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CONTROLLER REDUCTION FOR NONLINEAR PLANTS—AN L_2 APPROACH

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SUMMARY

This paper extends the H_∞ balanced truncation approach of Mustafa–Glover for linear plants to input affine nonlinear plants. Nonlinear H_∞ balanced truncation is used to obtain a reduced order controller. Conditions which ensure that this controller stabilizes the full order plant are derived. This is done by relating the model reduction problem to a robust stabilization problem with unstructured perturbation. In addition an upper bound on the performance of the closed loop system, with respect to the L_2 gain, is obtained. When specialized to linear plants this bound reduces to Mustafa–Glover's result. © 1997 by John Wiley & Sons, Ltd. Int. J. Robust Nonlinear Control, Vol. 7, 475–505 (1997)

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1. INTRODUCTION

When modern controller design algorithms are applied to high order plants the resulting controller may have order high enough to create problems with its implementation. The controller reduction problem is concerned with reducing the order of the controller so as to maintain the stability of the control system and preserve the control system performance.

Early research on this problem revealed that reducing the order of a full order stabilizing controller using open loop model reduction techniques was not sufficient to preserve closed loop stability. Moreover this drawback was also encountered when the low order controller was obtained by designing it to control a reduced order approximation of the plant. Recently in References 1 and 2 this difficulty has been overcome, in the linear case, by using H_∞ closed loop balancing,[2] to reduce the order of the plant. The controller obtained for this reduced order plant will then be of low order and still achieve both stability and an H_∞ bound when used with the reduced order plant. The preservation of stability when this controller is used with the full order plant is then obtained by resorting to robust stability theory.[3] This is done by viewing the change in the plant resulting from replacing the reduced order plant by the full order plant as an H_∞ bounded unstructured uncertainty.

The nonlinear balancing method and its use in model reduction was introduced in Reference 4 asymptotically stable plants and extended to unstable plants in Reference 5–7. The plant/controller reduction procedure based on nonlinear H_∞ balancing was studied in References 6 and 7. Properties of the H_∞ singular value functions and some properties of the reduced order

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plant were also investigated in References 6 and 7. However, conditions which ensure that the reduced order controller stabilizes not only the reduced order plant but also the full order plant were not considered in References 6 and 7. In addition, the degradation of the performance caused by using the reduced order controller with the full order plant was not considered in those papers. The objective of our paper is to solve these two problems.

After providing some basic definitions in the remaining part of this section, we review, in Section 2, the nonlinear balancing procedure,[4] based on general energy functions, together with the normalized $H_\infty(L_2)$ control problem,[8,9] which is the basis for nonlinear H_∞ balance truncation.[6,7] A candidate reduced order controller is obtained by solving the normalized $H_\infty(L_2)$ control problem for the reduced order plant.

In Section 3 the approach in Reference 2 for linear systems is used to provide a criterion for maintaining the stability of the closed loop system when the reduced order controller obtained in Section 2 is used to control the full order plant, i.e., when the reduced order plant is replaced by the full order plant. This is done by relating the model reduction problem to a robust stabilization problem with unstructured perturbation.[3,9] The degradation in performance which accompanies this substitution is considered in Section 4, where it is shown that the results in Reference 2 are recovered when the plant is assumed to be linear.

The notation used in standard with $\| z \|^2$ denoting the square of the Euclidean norm of the vector $z \in R^{p_1}$, i.e., $z^T z$. Moreover if $V(x)$ is a scalar function of a vector x then $(\partial V/\partial x)(x) = V_x(x)$ denotes the row vector of partial derivatives and $(\partial V_x/\partial x)(x) = (\partial^2 V/\partial x^2)(x)$ is referred to as the Hessian matrix of $V(x)$.

1.1. Preliminaries

A nonlinear system P having input affine state description is given as

$$P\colon \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R^m$ and $y \in R^p$ are the input and output vectors, respectively, with $f, g, h \in C^2$ and $f(0) = 0$, $h(0) = 0$. Notice that $f \in C^2$ implies that $(\partial f/\partial x)(x)$ and $(\partial^2 f/\partial x^2)(x)$ exist and are continuous.

Definition 1

A system P, (1), is said to be

(i) *reachable* (or $[f(x), g(x)]$ is said to be reachable) from the origin if for all $x \in R^n$ there exists a finite time T and an input $u \in L_2[0, T]$ such that the state is driven from $x(0) = 0$ to $x(t) = x$.
(ii) *zero-state observable* if for all $x \in R^n$, $u = 0$, $y = 0$ $\forall t \geqslant 0$ implies $x = 0$.

Two important energy functions which characterize the plant P, (1), if $\dot{x} = f(x)$ is asymptotically stable, are the controllability function and the observability function introduced in Reference 4.

Definition 2

The *controllability function* of system P, (1), is defined by

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0,\, x(0)=x_0}} \int_{-\infty}^{0} \| u(t) \|^2 \, dt$$

and is the unique positive definite solution of the Hamilton Jacobi equality (HJE)

$$\frac{\partial L_c}{\partial x}(x) f(x) + \frac{1}{4}\frac{\partial L_c}{\partial x}(x) g(x) g^{\mathrm{T}}(x)\frac{\partial^{\mathrm{T}} L_c}{\partial x}(x) = 0 \quad L_c(0) = 0 \tag{2}$$

Definition 3

The *observability function* of system P, (1), is defined by

$$L_o(x_0) = \int_0^\infty \| y \|^2 \, \mathrm{d}t \quad x(0) = x_0, \quad u(t) = 0, \quad t \geqslant 0$$

and is the unique non-negative smooth solution of the HJE

$$\frac{\partial L_c}{\partial x}(x) f(x) + h^{\mathrm{T}}(x) h(x) = 0 \quad L_o(0) \tag{3}$$

Definition 4

The *zero dynamics* of a system

$$\dot{x} = f(x) + g(x) u$$
$$y = h(x) + d(x) u$$

is defined as the set of state trajectories $\{x(t)\}$ generated by the set of inputs and initial conditions $\{u^*, x_{u^*}\}$ such that the output is null, i.e.,

$$\dot{x} = f(x) + g(x) u^* \qquad x(0) = x_{u^*}$$
$$0 = h(x) + d(x) u^* \qquad \forall t \geqslant 0$$

Definition 5

A nonlinear system P, (1), is said to have L_2 gain less than or equal γ,[10] if for initial state $x(0) = 0$,

$$\int_0^T \| y(t) \|^2 \, \mathrm{d}t \leqslant \gamma^2 \int_0^T \| u(t) \|^2 \, \mathrm{d}t \quad \forall u \in L_2[0, T], \quad \forall T \geqslant 0$$

Here $u \in L_2[0, T]$ implies that the L_2-norm of u is finite, i.e., $\| u \|_2 = (\int_0^T \| u(t) \|^2 \, \mathrm{d}t)^{1/2} < \infty$.

