends itself naturally to iterative algorithms that are decentralized with respect to the players.

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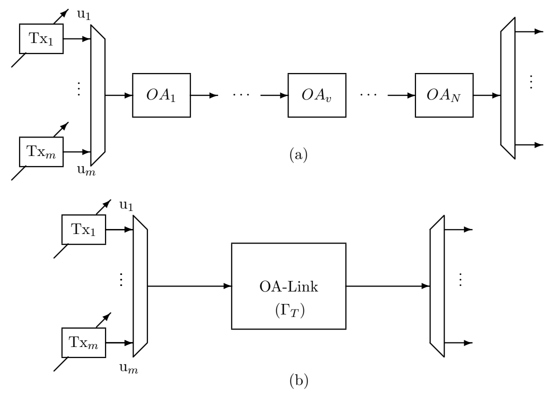

Fig. 1. Lumped OA-link model (Tx to Rx)

Along these lines, in [9] a noncooperative (Nash) game was formulated, with channel utility expressing the preference for higher OSNR. This game was solved explicitly and an iterative decentralized algorithm was proposed.

In this paper we extend the game approach in [9] to a more general configuration motivated as follows. In [9], an optical link is assumed to have only optical amplifiers (OA) as intermediary sites between transmitter and receiver, (Figure 1). Optical amplifiers provide simultaneous channel amplification and maintain a target total output power. In such an OA-link channel powers can be adjusted independently only at the transmitter sites, similar to wireless links [15]-[16]. The resulting OSNR model is an end-to-end model, lumped from Tx to Rx (Figure 1).

Recently dynamic optical filters have started to be used in optical networks, [24]-[27]. This provides the additional flexibility of being able to adjust channel powers at intermediary points, distributed across a link (see Figure 2). This case of distributed γ-spans in optical links is the more general case that we consider in this paper.

The following interesting question can be posed: can we take advantage of this inherent distributed γ-span structure? Specifically can we formulate a corresponding meaningful and tractable OSNR game that is also distributed with respect to γ-spans? By a meaningful game we understand a game with channel cost function being related to the overall OSNR

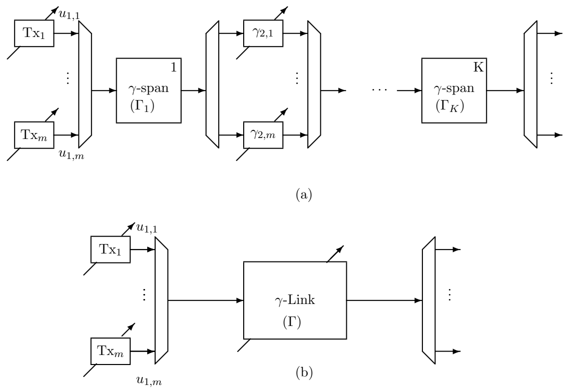

Fig. 2. Distributed γ-link model

performance. By a tractable game we understand a game with an explicitly computable NE solution, that leads to decentralized algorithms that are provable convergent to the NE solution.

We address these questions for a distributed optical γ-span configuration. A short version of this work appeared in [29]. For simplicity we consider a single link with multiple γ-spans, i.e., a distributed optical link. Our approach is based on first generalizing the channel utility function such that it captures the preference for lower OSNR degradation across the link from Tx to Rx. We formulate a nested Nash game with respect to both γ-spans and channels. At each γ-span channels are noncooperative players in a lower-level game. Minimizing channel cost function is related to minimizing OSNR degradation from one γ-span to another. Between all γ-spans we formulate a higher-level Nash game that is naturally in ladder-nested form: each player's action depends only on preceding players' actions, [23]. For each γ-span the NE solution of the channel game is found by applying the results in [9]. At each γ-span, channel power adjustment (player's action) depends on previous γ-spans' actions and on channel output powers. For the overall nested game, we take advantage of the ladder-nested form to develop a recursive procedure for computing the NE solution. This recursive computation is directly amenable to an iterative implementation, distributed with respect to both γ-spans and channels.

The paper is organized as follows. In Section II we review the OSNR model and channel game, [9]. In Section III we extend the OSNR model to a distributed γ- span link configuration. In Section IV we formulate a nested Nash game with respect to both γ-spans and channels. In Section V we show existence and uniqueness of the NE solution and give a recursive procedure for constructing it. We propose an iterative algorithm that is distributed with respect to both γ-spans and channels. A numerical example and conclusions are given in Section VI and VII.

II. BACKGROUND

A. OA-Link / Network OSNR Model

In the following we review the network OSNR model, [9], specialized here for an optical link. Let an OA-link be composed of N cascaded optical amplifiers (OAs), or OA-spans (Figure 1). OAs are used to amplify simultaneously the optical power of all channels, but introduce amplified spontaneous emission (ASE) noise, [4]. For the v^{th} OA in the link let $G_{v,i}$ and $ASE_{v,i}$ denote its gain profile and generated ASE noise power, respectively, both wavelength-dependent. As in [5], [6], [9], the following assumptions are used in an OA-link: all OA-spans have equal length L and all OAs have the same spectral shape, $G_{v,i} = G_i$. These assumptions are representative for typical cases used in industry, [4]. Amplifiers operate typically in automatic power control (APC) mode such that a specified target total power is launched into the next span. This mode compensates for variations in fiber-span loss across a link [6]. Moreover the target total power is selected to be below the threshold for nonlinear effects [5]. Since all spans have same length, this threshold power and hence the total power target P_0 is the same for all OA-spans.

There are m channels / wavelengths transmitted across a link, with $\mathcal{M} = \{1, ..., m\}$ denoting the channel set. For the i^{th} channel let u_i and $n_{0,i}$ denote the input signal and noise optical power (at Tx), respectively. Similarly, let $p_{N,i}$, $n_{N,i}^{out}$ be the output signal and noise optical power (at Rx), respectively. The i^{th} channel OSNR is defined as $OSNR_i = \frac{p_{N,i}}{n_{N,i}^{out}}$. The following result, [9], restated here as Lemma 1, gives the OSNR model for an OA-link.

Lemma 1: The optical signal and ASE noise power at the output of an OA-link are given as

$$
\begin{aligned}
p_{N,i} &= u_i \prod_{q=1}^{N} h_{q,i}, \\
n_{N,i}^{out} &= n_{0,i} \prod_{q=1}^{N} h_{q,i} + \sum_{v=1}^{N} ASE_{v,i} \prod_{q=v+1}^{N} h_{q,i}
\end{aligned}
$$

where

$$
\prod_{q=1}^{v} h_{q,i} = G_i^v \frac{P_0}{\sum_{j \in \mathcal{M}} G_j^v \; u_j}, \qquad \forall v = 1, \ldots, N
$$

The channel OSNR at the output of the link is given as

$$
OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j \in \mathcal{M}} \Gamma_{i,j} \, u_j}
$$

where $\mathbf{\Gamma} = [\Gamma_{i,j}]$ is the system matrix with

$$
\Gamma_{i,j} = \sum_{v=1}^{N} \frac{G_j^v}{G_i^v} \; \frac{ASE_{v,i}}{P_0}, \qquad \forall \, i, j \in \mathcal{M}
$$

B. End-to-end OSNR Game

Based on the OA-link model, a noncooperative channel game was formulated in [9]. We review here the main results. Let $\mathbf{u} = [u_1, \ldots, u_i, \ldots, u_m]^T$ denote the vector of channel powers at Tx, and $\mathbf{u}^{-\mathbf{i}} = [u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m]^T$ the vector obtained by deleting the i^{th} element from $\mathbf{u}$, $u_i \in [u_{min}, u_{max}]$. Each channel is a player that attempts to minimize an individual cost function J_i by adjusting its own action (transmission power) u_i, in response to other channels' (players') actions. The game settles at a Nash equilibrium (NE) solution $\mathbf{u}^*$, [22], [23], if

$$J_i(\mathbf{u}^*) = \inf_{u_i \in [u_{min}, u_{max}]} J_i(u_i, \mathbf{u}^{-\mathbf{i}*}) \qquad \forall i \in \mathcal{M}$$

or, equivalently if, $J_i(u_i^*, \mathbf{u}^{-\mathbf{i}*}) \leq J_i(u_i, \mathbf{u}^{-\mathbf{i}*})$, $\forall i \in \mathcal{M}$, for any given $\mathbf{u}^{-\mathbf{i}*}$.