Finally recall that the $H_\infty(L_2)$ control problem is concerned with the feedback configuration in Figure 1, where both the generalized plant G and the controller K have input affine state descriptions.

If the feedback system in Figure 1 is well posed,[11,12] then the fractional transformation $F_l(G, K)$ on the generalized plant G with coefficient K is defined[8] as the nonlinear operator such that

$$z = F_l(G, K) w \quad \text{for} \quad \begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}, \quad u = K y$$

In the following, the closed loop system shown in Figure 1 will be denoted as $F_l(G, K)$.

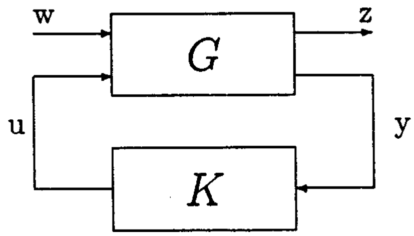

Figure 1. Feedback configuration: $H_\infty(L_2)$ control problem

The $H_\infty(L_2)$ control problem requires the determination of a controller K such that the system $F_l(G, K)$

(i) is asymptotically stable when w is null, i.e., the free system is asymptotically stable
(ii) has L_2 gain from w to z no larger than a prescribed constant γ

This problem was solved for restricted output structure plants in Reference 8. These restrictions on the output structure of the plant were removed recently by the authors in Reference 9.

2. NONLINEAR BALANCING AND MODEL REDUCTION

A reduced order model which approximates the full order plant P, (1), having general energy functions, denoted $E_c(x)$ and $E_o(x)$, can be obtained by using these functions to quantify the importance of each component of the state and truncating that part of the system corresponding to the least important components of the state.[4] If the plant P, (1), is internally asymptotically stable, the energy functions are the controllability and observability functions defined in the introduction. On the other hand if the plant is unstable the energy functions are defined according to a particular closed loop control problems, i.e., LQG control, H_∞ control, etc.

Once the energy functions are chosen, the nonlinear P, (1), is said to be transformed into input normal form if, in the new co-ordinates called input normal co-ordinates, the energy functions satisfy

$$\begin{aligned} \bar{E}_c(\bar{x}) &= \bar{x}^{\mathrm{T}}\bar{x} \\ \bar{E}_o(\bar{x}) &= \bar{x}^{\mathrm{T}}\,\mathrm{Diag}[\omega_1(\bar{x}), \ldots, \omega_n(\bar{x})]\,\bar{x} \end{aligned} \tag{4}$$

for $\bar{x} \in \mathscr{W}$, a neighbourhood of the origin. $\{\omega_i(\bar{x}), i = 1, \ldots, n\}$ is an ordered set of smooth functions of $\bar{x}$ called singular value functions. Conditions for these co-ordinates to exist[4] are reviewed in Appendix A. Thus, from the form (4) we see that it is always possible to arrange the input normal co-ordinates so that

$$\omega_i(\bar{x}) \geqslant \omega_{i+1}(\bar{x}), \quad i = 1, 2, \ldots, n-1 \quad \forall\, \bar{x} \in \mathscr{W} \tag{5}$$

Therefore assuming this arrangement, the state description for P in balanced co-ordinates can be obtained from the foregoing ordered input normal form through a co-ordinate transformation, with the two energy functions being given in balanced co-ordinates by

$$\begin{aligned} E_c(x) &= x^{\mathrm{T}}\,\mathrm{Diag}[\alpha_c^1(x_1), \ldots, \alpha_c^n(x_n)]\,x \\ E_o(x) &= x^{\mathrm{T}}\,\mathrm{Diag}[\beta_o^1(x), \ldots, \beta_o^n(x)]\,x \end{aligned} \tag{6}$$

where all the functions involved are smooth. Balanced co-ordinates for nonlinear systems were introduced in References 4–7 and are discussed further here in Appendix A.

Having obtained a state description for P, (1), which is balanced, we can obtain the reduced order balanced truncation approximation for P by setting $x_i = 0 : i = k+1,\ k+2, \ldots, n$ and truncating the vector functions in (1).

2.1. $H_\infty(L_2)$ Balancing and model reduction

The linear controller reduction method given in Reference 2 and its nonlinear extension,[6,7] rely on the solution to an $H_\infty(L_2)$ control problem referred to as the normalized $H_\infty(L_2)$ control problem. This problem requires the determination of a controller K such that the closed loop system shown in Figure 2

(C1): is asymptotically stable when w_1, w_2 are null
(C2): has L_2 gain from $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ no larger than a prescribed constant $\gamma > 1$

From Figure 2 and the state space description of P, (1), we see that solving the normalized $H_\infty(L_2)$ control problem is equivalent to solving the $H_\infty(L_2)$ control problem (Figure 1) for the generalized plant G, described by

$$G:\begin{cases} \dot{x} = f(x) + [g(x)\ \ 0]w + g(x)u \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h(x) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ y = h(x) + [0\ \ I]w \end{cases} \tag{7}$$

where w, z, u and y in Figure 1 correspond respectively to $[w_2^{\mathrm{T}}\ w_2^{\mathrm{T}}]^{\mathrm{T}}$, $[z_2^{\mathrm{T}}\ z_2^{\mathrm{T}}]^{\mathrm{T}}$, z_2 and e in Figure 2.

Notice that the foregoing generalized plant G, (7), has the same restricted output structure as the generalized plant in Reference 8. Therefore using the solution given in Reference 8 (or Reference 9) we see that the normalized $H_\infty(L_2)$ control problem of interest here has a solution as given in the following lemma.

Lemma 1[8,9]

Consider the configuration in Figure 2 where P, (1), is zero-state observable and $\gamma > 1$, $\beta^2 = 1 - \gamma^{-2}$. If

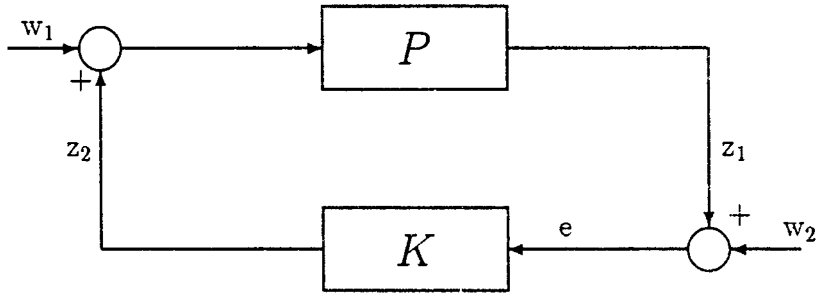

Figure 2. Feedback configuration: normalized $H_\infty(L_2)$ control problem

(i) there exists a positive definite solution $U_o(x)$, $U_o(0)=0$, for the HJE: $\mathscr{H}_{FI}(U_o, x)=0$, where

$$\mathscr{H}_{FI}(U_o, x)=\frac{\partial U_o}{\partial x}(x)f(x)-\frac{1}{4}\beta^2\frac{\partial U_o}{\partial x}(x)g(x)g^{\mathrm{T}}(x)\frac{\partial U_o^{\mathrm{T}}}{\partial x}(x)\ +h^{\mathrm{T}}(x)h(x) \tag{8}$$

(ii) there exists a positive definite solution $U_c(x)$, $U_c(0)=0$, for the HJE: $\mathscr{H}_{FC}(U_c, x)=0$ such that the Hessian matrix of $\mathscr{H}_{FC}(U_c, x)$ at $x=0$ is negative definite, where

$$\mathscr{H}_{FC}(U_c, x)=\frac{\partial U_c}{\partial x}(x)f(x)+\frac{1}{4}\frac{\partial U_c}{\partial x}(x)g(x)g^{\mathrm{T}}(x)\frac{\partial U_c^{\mathrm{T}}}{\partial x}(x)-\beta^2h^{\mathrm{T}}(x)h(x) \tag{9}$$

(iii) $U_c(x)-\gamma^{-2}U_o(x)$ is positive definite, and $T_2(x)$ is a solution of the following equation

$$\left(\frac{\partial U_c}{\partial x}(x)-\gamma^{-2}\frac{\partial U_o}{\partial x}(x)\right)T_2(x)=-2h^{\mathrm{T}}(x) \tag{10}$$

Then, a controller which solves the normalized $H_\infty(L_2)$ control problem for P, (1), is given by

$$K:\ \begin{cases}\dot{\tilde{x}}=A_K(\tilde{x})+B_K(\tilde{x})y\\ u=C_K(\tilde{x})\end{cases} \tag{11}$$

with

$$A_K(\tilde{x})=f(\tilde{x})-\frac{1}{2}\beta^2g(\tilde{x})g^{\mathrm{T}}(\tilde{x})\frac{\partial U_o^{\mathrm{T}}}{\partial x}(\tilde{x})+T_2(\tilde{x})h(\tilde{x})$$

$$B_K(\tilde{x})=-T_2(\tilde{x})$$

$$C_K(\tilde{x})=-\frac{1}{2}g^{\mathrm{T}}(\tilde{x})\frac{\partial U_o^{\mathrm{T}}}{\partial x}(\tilde{x})$$

In addition, the following relation holds along the trajectories of the closed loop system $F_l(G, K)$

$$\dot{Z}(x,\tilde{x})\leqslant\gamma^2\|w\|^2-\|z\|^2 \tag{12}$$

for the positive definite function

$$Z(x,\tilde{x})=\gamma^2U_c(\tilde{x}-x)-U_o(\tilde{x}-x)+U_o(x)$$

Now the function $U_c(x)$ and $U_o(x)$ introduced in the foregoing lemma play important roles in the $H_\infty(L_2)$ balanced truncation reduction of the plant. This is seen as follows.

Lemma 2[6,7]

The positive definite functions $U_c(x)$, $U_o(x)$ specified in Lemma 1 are given as

$$U_c(x)=\min_{\substack{u\in L_2(-\infty,0)\\ x(-\infty)=0,\,x(0)=x}}\int_{-\infty}^{0}(\beta^2\|y(t)\|^2+\|u(t)\|^2)\,\mathrm{d}t,\quad \gamma\neq 1$$

$$U_o(x)=\min_{\substack{u\in L_2(0,\infty)\\ x(0)=x,\,x(\infty)=0}}\int_{0}^{\infty}\left(\|y(t)\|^2+\frac{1}{\beta^2}\|u(t)\|^2\right)\mathrm{d}t,\quad \gamma>1$$

where $U_c(x)$ is called the $H_\infty(L_2)$-past energy function and $U_o(x)$ is called the $H_\infty(L_2)$-future energy function.

Based on these two energy functions, $U_c(x)$ and $U_o(x)$, the balancing procedure as described in Appendix A can be applied and a reduced order plant obtained. Then, based on Lemma 1, we can design a reduced order controller K_r which solves the normalized $H_\infty(L_2)$ control problem for this reduced order plant. At this point two questions arise which are answered in the remainder of this paper. These questions are

1. Under what conditions is the reduced order controller, which stabilizes the reduced order plant, a stabilizing controller for the full order plant P?
2. Assuming that K_r is designed such that when connected to the reduced order model the L_2 gain is less than or equal to γ, what is the new upper bound $\hat{\gamma}$ on the L_2 gain when the reduced controller K_r is connected to the full-order plant P?

The answer to these questions can be given by relating model reduction based on $H_\infty(L_2)$ balancing of P to model reduction based on balancing the normalized right coprime factorization of a scaled plant βP as shown in the following.