NE optimality means that no player has an incentive to change its action, since no further individual improvement in its cost is possible. Each cost function J_i is defined as the difference between a pricing P_i and a utility function U_i. The utility U_i is related to channel's performance criteria, while the pricing term P_i is used to penalize a channel for using too large an action (power). Minimizing the cost function is equivalent to maximizing the net utility, i.e., the difference between utility and pricing. In general a pricing mechanism is known to improve the NE efficiency, and a linear pricing is the simplest one, [15]. A simple linear pricing term was used in [9] such that each channel minimizes the cost function J_i

$$J_i(u_i, \mathbf{u}^{-\mathbf{i}}) = \alpha_i \, u_i - \beta_i U_i(u_i, \mathbf{u}^{-\mathbf{i}})$$

Here $\alpha_i > 0$, $\beta_i > 0$ are pricing parameters that capture the trade-off between penalizing a channel for using large power and its desire to maximize the utility. α_i and β_i are set by the network/link and the channel, respectively, and act as weighting factors, quantifying the trade-off between price and utility. Intuitively speaking, a large α_i means a large penalty put by the network on the use of power; a large β_i means a large weight put by the channel on maximizing its utility. Typically α_i, β_i are selected such that an NE solution is inner, or an interior point of the action set, (see [9], [16]). In what follows we assume that an NE solution is inner.

With OSNR as performance criteria, the utility U_i should reflect a channel's preference for high OSNR or for low OSNR degradation. In [9] a logarithmic utility function was used

$$U_i(u_i, \mathbf{u}^{-\mathbf{i}}) = \ln\left(1 + \frac{a_i}{\frac{1}{OSNR_i} - \Gamma_{i,i}}\right) \tag{1}$$

with $a_i > 0$ a channel parameter introduced for flexibility. Since U_i is monotone in $OSNR_i$, it captures the preference for higher OSNR. Using Lemma 1 we see that

$$U_i(u_i, \mathbf{u}^{-\mathbf{i}}) = \ln\left(1 + a_i \, \frac{u_i}{X^{-\mathbf{i}}}\right) \qquad a_i > 0 \tag{2}$$

where $X^{-\mathbf{i}} = \sum_{j \neq i} \Gamma_{i,j} \, u_j + n_{0,i}$. Therefore the channel cost function J_i is

$$J_i(u_i, \mathbf{u}^{-\mathbf{i}}) = \alpha_i u_i - \beta_i \ln\left(1 + a_i \, \frac{u_i}{X^{-\mathbf{i}}}\right) \tag{3}$$

or, equivalently

$$J_i(u_i, \mathbf{u}^{-\mathbf{i}}) = \alpha_i u_i - \beta_i \ln\left(1 + \frac{a_i}{\frac{1}{OSNR_i} - \Gamma_{i,i}}\right) \tag{4}$$

where α_i, β_i are selected such that an NE solution is inner. This cost function captures the preference towards higher OSNR and penalizes the use of large powers. Conditions for existence and uniqueness of NE solution are given in Theorem 1, [9], which is restated here.

Theorem 1: The m-player game with individual cost functions J_i, (3), admits a unique NE solution $\mathbf{u}^*$ if the selected a_i are such that

$$\sum_{j \neq i} \Gamma_{i,j} < a_i, \qquad \forall i \in \mathcal{M} \tag{5}$$

The NE solution $\mathbf{u}^*$ is given as $\mathbf{u}^* = \widetilde{\mathbf{\Gamma}}^{-1} \, \tilde{\mathbf{b}}$, where $\widetilde{\mathbf{\Gamma}} = [\widetilde{\Gamma}_{i,j}]$ and $\tilde{\mathbf{b}} = [\tilde{b}_i]$ are defined as

$$\widetilde{\Gamma}_{i,j} = \begin{cases} a_i, & j = i \\ \Gamma_{i,j}, & j \neq i \end{cases}, \qquad \tilde{b}_i = \frac{a_i \beta_i}{\alpha_i} - n_{0,i}$$

with $\Gamma_{i,j}$ as in Lemma 1.

Remark 1: In general, functions other than logarithmic may be used as utility, such as a linear function in OSNR as mentioned in [9]. Besides reflecting a channel's preference for high OSNR, a general utility function U_i should be twice continuously differentiable, monotone increasing and strictly concave in u_i, as this will guarantee existence of an NE solution, [23]. The advantage of the logarithmic utility function (2) is that it leads to an analytically tractable, closed-form NE solution (Theorem 1). For a given system matrix $\mathbf{\Gamma}$, parameters a_i offer the flexibility to be selected such that the diagonal dominance condition (5) holds, and hence the NE is unique. If $\mathbf{\Gamma}$ is itself diagonal dominant, then a possible choice is $a_i = \Gamma_{i,i}$. Once the system matrix changes (upon reconfiguration, for example), a_i can be adjusted to satisfy condition (5).

Some pricing strategies for selection of parameters α_i, β_i are discussed in [9]. In a proportional pricing scheme, α_i is proportional to the system matrix entry for channel i, $\Gamma_{i,i}$. For example $\alpha_i = \Gamma_{i,i} k_i$ and all β_i set to 1, with the scaling factors k_i selected such that each channel achieve a desired OSNR level.

III. Distributed γ-Link OSNR Model

In this section we extend the OA-link OSNR model to the case of a more general optical link, called γ-link (see Figure 2). In a γ-link every few OA-spans there is an intermediary site with a dynamic gain /adjustment element (DGE). Thus a γ-link is composed of K cascaded γ-spans, each γ-span denoting an optical span with one DGE and R cascaded OAs. In this case channel powers are adjustable not only at the link's input (Tx), but also at each γ-span's input.

DGEs are optical filters with spectrally adjustable attenuation, such that wavelength (channel) powers can be individually adjusted, [24]-[27]. Different spectral resolutions can be achieved depending on the technology, and even a channel decoupled response can be obtained. This justifies a generic DGE model that can be used independent of technology, such that

$$p_{out,i} = \gamma_i \, p_{in,i}, \qquad \forall i \in \mathcal{M} \tag{6}$$

where γ_i is the DGE filter attenuation per channel (adjustable), and $p_{in,i}$, $p_{out,i}$ are the input and output channel power, respectively, [7], [28]. The attenuation factor $\gamma_i \in [\gamma_{min}, \gamma_{max}]$, $0 < \gamma_{min} < \gamma_{max} \leq 1$. Due to insertion loss and cost, DGEs are not inserted at every optical amplifier (OA) site but every R cascaded OAs, thus realizing a γ-span.

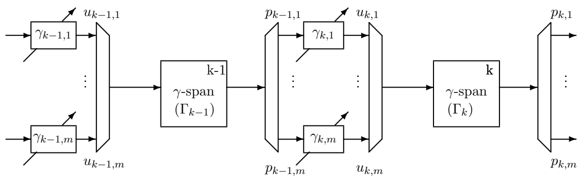

Fig. 3. Two consecutive γ-spans

Let $\mathcal{K} = \{1,...,K\}$ denote the set of γ-spans in a link, and $\mathcal{M} = \{1,...,m\}$ the set of channels. In what follows we let different γ-spans have different OA-span length, L_k, and OAs in different γ-spans have different gain profiles $G_{k,i}$, $k \in \mathcal{K}$. As before within a γ-span all R cascaded OAs have the same spectral shape $G_{k,i}$, and have the same total power target P_{0k}. We denote by $u_{k,i}$, $n_{k,i}^{in}$, the i^{th} channel signal and noise power at the input of k^{th} γ-span, $k \in \mathcal{K}$, $i \in \mathcal{M}$. Similarly, let $p_{k,i}$, $n_{k,i}^{out}$, denote the i^{th} channel signal and noise power, respectively, and $OSNR_{k,i}$,