2.2. Normalized right coprime factorization and model reduction

The concept of a normalized right coprime factorization (NRCF) for a nonlinear system was introduced in Reference 5. We say that two asymptotically stable nonlinear systems βN and M constitute a right factorization of βP, if $\beta P = (\beta N)M^{-1}$, i.e., if for every initial condition of βP there exist initial conditions of βN and M^{-1} such that the input–output behaviour of βP equals the input–output behaviour of $(\beta N)M^{-1}$. A right factorization βN, M is said to be coprime,[5] if the system $Q = \left[\begin{smallmatrix}\beta N\\ M\end{smallmatrix}\right]$ has asymptotically stable zero dynamics. A right coprime factorization is said to be normalized if the system Q is inner. As in the linear case a nonlinear system Q is inner,[5] if for null initial conditions the system $Q_v \circ Q$ is an identity where Q_v is the adjoint variational system,[13] given as

$$Q_v\colon \begin{cases} \dot{p} = -\left[\dfrac{\partial F_Q}{\partial x}\right]^{\mathrm{T}} p - \left[\dfrac{\partial H_Q}{\partial x}\right]^{\mathrm{T}} u_a \\ y_a = \left[\dfrac{\partial F_Q}{\partial u}\right]^{\mathrm{T}} p + \left[\dfrac{\partial H_Q}{\partial u}\right]^{\mathrm{T}} u_a \end{cases}$$

where

$$Q\colon \begin{cases} \dot{x} = F_Q(x, u) \\ y = H_Q(x, u) \end{cases}$$

The following lemma gives a characterization of an inner system in terms of its state description.

Lemma 3[5]

A system Q having state description

$$Q\colon \begin{cases} \dot{x} = f_Q(x) + g_Q(x)u \\ y = h_Q(x) + D_Q u \end{cases}$$

where $\dot{x} = f_Q(x)$ is asymptotically stable and D_Q is a constant matrix, is inner if

(i) $D_Q^{\mathrm{T}} D_Q = I$

(ii) $\frac{\partial L_{o,Q}}{\partial x}(x)\, g_Q(x) + 2h_Q^{\mathrm{T}}(x) D_Q = 0$

where $L_{o,Q}$ is the observability function of Q.

The following lemma gives a state space characterization of an NRCF.

Lemma 4[5]

Consider the nonlinear input affine system $\bar{P} = \beta P$

$$\bar{P}\colon \begin{cases} \dot{x} = f(x) + g(x)u \\ \bar{y} = \beta h(x) \end{cases} \tag{13}$$

and suppose $X(x) > 0\ \forall x \neq 0$, $X(0) = 0$, is a solution to the HJE

$$\begin{gathered} \mathscr{H}_{GC}(X, x) = 0 \\ \mathscr{H}_{GC}(X, x) = \frac{\partial X}{\partial x}(x) f(x) - \frac{1}{4}\frac{\partial X}{\partial x}(x) g(x) g^{\mathrm{T}}(x) \frac{\partial X^{\mathrm{T}}}{\partial x}(x) + \beta^2 h^{\mathrm{T}}(x) h(x) \end{gathered} \tag{14}$$

Then an NRCF of (13) is given by the systems

$$\begin{gathered} \beta N\colon \begin{cases} \dot{x} = f(x) + g(x)F(x) + g(x)\xi \\ \bar{y} = \beta h(x) \end{cases} \\ M\colon \begin{cases} \dot{x} = f(x) + g(x)F(x) + g(x)\xi \\ u = F(x) \qquad\qquad + \xi \end{cases} \end{gathered} \tag{15}$$

where

$$F(x) = -\frac{1}{2} g^{\mathrm{T}}(x) \frac{\partial X^{\mathrm{T}}}{\partial x}(x)$$

Our goal is the determination of an NRCF for the scaled plant βP, with $\beta = \sqrt{1 - \gamma^{-2}}$. This can be done by solving a normalized $H_\infty(L_2)$ control problem as indicated in the following lemma.

Lemma 5[6]

Consider the nonlinear input affine plant P, (1), and the HJE: $\mathscr{H}_{FI}(U_o, x) = 0$,(8). Then an NRCF of the scaled plant $\bar{P} = \beta P$, (13), with $\beta = \sqrt{1 - \gamma^{-2}}$, is given by (15) where

$$F(x) = -\frac{1}{2}\beta^2 g^{\mathrm{T}}(x) \frac{\partial U_o^{\mathrm{T}}}{\partial x}(x) \tag{16}$$

In the previous section we showed that a reduced order approximation for P, (1), can be obtained by using the $H_\infty(L_2)$ energy functions, $U_c(x)$, $U_o(x)$. We will show that an equivalent reduced order plant, denoted P_r, can be obtained by using balanced truncation on the inner system Q, where

$$Q = \begin{bmatrix} \beta N \\ M \end{bmatrix} \quad \beta P = (\beta N) M^{-1} \tag{17}$$

Since Q, (15), is internally asymptotically stable, the role played by $U_c(x)$, $U_o(x)$ in obtaining a reduced order model for P, (1), can equally well be played by the controllability and observability functions $L_{c,Q}(x)$ and $L_{o,Q}(x)$ in getting a balanced truncation approximate to Q. These functions are simply related to $U_c(x)$, $U_o(x)$ as shown in the following lemma.

Lemma 6[6,7]

The controllability and observability functions $L_{c,Q}(x)$ and $L_{o,Q}(x)$ of Q, (15), are related to $U_c(x)$, $U_o(x)$, (Lemma 2), by the following relations

$$L_{c,Q}(x) = U_c(x) + \beta^2 U_o(x) \quad L_{o,Q}(x) = \beta^2 U_o(x) \tag{18}$$

Next recall that a system is said to be in input normal co-ordinates when its energy functions satisfy (4). Now the energy functions for Q are $L_{c,Q}(x)$ and $L_{o,Q}(x)$ whereas those for P are $U_c(x)$ and $U_o(x)$. The simplicity of (18) enables Q, (15), (17), to be put in input normal co-ordinates once P, (1), has been put in input normal co-ordinates. This fact, which is described in the following lemma, is of importance in showing the equivalence between the reduced order models for the plant obtained by either balanced truncation of Q, using $L_{c,Q}$ and $L_{o,Q}$, or balanced truncation of P, using U_c, U_o.