$$OSNR_{k,i} = \frac{p_{k,i}}{n_{k,i}^{out}}$$

denote channel i^{th} OSNR at the output of k^{th} γ-span, $k \in \mathcal{K}$, $i \in \mathcal{M}$. At the input of the k^{th} γ-span, channel powers can be adjusted individually, so that from (6) we can write recursively,

$$u_{k,i} = \gamma_{k,i}\, p_{k-1,i} \quad \forall k \in \mathcal{K}, \quad \forall i \in \mathcal{M} \tag{7}$$

where $\gamma_{k,i}$ is the channel power adjustment and $p_{k-1,i}$ is the output signal power of the previous γ-span. For $k = 1$, $u_{1,i}$ is the Tx power (input to first span) equal by convention to $\gamma_{1,i}p_{0,i}$, with $p_{0,i}$ as the initial condition. Similarly, for the input noise $n_{k,i}^{in}$ to the k^{th} γ-span we have

$$n_{k,i}^{in} = \gamma_{k,i}\; n_{k-1,i}^{out} \quad \forall k \in \mathcal{K}, \quad \forall i \in \mathcal{M} \tag{8}$$

where $n_{k-1,i}^{out}$ is the output noise of the previous γ-span. The assumption in (8) is that $\gamma_{k,i}$ affects both signal and noise; this is reasonable since an actual DGE filter cannot separate them.

We can use the OA-link model in Lemma 1 together with the connection relations (7) to obtain a recursive γ-span model, for both output signal power and channel OSNR.

Lemma 2: Consider K cascaded γ-spans (Figure 3), each being composed of R cascaded OAs. The following recursive relations hold for any $k \in \mathcal{K}$ and $i \in \mathcal{M}$:

(i) The signal power of the i^{th} channel at the output of the k^{th} γ-span is given recursively as

$$p_{k,i} = \frac{P_{0k}\, G_{k,i}^R}{\sum_{j \in \mathcal{M}} G_{k,j}^R\, \gamma_{k,j}\, p_{k-1,j}}\, \gamma_{k,i} p_{k-1,i} \quad \forall k \in \mathcal{K}$$

(ii) The OSNR of the i^{th} channel at the output of the kth γ-span is given recursively as

$$\frac{1}{OSNR_{k,i}} = \frac{1}{OSNR_{k-1,i}} + \sum_{j \in \mathcal{M}} \Gamma_{k_{i,j}}\, \frac{\gamma_{k,j}\, p_{k-1,j}}{\gamma_{k,i}\, p_{k-1,i}}$$

where $\mathbf{\Gamma_k} = [\Gamma_{k_{i,j}}]$ is the k^{th} γ-span matrix, defined as

$$\Gamma_{k_{i,j}} = \sum_{r=1}^{R} \frac{G_{k,j}^r}{G_{k,i}^r}\, \frac{ASE_{r,i}}{P_{0k}}, \qquad \forall i,j \in \mathcal{M} \tag{9}$$

Proof: Appendix I.

Remark 2: Note that signal power $p_{k,i}$, for the i^{th} channel at the output of the k^{th} γ-span, depends nonlinearly on the corresponding channel power at the output of the $(k-1)^{th}$ span $p_{k-1,i}$ and on the adjustable factor $\gamma_{k,i}$. Also $p_{k,i}$ is coupled to all other channels' powers $p_{k-1,j}$, $j \neq i$. Let $\mathbf{p_k} = [p_{k,1}, \ldots, p_{k,m}]^T$ and $\gamma_\mathbf{k} = [\gamma_{k,1}, \ldots, \gamma_{k,m}]^T$, so that in vector notation

$$\mathbf{p_k} = \mathbf{F_k}(\mathbf{p_{k-1}}, \gamma_\mathbf{k}) \tag{10}$$

with the vector function $\mathbf{F_k}$ defined component-wise by the right-hand side of Lemma 2 (i).

In the following we use the recursive γ-span model in Lemma 2 to obtain the end-to-end γ-link OSNR model (Figure 2) and relate it to the OA-link OSNR model in Lemma 1.

Lemma 3: Consider a γ-link with K γ-spans. The following end-to-end relation holds

$$OSNR_{K,i} = \frac{u_{1,i}}{n_{0,i} + \sum_{j=1}^{m} \Gamma_{\gamma_{i,j}} u_{1,j}}$$

where $\mathbf{\Gamma}_\gamma = [\Gamma_{\gamma_{i,j}}]$, is given as

$$\Gamma_{\gamma_{i,j}} = \sum_{k=1}^{K} \Gamma_{k_{i,j}} \left(\prod_{q=1}^{k-1} \frac{G_{q,j}^R}{G_{q,i}^R} \right) \left(\prod_{r=2}^{k} \frac{\gamma_{r,j}}{\gamma_{r,i}} \right)$$

$u_{1,i} = \gamma_{1,i}\; p_{0,i}$ and $\Gamma_{k_{i,j}}$ as in Lemma 2, (ii).

Proof: Appendix I.

Lemma 3 gives the end-to-end OSNR at the output of a γ-link as a function of input powers at the beginning of the link (Tx) and all the adjustable factors $\gamma_{k,i}$, $k \geq 2$. Note that in [9], only end-to-end adjustment was considered, i.e., only Tx power $u_{1,i}$ (or $\gamma_{1,i}$) is adjustable and $\gamma_{k,i} = 1$, $\forall k \geq 2$. It can be shown that the end-to-end lumped results in Lemma 1 are recovered as a special case of Lemma 3, for $\gamma_{k,i} = 1$, $\forall k \geq 2$ and $N = KR$.

IV. Nested Nash Game Formulation

In this section we formulate a nested Nash game with respect to both K γ-spans and m channels. Let $\mathbf{u}_k = [u_{k,1}, \ldots, u_{k,m}]^T$, $\gamma_k = [\gamma_{k,1}, \ldots, \gamma_{k,m}]^T$ denote the vector of channel input powers and channel actions (adjustments) at the k^{th} γ-span, respectively. From (7) we have

$$\mathbf{u}_k = Diag(\gamma_k)\, \mathbf{p}_{k-1} \qquad k \in \mathcal{K} \tag{11}$$

or equivalently, $\mathbf{u}_k = Diag(\mathbf{p}_{k-1})\, \gamma_k$, where $\mathbf{p}_{k-1}$ is the vector of output powers from the $(k-1)^{th}$ γ-span ((10) or Lemma 2 (i)).

Let $\gamma_k^{-\mathbf{i}} = (\gamma_{k,1}, \ldots, \gamma_{k,i-1}, \gamma_{k,i+1}, \ldots, \gamma_{k,m})$ be the vector obtained by deleting the i^{th} element from γ_k, so that $\gamma_k = (\gamma_{k,i}, \gamma_k^{-\mathbf{i}})$, with $\gamma_{k,i} \in U$, $\gamma_k \in U^m$, $U = [\gamma_{min}, \gamma_{max}]$. Let $\widehat{\gamma} = (\gamma_1, \ldots\ldots \gamma_K)$be the K-tuple of all γ_k, and $\widehat{\gamma}^{-\mathbf{k}} = (\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_K)$ denote the $(K-1)$-tuple obtained by deleting the k^{th} element from $\widehat{\gamma}$. Then $\widehat{\gamma} = (\gamma_k, \widehat{\gamma}^{-\mathbf{k}}) \in U^{mK}$.

We formulate a K-player game between the γ-spans. Each γ-span is a player $\mathbf{P}_k$ with action γ_k, the adjustments of all channels' powers at the k^{th} γ-span. Each γ-span minimizes its own cost function $\mathbf{J}_k$ that depends on the actions of all other players, i.e., on $\widehat{\gamma}$. The game settles at a Nash equilibrium (NE) if one exists, which is defined as follows, [22], [23].

Definition 1: Consider a K-player game between γ-spans. Each player minimizes the cost function $\mathbf{J}_k : U^{mK} \to R$, over $\gamma_k \in U^m$. A vector $\widehat{\gamma}^* = (\gamma_1^*, \ldots, \gamma_K^*)$, $\widehat{\gamma}^* \in U^{mK}$, is called a Nash equilibrium (NE) solution of this game if $\mathbf{J}_k(\widehat{\gamma}^*) = \inf_{\gamma_k \in U^m} \mathbf{J}_k(\gamma_k, \widehat{\gamma}^{-\mathbf{k}*})$, $\forall k \in \mathcal{K}$, or, equivalently, if for any given $\widehat{\gamma}^{-\mathbf{k}*}$,

$$\mathbf{J}_k(\gamma_k^*, \widehat{\gamma}^{-\mathbf{k}*}) \leq \mathbf{J}_k(\gamma_k, \widehat{\gamma}^{-\mathbf{k}*}), \quad \forall \gamma_k \in U^m, \quad \forall k \in \mathcal{K}$$

Definition 1 specifies that $\widehat{\gamma}^*$ is an NE when γ_k^* is solution to the individual optimization problem $\mathbf{J}_k$ for γ-span k, given all γ-spans have equilibrium power levels, $\widehat{\gamma}^{-\mathbf{k}*}$. Existence of an NE solution depends on existence of a common intersection point for the reaction curves of all K players. It can be seen that the K-player game has a fixed order of play, based on the order of precedence for γ-spans in a link. Thus this K-player game is in ladder-nested form, [23]: each player $\mathbf{P}_k$ has access to the information acquired by all his precedents, and the difference between $\mathbf{P}_k$'s information and his precedent $\mathbf{P}_{k-1}$'s information involves only actions of $\mathbf{P}_{k-1}$. Games in ladder-nested form can be recursively decomposed into simpler structures, and Nash equilibria can be obtained recursively. We exploit this in the following.

Each γ-span's cost function is to be minimized in an NE sense, between all channels that share it. Thus at each k^{th} γ-span we define an m-player game between channels, with individual cost $J_{k,i}$. In effect we formulate a nested two-level (K x m) noncooperative game.

Definition 2: For each $k \in \mathcal{K}$, consider an m-player game between channels, with each channel minimizing the cost $J_{k,i} : U^m \to R$, over $\gamma_{k,i} \in U$. A vector (m-tuple) $\gamma_k^* = (\gamma_{k,1}^*, \ldots, \gamma_{k,m}^*) \in U^m$ is called an NE solution of this game if $J_{k,i}(\gamma_k^*) = \inf_{\gamma_{k,i} \in U} J_{k,i}(\gamma_{k,i}, \gamma_k^{-\mathbf{i}*})$, $\forall i \in \mathcal{M}$, or, equivalently, if for each $k \in \mathcal{K}$ and for any given $\gamma_k^{-\mathbf{i}*}$,

$$J_{k,i}(\gamma_{k,i}^*, \gamma_k^{-\mathbf{i}*}) \leq J_{k,i}(\gamma_{k,i}, \gamma_k^{-\mathbf{i}*}), \quad \forall \gamma_{k,i} \in U, \quad \forall i \in \mathcal{M}$$

At each k^{th} γ-span we specify a channel cost function $J_{k,i}$. We start by generalizing the utility (1) , and introduce at each k^{th} γ-span a channel utility function $U_{k,i}$

$$U_{k,i} = \ln(1 + \frac{a_{k,i}}{\frac{1}{\delta Q_{k,i}} - \Gamma_{k_{i,i}}}) \tag{12}$$

where

$$\frac{1}{\delta Q_{k,i}} = \frac{1}{OSNR_{k,i}} - \frac{1}{OSNR_{k-1,i}} \tag{13}$$

Note that $1/\delta Q_{k,i}$, (13) is a measure of the OSNR degradation from one γ-span to another, and the utility $U_{k,i}$, (12), is monotone decreasing in $1/\delta Q_{k,i}$. Thus channel utility $U_{k,i}$ reflects the preference for lower OSNR degradation in the k^{th} γ-span. As in Section II.B, at each k^{th} γ-span we define channel cost $J_{k,i}$ as the net utility in the presence of pricing, i.e., as the difference between a linear pricing term and the foregoing logarithmic utility,

$$J_{k,i} = \quad \alpha_{k,i} u_{k,i} - \beta_{k,i} \ln(1 + \frac{a_{k,i}}{\frac{1}{\delta Q_{k,i}} - \Gamma_{k_{i,i}}}) \tag{14}$$

At each k^{th} γ-span, $\alpha_{k,i}$ and $\beta_{k,i}$ are parameters selected such that an NE solution of the channel game is inner. Thus minimizing channel cost function $J_{k,i}$ is related to minimizing OSNR degradation from one γ-span to another.

Using Lemma 2 (ii), we can express $1/\delta Q_{k,i}$, (13) as

$$\frac{1}{\delta Q_{k,i}} = \sum_{j \neq i} \Gamma_{k_{i,j}} \frac{\gamma_{k,j}}{\gamma_{k,i}} \frac{p_{k-1,j}}{p_{k-1,i}} + \Gamma_{k_{i,i}}$$

Substituting this together with (7) into (14) yields

$$J_{k,i}(\gamma_k) = \quad \alpha_{k,i}\, \gamma_{k,i}\, p_{k-1,i} - \beta_{k,i} \ln(1 + a_{k,i} \frac{\gamma_{k,i}}{\widetilde{X}_{0k}^{-\mathbf{i}}}) \tag{15}$$

where $\widetilde{X}_{0k}^{-\mathbf{i}} = \sum_{j \neq i} \Gamma_{k_{i,j}}\, \gamma_{k,j} \frac{p_{k-1,j}}{p_{k-1,i}}$. Since $p_{k-1,i}$ is given as in Lemma 2 (i),

$$p_{k-1,i} = \frac{P_{0k-1}\, G_{k-1,i}^R}{\sum_{j \in \mathcal{M}} G_{k-1,j}^R \gamma_{k-1,j}\, p_{k-2,j}} \gamma_{k-1,i}\, p_{k-2,i}$$

we see that $J_{k,i}$ (15) depends on $\gamma_{k,i}$, the adjustable parameter (action) at the k^{th} γ-span, and also implicitly on previous γ-span's actions such as $\gamma_{k-1,i}$.

Next for the K player game, we define an appropriate k^{th} γ-span cost $\mathbf{J}_k$, that is related to the cost function $J_{k,i}$ in the channel game. We make use of the following "system-like" cost function interpretation (see (4.10), p. 176 in [23]). Definition 2 involves a set of m inequalities that have to be satisfied simultaneously. It can be equivalently formulated by a two-argument cost function, $\widehat{J}_k(\cdot;\cdot)$, $\widehat{J}_k : U^m \times U^m \to R$, defined as

$$\widehat{J}_k(\gamma_k; \gamma_k^*) := \sum_{i=1}^{m} J_{k,i}(\gamma_{k,i}, \gamma_k^{-\mathbf{i}*}), \quad \forall k \in \mathcal{K} \tag{16}$$

Typically $J_{k,i}$ depends implicitly on the actions taken at previous γ-spans, so that we write

$$J_{k,i}(\gamma_{k,i}, \gamma_k^{-\mathbf{i}*}, \widehat{\gamma}^{-\mathbf{k}*}), \qquad J_{k,i} : U^{mK} \to R \tag{17}$$

For simplicity of notation, we sometimes drop the last argument. From (16, 17), it follows that

$$\widehat{J}_k(\gamma_k; \gamma_k^*, \widehat{\gamma}^{-\mathbf{k}*}) := \sum_{i=1}^{m} J_{k,i}(\gamma_{k,i}, \gamma_k^{-\mathbf{i}*}, \widehat{\gamma}^{-\mathbf{k}*}) \tag{18}$$

The cost function of each k^{th} γ-span in the K-player game is taken as $\widehat{J}_k$, (18), which can be seen as being the net utility over channels at each γ-span, with $J_{k,i}$ as in (15) or (14). The overall $K \times m$ game process is such that at a γ-span each channel plays to minimize its cost (over m channels), and then the γ-span plays its game over the remaining γ-spans. In other words, the K-player game between the γ-spans occurs one by one, first γ-span, second γ-span and so on (fixed-order of play, or precedence order as mentioned before).