Lemma 7[6,7]

Assume that U_c and U_o have input normal form as in (4) with the H_∞ singular value functions $\tau_i(\bar{x})$, i.e., $\tau_i(\bar{x}) = \omega_i(\bar{x})$ in (4). Then the co-ordinate transformation $\phi(\bar{x})$ needed to bring $L_{c,Q}$ and $L_{o,Q}$ into input normal form is

$$\hat{x} = \phi(\bar{x}), \quad \phi(\bar{x}) = [\phi_1(\bar{x}) \cdots \phi_n(\bar{x})]^{\mathrm{T}}$$

where

$$\phi_i(\bar{x}) = [1 + \beta^2 \tau_i(\bar{x})]^{1/2} \bar{x}_i \quad i = 1, \ldots, n$$

and $L_{c,Q}$, $L_{o,Q}$ are given by

$$L_{c,Q}(\hat{x}) = \hat{x}^{\mathrm{T}} x \quad L_{o,Q}(\hat{x}) = \hat{x}^{\mathrm{T}}\, \mathrm{Diag}[\sigma_1(\hat{x}), \ldots, \sigma_n(\hat{x})]\, \hat{x}$$

where the H_∞ singular value functions of P, $\tau_i(\bar{x})$, are related to the singular value functions of Q, $\sigma_i(\phi(\bar{x}))$, as

$$\tau_i(\bar{x}) = \beta^{-2} \frac{\sigma_i(\phi(\bar{x}))}{1 - \sigma_i(\phi(\bar{x}))} \quad i = 1, \ldots, n$$

Notice that $\tau_1(\bar{x}) \geqslant \cdots \geqslant \tau_n(\bar{x})$ is equivalent to $\sigma_1(\phi(\bar{x})) \geqslant \cdots \geqslant \sigma_n(\phi(\bar{x}))$. Notice also that the form of the transformation $\hat{x} = \phi(\bar{x})$ given in the lemma is such that $\hat{x}_k = 0$ is equivalent to $\bar{x}_k = 0$. Hence, using either pair of energy functions leads to reduced models having the same order.

Assume from now on that Q, (15), (17), is in balanced co-ordinates with $L_{c,Q}$ and $L_{o,Q}$ in the same form as (6). Then partitioning the state vector together with the corresponding vector functions as

$$x = (x^r, x^q) \quad \text{with} \quad x^r = (x_1, \ldots, x_k) \quad x^q(x_{k+1}, \ldots, x_n)$$

$$f(x) = \begin{pmatrix} f^r(x^r, x^q) \\ f^q(x^r, x^q) \end{pmatrix} \quad g(x) = \begin{pmatrix} g^r(x^r, x^q) \\ g^q(x^r, x^q) \end{pmatrix} \quad h(x) = h(x^r, x^q)$$

and setting $x^q = 0$, gives a reduced order approximation for Q, Q_r, having state description

$$Q_r = \begin{bmatrix} \beta N_r \\ M_r \end{bmatrix} : \begin{cases} \dot{x}^r = \hat{f}(x^r) + \hat{g}(x^r) F_r(x^r) + \hat{g}(x^r)\xi \\ \begin{bmatrix} \bar{y}_r \\ u_r \end{bmatrix} = \begin{bmatrix} \beta \hat{h}(x^r) \\ F_r(x^r) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \xi \end{cases} \tag{19}$$

and a reduced order approximation, $\bar{P}_r$, of the scaled plant, βP, (13), having state description

$$\bar{P}_r\colon \begin{cases} \dot{x}^r = \hat{f}(x^r) + \hat{g}(x^r) u_r \\ \bar{y}_r = \beta \hat{h}(x^r) \end{cases} \qquad \bar{P}_r = (\beta N_r) M_r^{-1} \tag{20}$$

$$\hat{f}(x^r) = f(x^r, 0), \quad \hat{g}(x^r) = g^r(x^r, 0), \quad \hat{h}(x^r) = h(x^r, 0), \quad F_r(x^r) = F(x^r, 0)$$

Denote the error system between Q, (15), and Q_r, (19), by

$$\begin{bmatrix} \beta \Delta N \\ \Delta_M \end{bmatrix} = \begin{bmatrix} \beta N \\ M \end{bmatrix} - \begin{bmatrix} \beta N_r \\ M_r \end{bmatrix} \tag{21}$$

and denote the upper bound on the L_2 gain of (21) by $\tilde{\varepsilon} = \beta\varepsilon$ with $\beta = \sqrt{1 - \gamma^{-2}}$, i.e.,

$$\left\| \begin{bmatrix} \beta \Delta N \\ \Delta_M \end{bmatrix} (v) \right\|_2 \leqslant \tilde{\varepsilon} \| v \|_2 \quad \text{with } \varepsilon = \beta\varepsilon \tag{22}$$

The main result for this section is given now in the following theorem.

Theorem 1

If Q, (15), is in balanced form and if

$$\frac{\partial L_{o,Q}}{\partial x^q}(x^r, 0) f^q(x^r, 0) = 0$$

$$\frac{\partial L_{o,Q}}{\partial x^q}(x^r, 0)\, g^q(x^r, 0) = 0$$

then $X_r(x^r) = \beta^2 U_o(x^r, 0)$ is a solution to

$$\mathscr{H}_{GC}(X_r, x^r) = 0 \tag{23}$$

$$\mathscr{H}_{GC}(X_r, x^r) = \frac{\partial X_r}{\partial x^r}(x^r)\hat{f}(x^r) - \frac{1}{4}\frac{\partial X_r}{\partial x^r}(x^r)\hat{g}(x^r)\hat{g}^{\mathrm{T}}(x^r)\frac{\partial X_r^{\mathrm{T}}}{\partial x^r}(x^r) + \beta^2 \hat{h}^{\mathrm{T}}(x^r)\hat{h}(x^r)$$

and the reduced order approximation Q_r, (19), of Q, (15), is inner where $F(x)$, (16), becomes

$$F_r(x^r) = -\frac{1}{2}\beta^2 \hat{g}^{\mathrm{T}}(x^r)\frac{\partial U_o^{\mathrm{T}}}{\partial x^r}(x^r, 0) \tag{24}$$