Remark 3: Note that using a system-like cost interpretation as in (16), the overall $K \times m$ game between γ-spans and channels has a cost function

$$J_t = \sum_{k=1}^{K} \widehat{J}_k \tag{19}$$

Using (18) yields $J_t = \sum_{k=1}^{K}\sum_{i=1}^{m} J_{k,i}$, which can be interpreted as the sum of net utilities over all γ-spans and over channels. After changing the summation order this gives

$$J_t = \sum_{i=1}^{m} \tilde{J}_i, \qquad with \qquad \tilde{J}_i = \sum_{k=1}^{K} J_{k,i} \tag{20}$$

Thus J_t can be interpreted as the sum of net utilities over channels, each channel with cost $\tilde{J}_i$

$$\tilde{J}_i = \sum_{k=1}^{K} [\alpha_{k,i}\, u_{k,i} - \beta_{k,i} \ln(1 + \frac{a_{k,i}}{\frac{1}{\delta Q_{k,i}} - \Gamma_{k_{i,i}}})]$$

where (14) was used. An interesting parallel can be made between $\tilde{J}_i$ and the cost J_i, (4) used in the end-to-end game in Section II.B. This relation is more evident if we consider the case of identical γ-spans. In this case the parameters $\alpha_{k,i}$, $\beta_{k,i}$, $a_{k,i}$ can be taken to be the same for all γ-spans, $\alpha_{k,i} = \alpha_i$, $\beta_{k,i} = \beta_i$, $a_{k,i} = a_i$ and $\tilde{J}_i$ can be re-written as

$$\tilde{J}_i = \alpha_i \sum_{k=1}^{K} u_{k,i} - \quad \beta_i \sum_{k=1}^{K} \ln(1 + \frac{a_i}{\frac{1}{\delta Q_{k,i}} - \Gamma_{i,i}}) \tag{21}$$

Comparing this with J_i, (4), we see that $\tilde{J}_i$, (21), is similarly expressed as the difference between a linear pricing term and a logarithmic utility term. In fact $\tilde{J}_i$, (21), generalizes J_i, (4), since it captures in pricing and utility the contribution of each γ-span from Tx to Rx.

V. CHARACTERIZATION OF NE SOLUTION AND AN ITERATIVE ALGORITHM

In this section we give conditions for existence and uniqueness of a Nash equilibrium (NE) for the overall game between γ-spans and channels.

Theorem 2: Assume that $J_{k,i}$, $k \in \mathcal{K}$, is defined as in (15). Then the K-player game between γ-spans, with cost function $\widehat{J}_k$, (18), admits an NE solution, $\widehat{\gamma}^*$. If the selected $a_{k,i}$ are such that

$$\sum_{j \neq i} \Gamma_{k_{i,j}} < a_{k,i} \qquad \forall i \in \mathcal{M} \tag{22}$$

with $\Gamma_{k_{i,j}}$ as in Lemma 2, then the NE solution is unique.

Proof: Appendix II.

As in Remark 1, note that for a given γ-span matrix $\mathbf{\Gamma}_k$, the selected $a_{k,i}$ factors can be set such that the diagonal dominance condition (22) holds. If $\mathbf{\Gamma}_k$ is itself diagonal dominant, then a possible choice is $a_{k,i} = \Gamma_{k_{i,i}}$. If all γ-spans are identical then the same $a_{k,i} = a_i$ can be used.

Remark 4: We started the formulation from a K-player game with action space in R^m that was further decomposed. This formulation has a computational advantage when compared to an overall link m-player game with action space in R^K, and a possible cost $\tilde{J}_i$, (21). For the latter game formulation one cannot find an explicit, analytically tractable NE solution, due to the coupling between channel powers at one γ-span and all previous spans (see Lemma 2). Alternatively, the game formulation as developed here corresponds to a ladder-nested structure: in the game between the γ-spans the k^{th} player decision is taken after the $(k-1)^{th}$ player's action. The overall NE solution is still coupled but it has decoupled necessary existence conditions. Moreover, the specific triangular structure of these conditions enables a recursive computation of the unique NE solution (see (36, 37) in Appendix II).

Next we consider the following recursive algorithm for updating channel power level

$$u_{k,i}(n+1) = \frac{\beta_{k,i}}{\alpha_{k,i}} - \frac{X_{0k}^{-\mathbf{i}}(n)}{a_{k,i}} \qquad \forall i \in \mathcal{M} \tag{23}$$

which is based on (35) (Appendix II), where

$$X_{0k}^{-\mathbf{i}} = \sum_{j \neq i} \Gamma_{k_{i,j}} u_{k,j}$$

Note that (23) corresponds to a parallel adjustment scheme, whereby each player responds optimally to the previously selected action of the other players, [23]. Also $X_{0k}^{-\mathbf{i}}$ needs measurements of all input channel powers, $u_{k,j}$, of k^{th} γ-span and all channel gains, hence centralized information. However, using Lemma 2, (ii) and (7) we can write

$$X_{0k}^{-\mathbf{i}} = u_{k,i}\left(\frac{1}{OSNR_{k,i}} - \frac{1}{OSNR_{k-1,i}} - \Gamma_{k_{i,i}}\right)$$

so that $X_{0k}^{-\mathbf{i}}$ is expressed in terms of OSNR at the output of k^{th} γ-span and $(k-1)^{th}$ γ-span (which can be measured in real-time). Using the foregoing relation into (23) together with

$$u_{k,i}(n) = \gamma_{k,i}(n)\, p_{k-1,i}(n-1)$$

based on (7), yields the following channel adjustment $\gamma_{k,i}$ at the input of the k^{th} span

$$\begin{aligned} \gamma_{k,i}(n+1) = {} & \frac{\beta_{k,i}}{\alpha_{k,i}} \frac{1}{p_{k-1,i}(n)} \\ & - \frac{\gamma_{k,i}(n)}{a_{k,i}} \left(\frac{1}{OSNR_{k,i}(n)} - \Gamma_{k_{i,i}}\right) \frac{p_{k-1,i}(n-1)}{p_{k-1,i}(n)} \\ & + \frac{\gamma_{k,i}(n)}{a_{k,i}} \frac{1}{OSNR_{k-1,i}(n)} \frac{p_{k-1,i}(n-1)}{p_{k-1,i}(n)} \end{aligned} \tag{24}$$

Lemma 4: If (22) holds then algorithm (24) converges to the unique NE solution.

Proof Appendix III.

Algorithm (24) is distributed with respect to both channels and γ-spans. The only information fedback is local: individual channel $OSNR$ and channel power from the current and the previous γ-span, and the channel "gain" $\Gamma_{k_{i,i}}$. Moreover, the algorithm is fast and has geometric convergence. As in [11] an additional update factor could be added to allow for a trade-off between speed of convergence and smoother transient response (smaller change at each iteration).

VI. NUMERICAL EXAMPLE

In this section we describe a MATLAB simulation used to exemplify the iterative algorithm. We considered a link with three identical γ-spans (Fig. 2) and eight channels. Each optical amplifier in a γ-span has a parabolic spectral gain profile, a noise figure of 5.2 dB and a total output power of 8

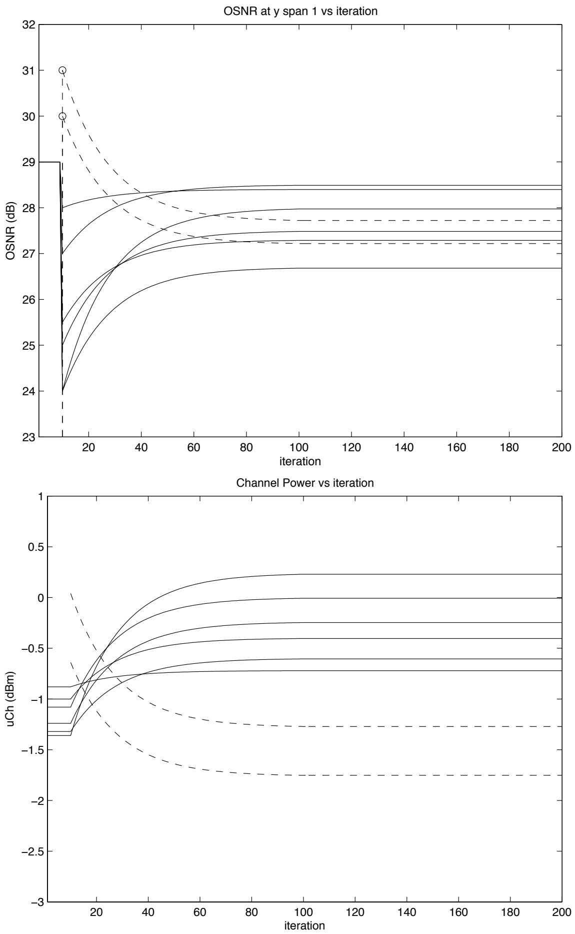

Fig. 4. Evolution of OSNR and input channel power at γ-span 1

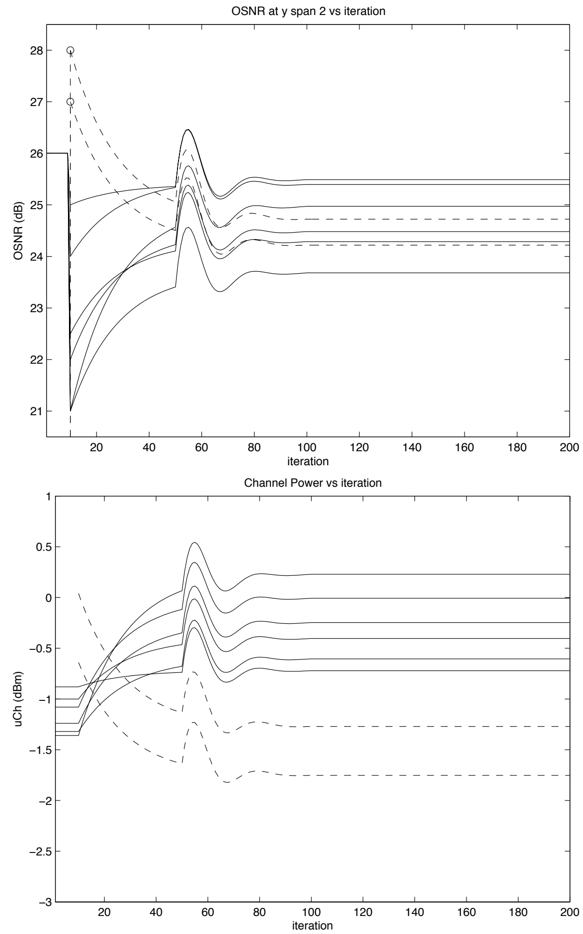

Fig. 5. Evolution of OSNR and input channel power at γ-span 2

dBm, for a span loss of 15 dB. We consider initially that only the first 6 channels were present and the optimal powers set for equalized OSNR, with channel power distributed around the -1dBm. At step $t = 10$ the network is reconfigured such that two new channels, 7 and 8, are added. With transmitter powers maintained at the same level as before the add event, the OSNR for the existing channels has a sudden drop, due to the extra channels that share the link. Channel powers at the input of each of three γ-spans are adjusted using the iterative algorithm. Since the three γ-spans are identical, the same parameters are taken for all. We used $\beta_i = 1$, $a_i = \Gamma_{i,i}$ and α_i proportional to $\Gamma_{i,i}$ (proportional pricing as in Remark 1). Fig. 4, 5 and 6 show the evolution in time of OSNR and input channel power at each of the three γ-spans, respectively. The effect of adaptation and simultaneous change in power at the previous spans is evident in the transient response seen at γ-span 2 and 3 (see Fig. 5 and 6). It can be seen that the channel OSNR at Rx (output of the third γ-span) converge to new steady-state values, while the channel powers are more spread out (Fig. 6).

VII. CONCLUSIONS

In this paper we considered a game formulation for optimizing OSNR, or minimizing the OSNR degradation, in distributed WDM optical links. The starting point was the recent OSNR model in [9], developed for optically amplified links, where channel powers can be adjusted independently only at the transmitter sites. In this paper we considered a more general configuration that allows for adjustable channel powers at intermediary dynamic sites. For this inherent distributed configuration specific to optical networks, a nested Nash game was formulated towards minimizing OSNR degradation from Tx to Rx, with respect to both spans and channels. Existence and uniqueness of the NE solution was shown and a recursive procedure for constructing this NE solution was given. An iterative algorithm was proposed that is distributed with respect to both channels and γ-spans. This algorithm has fast convergence and is based on local feedback of channel parameters from neighboring spans.

OSNR games and games studies in optical networks represent a challenging new research direction and there are many open questions. A possible future direction is to consider other physical impairments in the game and algorithm framework; an example is to incorporate total power constraints that limit nonlinearities as in [10], or adjusting pricing parameters using tools as in [30]. Other open issues are extension to multi-links, studies of pricing mechanisms to analyze the influence of one channel's adjustment on the others' signal quality. Also interesting to explore are studies on incorporating different

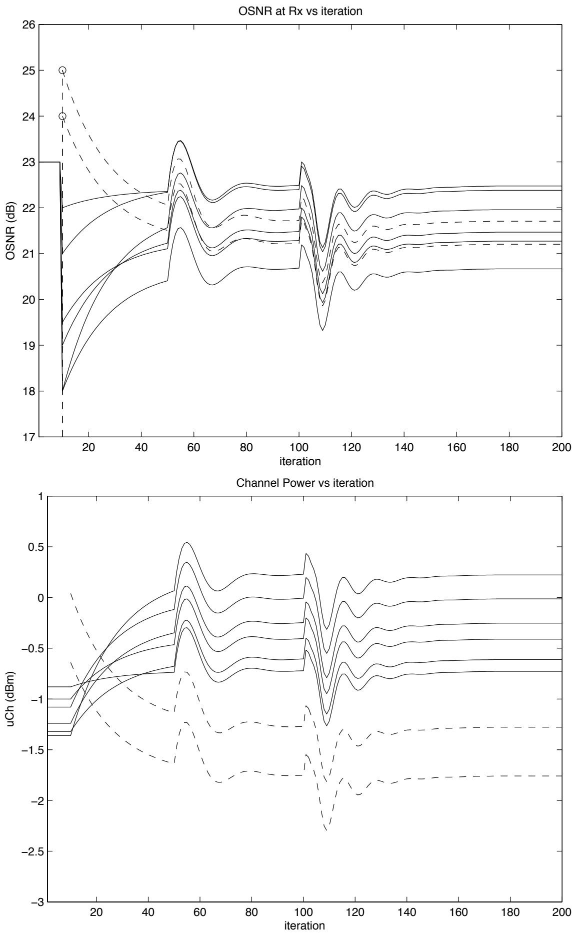

Fig. 6. Evolution of OSNR and input channel power at γ-span 3 (Rx)

OSNR models in the game framework, that are closer to the physical layer, as the one in [31].

Appendix I
Proof of Lemma 2 and Lemma 3

Proof: of Lemma 2

(i) Applying Lemma 1 to the k^{th} γ-span, we have

$$lllp_{k,i} = u_{k,i} \prod_{q=1}^{R} h_{q,i}$$
$$n_{k,i}^{out} = n_{k,i}^{in} \prod_{q=1}^{R} h_{q,i} + \sum_{r=1}^{R} ASE_{r,i} \prod_{q=r+1}^{R} h_{q,i}$$

where

$$\prod_{q=1}^{r} h_{q,i} = G_{k,i}^{r} \frac{P_{0k}}{\sum_{j\in\mathcal{M}} G_{k,j}^{r}\ u_{k,j}}, \qquad \forall r = 1,\dots,R$$

Using the γ-span connection relation (7) and $r = R$ yields (i).