In addition the following reduced order approximation of the original plant P, (1), is obtained

$$P_r\colon \begin{cases} \dot{x}^r = \hat{f}(x^r) + \hat{g}(x^r) u \\ y = \hat{h}(x^r) \end{cases} \tag{25}$$

where P_r, (25), satisfies

$$P_r = N_r M_r^{-1}$$

and the reduced order controller which solves the normalized $H_\infty(L_2)$ control problem for P_r, (25), is given by

$$K_r\colon \begin{cases} \dot{\tilde{x}}^r = \hat{A}_K(x^r) + \hat{B}_K(\tilde{x}^r)\,y \\ u = \hat{C}_K(\tilde{x}^r) \end{cases} \tag{26}$$

with

$$\hat{A}_K(\tilde{x}^r) = \hat{f}(\tilde{x}^r) - \frac{1}{2}\beta^2 \hat{g}(\tilde{x}^r)\hat{g}^{\mathrm{T}}(\tilde{x}^r)\frac{\partial U_{or}^{\mathrm{T}}}{\partial x^r}(\tilde{x}^r) + T_2^r(\tilde{x}^r)\hat{h}(\tilde{x}^r)$$

$$\hat{B}_K(\tilde{x}^r) = -\,T_2^r(\tilde{x}^r)$$

$$\hat{C}_K(\tilde{x}^r) = -\frac{1}{2}\hat{g}^{\mathrm{T}}(\tilde{x}^r)\frac{\partial U_{or}^{\mathrm{T}}}{\partial x^r}(\tilde{x}^r)$$

where $T_2^r(\tilde{x}^r)$ is a solution to

$$\left(\frac{\partial U_{cr}}{\partial x^r}(x^r) - \gamma^{-2}\frac{\partial U_{or}}{\partial x^r}(x^r)\right)T_2^r(x^r) = -\,2\hat{h}^{\mathrm{T}}(x^r)$$

and

$$U_{or}(x^r) = U_o(x^r, 0) \quad U_{cr}(x^r) = U_c(x^r, 0)$$

Finally, if the system $\dot{x} = f(x) - \beta^2 g(x) g^{\mathrm{T}}(x)(\partial U_o^{\mathrm{T}}/\partial x)(x)$ is exponentially asymptotically stable then the error system (21) is locally asymptotically stable and has L_2 gain $\leqslant \tilde{\varepsilon}$, (22), for any $\tilde{\varepsilon} > 2(\delta_{k+1}(0) + \cdots + \delta_n(0))$, where $\delta_i(0) = \sigma_i^{1/2}(0)$, $i = 1, \ldots, n$.

Proof. Appendix B □

Having defined the error system as (21) we see that the original unscaled plant can be recovered by multiplying the scaled perturbed plant by β^{-1}, i.e.,

$$P = NM^{-1} = (N_r + \Delta_N)(M_r + \Delta_M)^{-1} \tag{27}$$

where $P = \beta^{-1}\bar{P}$ and

$$\bar{P} = \beta NM^{-1} = (\beta N_r + \beta\Delta_N)(M_r + \Delta_M)^{-1}$$

Consequently question 1 posed at the end of the previous section leads to a type of unstructured robust stabilization problem. In this problem the original full order plant is one of the elements of the set of plants generated by perturbing the factors of a normalized coprime factorization of the scaled reduced order plant. This type of robust stabilization problem,[3] was solved recently for nonlinear plants,[9] by relating it to a generic robust stabilization problem. This approach is used now to characterize the stability of the closed loop system formed from the original full order plant and the reduced order controller.

3. CONSERVATION OF STABILITY

This section deals mainly with the first question posed at the end of Section 2.1. The answers to both questions posed there are obtained by generalizing the approach in Reference 2 for linear plants. In that approach the full order plant is viewed as a coprime perturbation on a scaled version of the reduced order plant. The resulting normalized L_2 feedback configuration is shown in Figure 3.

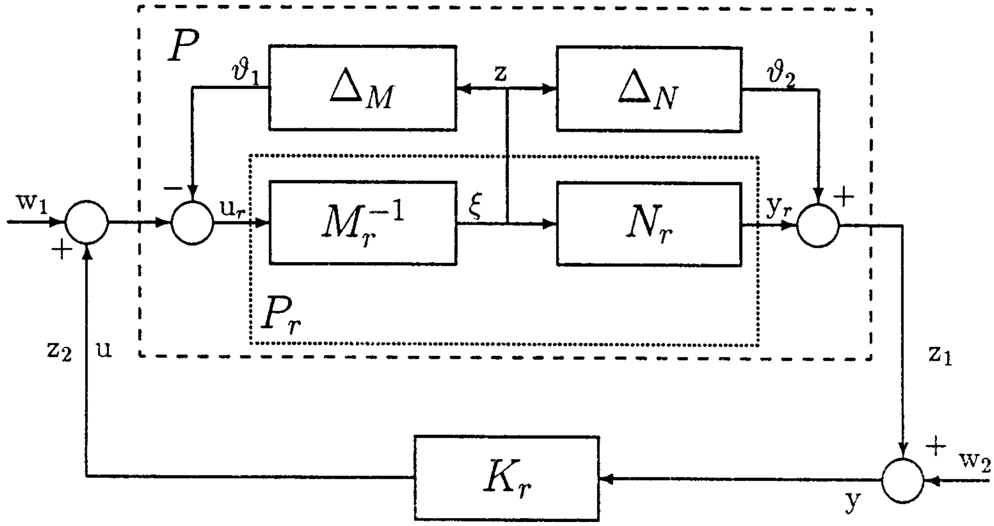

Figure 3. Feedback configuration: normalized $H_\infty(L_2)$ control problem: reduced order controller with full order plant

In the previous section it was shown that the scaled reduced order plant, $\bar{P}_r = \beta P_r$, (20), has a normalized right coprime factorization given as

$$\beta P_r = (\beta N_r) M_r^{-1} \quad \text{such that} \quad \begin{bmatrix} \beta N_r \\ M_r \end{bmatrix}, \text{(19), is inner}$$

Having designed K_r, (26), to stabilize P_r, (25), the stability of the closed loop system in Figure 3 depends on the L_2 norm of the perturbation needed to recover the full order plant. More specifically it will be shown that the stability is preserved provided

$$(\gamma + \beta)\varepsilon < 1$$

where $\beta = \sqrt{1 - \gamma^{-2}}$ and ε is defined in (22). This fact is developed in the remainder of this section by using robust stabilization theory.