(ii) For the channel OSNR, from Lemma 1 applied to the k^{th} γ-span we have

$$OSNR_{k,i} = \frac{u_{k,i}}{n_{k,i}^{in} + \sum_{j\in\mathcal{M}} \Gamma_{k_{i,j}}\ u_{k,j}} \qquad \forall i \in \mathcal{M}$$

where $\mathbf{\Gamma_k} = [\Gamma_{k_{i,j}}]$ is given in (9). Then

$$\frac{1}{OSNR_{k,i}} = \frac{n_{k,i}^{in}}{u_{k,i}} + \sum_j \Gamma_{k_{i,j}} \frac{u_{k,j}}{u_{k,i}} \qquad \forall i \in \mathcal{M}$$

Using (7, 8) we have

$$\frac{u_{k,i}}{n_{k,i}^{in}} = \frac{\gamma_{k,i} p_{k-1,i}}{\gamma_{k,i} n_{k-1,i}^{out}} = OSNR_{k-1,i}$$

so that from the foregoing we obtain

$$\frac{1}{OSNR_{k,i}} = \frac{1}{OSNR_{k-1,i}} + \sum_j \Gamma_{k_{i,j}} \frac{u_{k,j}}{u_{k,i}}$$

Then (ii) follows immediately by using (7) again. ■

Proof: of Lemma 3

Using Lemma 2, (ii), recursively after k, we have for the end-to-end output OSNR, $\forall i \in \mathcal{M}$

$$\frac{1}{OSNR_{K,i}} = \frac{1}{OSNR_{0,i}} + \sum_{k=1}^{K} \sum_{j=1}^{m} \Gamma_{k_{i,j}} \frac{\gamma_{k,j}}{\gamma_{k,i}} \frac{p_{k-1,j}}{p_{k-1,i}} \qquad (25)$$

Since the output of span 0 is actually the input to span 1, $OSNR_{0,i}$ is the OSNR at Tx, i.e.,

$$OSNR_{0,i} = \frac{u_{1,i}}{n_{0,i}}$$

where $n_{0,i}$ is usually negligible (virtually no ASE at Tx). From Lemma 2 (i) we can write

$$\frac{p_{k,i}}{p_{k,j}} = \frac{G_{k,i}^{R}\,\gamma_{k,i}}{G_{k,j}^{R}\,\gamma_{k,j}} \frac{p_{k-1,i}}{p_{k-1,j}} \qquad (26)$$

so that the ratio of any two channel powers at the k^{th} γ-span is linearly related to the corresponding ratio at the previous γ-span. Using (26) into (25), we obtain after recursive manipulation

$$\frac{1}{OSNR_{K,i}} = \frac{n_{0,i}}{u_{1,i}} + \sum_{k=1}^{K} \sum_{j=1}^{m} \Gamma_{k_{i,j}} \prod_{q=1}^{k-1} \frac{G_{q,j}^{R}}{G_{q,i}^{R}} \left(\prod_{r=2}^{k} \frac{\gamma_{r,j}}{\gamma_{r,i}} \right) \frac{u_{1,j}}{u_{1,i}}$$

which completes the proof. ■

Appendix II
Proof of Theorem 2

We prove the result by applying twice Theorem 4.3, p. 173 in [23]. Consider the cost $\widehat{J}_k$ (18) with $J_{k,i}$ as in (15). It can be seen that $\widehat{J}_k$ is continuous in all its arguments, and is separable in $\gamma_{k,i}$, for every given $\widehat{\gamma}^{-\mathbf{k}}$ and γ_k^*. The gradient of $\widehat{J}_k$ with respect to γ_k is the row vector with the i^{th} component given as

$$\frac{\partial \widehat{J}_k}{\partial \gamma_{k,i}} = \frac{\partial J_{k,i}}{\partial \gamma_{k,i}} \qquad (27)$$

The Hessian of $\widehat{J}_k$ is the symmetric matrix with the $(i,j)^{th}$ element $\frac{\partial^2 \widehat{J}_k}{\partial \gamma_k^2} = [\frac{\partial^2 \widehat{J}_k}{\partial \gamma_{k,j} \partial \gamma_{k,i}}]$ given as

$$\frac{\partial^2 \widehat{J}_k}{\partial \gamma_{k,j} \partial \gamma_{k,i}} = \begin{cases} 0, & j \neq i \\ \frac{\partial^2 J_{k,i}}{\partial \gamma_{k,i}^2}, & j = i \end{cases} \qquad (28)$$

From (15), we see that $\frac{\partial^2 J_{k,i}}{\partial \gamma_{k,i}{}^2} > 0$, for any given $\gamma_{k,j}$ and $p_{k-1,i} \neq 0$. Then for each $k \in \mathcal{K}$, there exists a $\gamma_{k,i}^*$ minimizing $J_{k,i}$ on the closed and bounded (compact) set U, i.e., such that $\forall i \in \mathcal{M}$

$$J_{k,i}(\gamma_{k,i}^*, \gamma_k^{-\mathbf{i}}) < J_{k,i}(\gamma_{k,i}, \gamma_k^{-\mathbf{i}}), \quad \forall \gamma_{k,i} \neq \gamma_{k,i}^* \tag{29}$$

for every given $\gamma_k^{-\mathbf{i}}$. Then for each k, by Theorem 4.3 in [23], there exists a vector solution γ_k^* to the set of m inequalities (29), which is an NE solution to the m-player game and is inner. With the full notation in (17) we obtain

$$J_{k,i}(\gamma_{k,i}^*, \gamma_k^{-\mathbf{i}}, \widehat{\gamma}^{-\mathbf{k}}) < J_{k,i}(\gamma_{k,i}, \gamma_k^{-\mathbf{i}}, \widehat{\gamma}^{-\mathbf{k}}) \tag{30}$$

for every given $\widehat{\gamma}^{-\mathbf{k}}$ and $\forall \gamma_{k,i} \neq \gamma_{k,i}^*$.

Now for the K-player game, from (28) we see that the Hessian of the cost $\widehat{J}_k$ is positive definite, for every given $\widehat{\gamma}^{-\mathbf{k}}$ and γ_k^*. Then $\widehat{J}_k$ is strictly convex in γ_k on the compact and convex set U^m, and there exists a unique mapping T_k minimizing $\widehat{J}_k$, for every given $\widehat{\gamma}^{-\mathbf{k}}$ and γ_k^*. This mapping is the reaction function of the k^{th} player defined as

$$T_k(\gamma_k^*, \widehat{\gamma}^{-\mathbf{k}}) = \{\gamma_k | \widehat{J}_k(\gamma_k; \gamma_k^*, \widehat{\gamma}^{-\mathbf{k}}) < \widehat{J}_k(\mathbf{v}_k; \gamma_k^*, \widehat{\gamma}^{-\mathbf{k}}), \forall \mathbf{v}_k \in U^m\},$$

for any given γ_k^*, and $\widehat{\gamma}^{-\mathbf{k}}$. Applying again Theorem 4.3 in [23], it follows that there exists a vector solution $\widehat{\gamma}^*$ to the set of K foregoing inequalities, which is an NE solution to the K-player game. This $\widehat{\gamma}^*$ is found at the intersection of all reaction functions T_k, i.e., is a fixed-point of T

$$\widehat{\gamma}^* = T(\widehat{\gamma}^*) \tag{31}$$

where $\widehat{\gamma}^* = [\gamma_k^*]$, $T = [T_k]$ in vector notation. In fact using (18, 30) for $\gamma_k = \gamma_k^*$ yields

$$\widehat{J}_k(\gamma_k^*; \gamma_k^*, \widehat{\gamma}^{-\mathbf{k}*}) < \widehat{J}_k(\gamma_k; \gamma_k^*, \widehat{\gamma}^{-\mathbf{k}*}), \quad \forall \gamma_k \in U^m \tag{32}$$

for any given $\widehat{\gamma}^{-*\mathbf{k}}$, for $k \in \mathcal{K}$. Therefore, given a $\gamma_{k,i}^*$ that minimizes $J_{k,i}$, (15), as in (30), we see that the vector $\gamma_k^* = [\gamma_{k,i}^*]$, $i \in \mathcal{M}$, minimizes $\widehat{J}_k$, (18), as in (32). Hence, γ_k^* are the individual components of an NE solution $\widehat{\gamma}^*$ to the K-player game.

Next we prove the uniqueness of this inner NE solution. To find $\widehat{\gamma}^*$, or its components γ_k^*, we solve the necessary conditions

$$\frac{\partial \widehat{J}_k}{\partial \gamma_k} = 0, \qquad \forall k \in \mathcal{K}$$

which defines the k^{th} player's reaction curve, T_k. The vector solution of this set of equations is an NE solution to the K-player game. Recalling (27), this reduces to

$$\frac{\partial J_{k,i}}{\partial \gamma_{k,i}} = 0 \qquad \forall k \in \mathcal{K}, \qquad \forall i \in \mathcal{M} \tag{33}$$

We show next that (33) admits a unique solution $\widehat{\gamma}^*$. For each $k \in \mathcal{K}$, (33) are necessary conditions for finding an NE solution to the m-player game with costs $J_{k,i}$, (15). For each $k \in \mathcal{K}$, $J_{k,i}$ (15) can be equivalently written as

$$J_{k,i}(u_{k,i}, \mathbf{u_k}^{-\mathbf{i}}) = \alpha_{k,i}\, u_{k,i} - \beta_{k,i} \ln \left(1 + a_{k,i} \frac{u_{k,i}}{X_{0k}^{-\mathbf{i}}}\right) \tag{34}$$

with $u_{k,i} = \gamma_{k,i}\, p_{k-1,i}$, (7) and $X_{0k}^{-\mathbf{i}} = \sum_{j \neq i} \Gamma_{k_{i,j}}\, u_{k,j}$. It can be seen that $J_{k,i}$ (34) is similar to J_i (3). For each $k \in \mathcal{K}$, we use Theorem 1 to characterize the NE solution to the m-player game with costs $J_{k,i}$ (34). We will express this NE solution in terms of $\gamma_{k,i}$. Using (7), the necessary conditions (33) become

$$\frac{\partial J_{k,i}}{\partial \gamma_{k,i}} = \frac{\partial J_{k,i}}{\partial u_{k,i}} p_{k-1,i} = 0$$

which leads to $\frac{\partial J_{k,i}}{\partial u_{k,i}} = 0$ since $p_{k-1,i} > 0$. Then using (34) yields

$$a_{k,i}\, u_{k,i}^* + X_{0k}^{-\mathbf{i}*} = \frac{a_{k,i}\beta_{k,i}}{\alpha_{k,i}} \qquad \forall i \tag{35}$$

Since (22) holds for each k, from Theorem 1 it follows that the set of m equations (35) and hence this game admits a unique NE solution in terms of $\mathbf{u}_k$, $\mathbf{u}_k^* = [u_{k,i}^*]$,

$$\mathbf{u}_k^* = \widetilde{\mathbf{\Gamma}}_k^{-1}\, \tilde{\mathbf{b}}_k$$

where $\widetilde{\mathbf{\Gamma}}_\mathbf{k} = [\widetilde{\Gamma}_{i,j}]$ and $\tilde{\mathbf{b}}_\mathbf{k} = [\tilde{b}_{k,i}]$ are defined as

$$\widetilde{\Gamma}_{k_{i,j}} = \begin{cases} a_{k,i}, & j = i \\ \Gamma_{k_{i,j}}, & j \neq i \end{cases} \quad \tilde{b}_{k,i} = \frac{a_{k,i}\beta_{k,i}}{\alpha_{k,i}}$$

Equivalently using (11) this unique NE solution is expressed in terms of γ_k as

$$\gamma_k^* = Diag(\mathbf{v}_{k-1})\widetilde{\mathbf{\Gamma}}_k^{-1}\, \tilde{\mathbf{b}}_k \tag{36}$$

for each given $\mathbf{p}_{k-1}$, where $\mathbf{v}_{k-1} = [1/p_{k-1,1}, \ldots, 1/p_{k-1,m}]$, also denoted as $\mathbf{v}_{k-1} = 1./\mathbf{p}_{k-1}$. Recall that $\mathbf{p}_{k-1}$ is given recursively as in Lemma 2, (i), or compactly, as in (10),

$$\mathbf{p_{k-1}} = \mathbf{F_{k-1}}(\mathbf{p_{k-2}}, \gamma_{\mathbf{k-1}}) \tag{37}$$

Therefore the optimal γ_k^* (36) depends recursively on γ_{k-1}^* via $\mathbf{p}_{k-1}$ (37), and hence depends on $\widehat{\gamma}^{-\mathbf{k}}$. Then the full NE solution $\widehat{\gamma}^*$ with components γ_k^* is the mK vector solution of the set of equations (36, 37) for all k. It can be seen that (36, 37) has a triangular structure. Hence for any given $\mathbf{p}_0$, the unique solution $\widehat{\gamma}^*$ can be found component-wise, by forward substitution. ■

APPENDIX III
PROOF OF LEMMA 4

We prove convergence using the same approach as in [9], based on the equivalence between (24) and (23). Let $e_{k,i}(n) = u_{k,i}(n) - u_{k,i}^*$ where $[u_{k,i}^*]$ is the NE solution given component-wise as in (35). Using (35) and (23) we can write

$$e_{k,i}(n+1) = -\frac{1}{a_{k,i}} \sum_{j \neq i} \Gamma_{k_{i,j}}\, e_{k,j}(n), \qquad \forall i \in \mathcal{M} \tag{38}$$

Let $\mathbf{e}_k(n) = [e_{k,1}(n) \ldots e_{k,m}(n)]^T$ and $\| \mathbf{e}_k(n) \|_\infty := \max_i |e_{k,i}(n)|$. Using the fact that $\mathbf{\Gamma}_k$ is component-wise positive, from the foregoing it follows that for all k

$$\begin{aligned} \| \mathbf{e}_k(n+1) \|_\infty &= \max_i |e_{k,i}(n+1)| \\ &\leq \max_i \Big(\frac{1}{a_{k,i}} \sum_{j \neq i} \Gamma_{k_{i,j}}\, |e_{k,j}(n)|\Big) \end{aligned}$$

Since $|e_{k,j}(n)| \leq \parallel \mathbf{e}_k(n) \parallel_\infty$, $\forall j$, from the foregoing we see that

$$\parallel \mathbf{e}_k(n+1) \parallel_\infty \leq \max_i c_{k,i} \parallel \mathbf{e}_k(n) \parallel_\infty$$

where

$$c_{k,i} := \frac{1}{a_{k,i}} \sum_{j \neq i} \Gamma_{k_{i,j}}$$

From (22) we have $0 \leq c_{k,i} < 1$, $\forall i$, so that $c_{k,max} := \max_i c_{k,i} < 1$. Then, denoting $c_0 := \max_k c_{k,max} < 1$, it follows that from the foregoing inequality we can write

$$\parallel \mathbf{e}_k(n+1) \parallel_\infty \leq c_0 \parallel \mathbf{e}_k(n) \parallel_\infty, \forall k \in \mathcal{K}, \forall n = 0, 1, 2, \ldots$$

where $0 \leq c_0 < 1$. Using again a vector notation $\widehat{\mathbf{e}}(n) = [\mathbf{e}_1(n) \ldots \mathbf{e}_K(n)]^T$ and

$$\parallel \widehat{\mathbf{e}}(n) \parallel_\infty = \max_k \parallel \mathbf{e}_k(n) \parallel_\infty$$

from the K foregoing inequalities it can be seen that

$$\parallel \widehat{\mathbf{e}}(n+1) \parallel_\infty \leq c_0 \parallel \widehat{\mathbf{e}}(n) \parallel_\infty \qquad \forall n = 0, 1, 2, \ldots$$

where $0 \leq c_0 < 1$. Using this recursively we obtain,

$$\parallel \widehat{\mathbf{e}}(n) \parallel_\infty \leq c_0^n \parallel \widehat{\mathbf{e}}(0) \parallel_\infty$$

Since $0 \leq c_0 < 1$, which shows that $\widehat{\mathbf{e}}(n)$ converges geometrically to 0. ■

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