3.1. Generic robust stabilization

In Reference 9 it was shown that the robust stabilization of several classes of unstructured uncertainty could be obtained from the solution to a generic robust stabilization problem. A controller solving this problem makes the closed loop system shown in Figure 4 internally stable for a specified generalized nominal plant G and for all perturbational system $\Delta \in \mathscr{F}\mathscr{G}(\tilde{\varepsilon})$, where

$$\mathscr{F}\mathscr{G}(\tilde{\varepsilon}) = \{\Delta \,|\, \Delta \text{ is internally asymptotically stable and has } L_2 \text{ gain} \leqslant \tilde{\varepsilon}\}$$

The generalized plant G, the controller K and the perturbational system Δ, connected as shown in Figure 4, are each assumed to have nonlinear input affine state descriptions such that

(AS1): The closed loop system G_Δ, obtained by disconnecting K, is well-posed.[10]
(AS2): The nominal closed loop system obtained by disconnecting Δ, denoted as $F_l(G, K)$, is well posed and has smooth available storage function.[10]

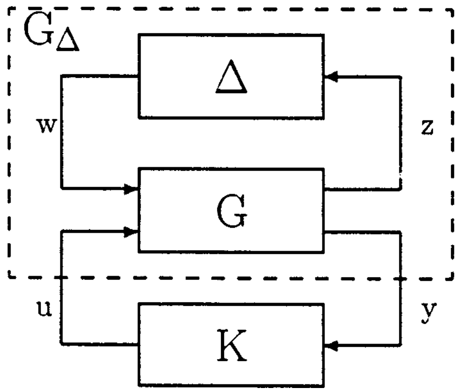

Figure 4. Feedback configuration: generic robust stabilization problem

(AS3): The perturbational system Δ is reachable with smooth available storage function.[10]

The following theorem gives sufficient conditions for solving the generic robust stabilization problem.

Theorem 2

Under the assumptions (AS1)–(AS3), a controller K asymptotically stabilizes the perturbed closed loop system in Figure 4 for all uncertainties $\Delta \in \mathscr{F}\mathscr{G}(\tilde{\varepsilon})$ if the following conditions hold

(S1): K internally stabilizes G
(S2): the nominal closed loop system $F_l(G, K)$ has L_2 gain $\leqslant \tilde{\gamma}$, with $0 < \tilde{\gamma} < 1/\tilde{\varepsilon}$.

Proof. Appendix C. □

Note that Theorem 2, although similar to the small gain theorem,[11,12,14] cannot be derived directly from it. The small gain theorem is concerned with input–output stability, not internal stability. Internal stability can be shown if both the nominal plant G and the perturbational system Δ are zero-state observable. This assumption is not used in Theorem 2.

3.2. Full order plant stabilization

The solution to the generic robust stabilization problem is used now to find conditions which ensure that the closed loop system shown in Figure 3 is asymptotically stable. This is done by relating the normalized $H_\infty(L_2)$ control problem depicted in Figure 3 to an equivalent normalized $H_\infty(L_2)$ control problem depicted in Figure 5 and involving an output scaled plant and input scaled controller.

Notice that corresponding signals without overbar in Figures 5 and 3 are equal. However, signals in Figure 5 that have an overbar, e.g., $\bar{\vartheta}_2$, are related to the corresponding signal without overbar. e.g. ϑ_2 in Figure 3 as $\bar{\vartheta}_2 = \beta \vartheta_2$. Moreover the input scaled controller $\bar{K}_r = K_r \beta^{-1}$ is seen from (26) to have state space description

$$\bar{K}_r\colon \begin{cases} \dot{\tilde{x}}^r = \hat{A}_K(\tilde{x}) + \hat{B}_K(\tilde{x}^r)\beta^{-1}\bar{y} \\ u = \hat{C}_K(\tilde{x}^r) \end{cases} \tag{28}$$

with $\hat{A}_K(\tilde{x}^r)$, $\hat{B}_K(\tilde{x}^r)$ and $\hat{C}_K(\tilde{x}^r)$ defined in (26).

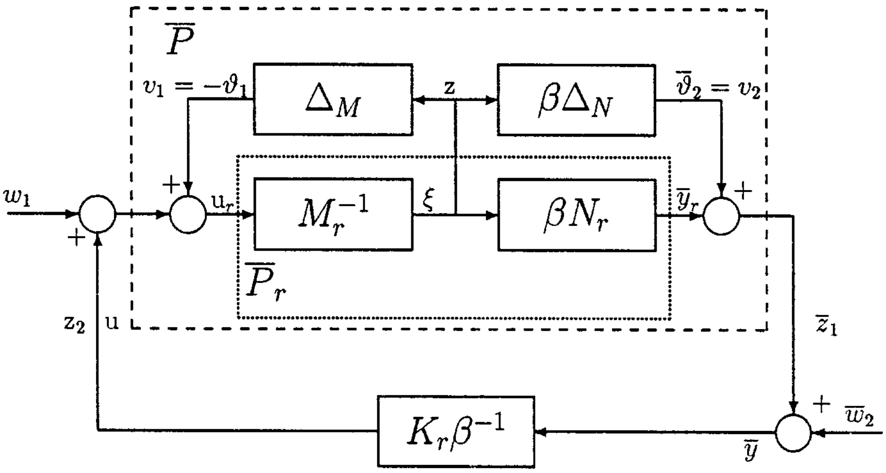

Figure 5. Feedback configuration: equivalent scaled normalized $H_\infty(L_2)$ control problem

Now in order to study the stability of the feed back configuration in Figure 5, for $w_1 = 0$ and $\bar{w}_2 = 0$, we will apply Theorem 2. This will be done by relating the generalized plant G in Figure 4 to the scaled reduced order plant $\bar{P}_r$ in Figure 5, so that the scaled full order plant P in Figure 5 is the same as G_Δ in Figure 4.

Notice that the perturbational (error) system $\begin{bmatrix} \beta\Delta_N \\ \Delta_M \end{bmatrix}$ maps ξ to v_2 and v_1. Therefore, comparing Figure 4 with Figure 5, we see that we have to find the generalized plant, denoted $G^r_{\mathscr{I}}$, which maps $\begin{bmatrix} v \\ u \end{bmatrix}$ to $\begin{bmatrix} \xi \\ \bar{y} \end{bmatrix}$, where $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Now the connections in Figure 5 imply that

$$u_r = v_1 + u \qquad \bar{y} = \bar{y}_r + v_2 \tag{29}$$

Recall the state description of $\bar{P}_r$, (20), and notice, from (19), that a state description of M_r^{-1} is given as

$$M_r^{-1}\colon \begin{cases} \dot{x}^r = \hat{f}(x^r) + \hat{g}(x^r)u_r \\ \xi = -F_r(x^r) + u_r \end{cases} \tag{30}$$

with $F_r(x^r)$ defined in (24). Therefore, using (29) in (20), (30), we see that the generalized plant $G^r_{\mathscr{I}}$ needed to apply Theorem 2 to the feedback situation depicted in Figure 5 is given as

$$G^r_{\mathscr{I}}\colon \begin{cases} \dot{x}^r = \hat{f}(x^r) + [\hat{g}(x^r)\ \ 0]v + \hat{g}(x^r)u \\ \xi = -F_r(x^r) + [I\ \ 0]v + u \\ \bar{y} = \beta\hat{h}(x^r) + [0\ \ I]v \end{cases} \tag{31}$$

where $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\vartheta_1 \\ \bar{\vartheta}_2 \end{bmatrix}$.

Then using Theorem 2 we see that the controller $\bar{K}_r$, (28), stabilizes the full order scaled system $\bar{P}$ in Figure 5 if

(S1′): the closed loop system $F_l(G^r_{\mathscr{I}}, \bar{K}_r)$, with $G^r_{\mathscr{I}}$ defined in (31), is asymptotically stable for $v = 0$

(S2′): the L_2 gain of $F_l(G^r_{\mathscr{I}}, \bar{K}_r)$, from v to ξ, is less than or equal to $\tilde{\gamma}$, where $\tilde{\gamma}\tilde{\varepsilon} < 1$ and, from (22), $\tilde{\varepsilon} = \beta\varepsilon$.

Therefore since Figure 3 and 5 are equivalent, we see that satisfaction of the conditions (S1′, S2′) ensures that the controller K_r stabilizes P in Figure 3.

In order to develop conditions which ensure satisfaction of (S1′, S2′) we will use an additional generalized plant, $G^r_{\bar{\mathscr{B}}}$ with state space description given as

$$G^r_{\bar{\mathscr{B}}}:\begin{cases} \dot{x}^r = \hat{f}(x^r) + [\hat{g}(x^r)\ \ 0]v + \hat{g}(x^r)u \\ \begin{bmatrix}\bar{y}_r\\ u_r\end{bmatrix} = \begin{bmatrix}\beta\hat{h}(x^r)\\ 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ I & 0\end{bmatrix}v + \begin{bmatrix}0\\ I\end{bmatrix}u \\ \bar{y} = \beta\hat{h}(x^r) + [0\ \ I]v \end{cases} \tag{32}$$

Notice that $G^r_{\bar{\mathscr{A}}}$, (31), and $G^r_{\bar{\mathscr{B}}}$, (32), have the same state differential equation and measured output equation, but different controlled output equations.

One way of justifying the utility of $G^r_{\bar{\mathscr{B}}}$, (32), in developing our stability results is to resort to the linear counterpart. Thus in the case when the plant is linear it can be easily shown that the closed loop transfer function from v to $\left[\begin{smallmatrix}\bar{y}_r\\ u_r\end{smallmatrix}\right]$, $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)(s)$, and the closed loop transfer function from v to ξ, $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)(s)$, are related through the inner system $\left[\begin{smallmatrix}\beta N_r(s)\\ M_r(s)\end{smallmatrix}\right]$, viz.,

$$F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)(s) = \begin{bmatrix}\beta N_r(s)\\ M_r(s)\end{bmatrix} F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)(s)$$

Since pre-multiplication by an inner matrix is H_∞-norm preserving, it follows from $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)(s)$ and $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)(s)$ have the same H_∞-norm.

Similarly, in the nonlinear case it can be easily shown that, for the same initial conditions of $[\beta N_r/M_r]$, (19), $G^r_{\bar{\mathscr{A}}}$, (31), and $G^r_{\bar{\mathscr{B}}}$, (32), the input–output behaviour of the closed loop system $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)$, is the same as the input–output behaviour of $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)$ followed by $\left[\begin{smallmatrix}\beta N_r\\ M_r\end{smallmatrix}\right]$, i.e, symbolically,

$$F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r) = \begin{bmatrix}\beta N_r\\ M_r\end{bmatrix} F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)$$

Recall now, from Theorem 1, that $\left[\begin{smallmatrix}\beta N_r\\ M_r\end{smallmatrix}\right]$, (19), is inner. As in the linear case, this property is essential in proving the following result which gives conditions for the equality of the L_2 gains of $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)$ and of $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)$.

Lemma 8

If $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)$ has L_2 gain from v to $\left[\begin{smallmatrix}\bar{y}\\ u_r\end{smallmatrix}\right]$ less than or equal to $\tilde{\gamma}$, i.e., there exists $W(x^r, \tilde{x}^r) \geqslant 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}W(x^r, \tilde{x}^r) \leqslant \tilde{\gamma}^2 \| v \|^2 - \left\| \begin{bmatrix}\bar{y}_r\\ u_r\end{bmatrix} \right\|^2 \tag{33}$$

and if $W(x^r, \tilde{x}^r) - \beta^2 U_{or}(x^r) \geqslant 0$, where $\beta^2 U_{or}(x^r)$ is a solution to (23), then $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)$ has L_2 gain from v to ξ less than or equal to $\tilde{\gamma}$.

Proof. Appendix D. □

In the following we show that all the conditions in Lemma 8 hold. Recall that K_r, (26), was designed to solve the normalized $H_\infty(L_2)$ control problem for the reduced order plant P_r, (25), or equivalently, K_r was designed to solve the $H_\infty(L_2)$ control problem for the generalized plant G_r

$$G_r\colon \begin{cases} \dot{x}^r = \hat{f}(x^r) + [\hat{g}(x^r)\ 0]w + \hat{g}(x^r)u \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \hat{h}(x^r) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ y = \hat{h}(x^r) + [0\ I]w \end{cases} \tag{34}$$

Thus we have the closed loop system $F_l(G_r, K_r)$

(C1′): is asymptotically stable for $w = 0$
(C2′): has L_2 gain, from w to z, less than or equal to γ

Now the additional system $G^r_{\bar{\mathscr{B}}}$, (32), which is related to $G^r_{\bar{\mathscr{A}}}$, (31), as shown in Lemma 8, is related also to G_r, (34). More specifically, the solution K_r to the $H_\infty(L_2)$ control problem for G_r, i.e., satisfaction of (C1′, C2′), implies the following facts about the system $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)$.

Lemma 9

If the controller K_r satisfies (C1′, C2′) then $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)$

(i) is asymptotically stable for $v = 0$
(ii) has L_2 gain from v to $[\begin{smallmatrix} \bar{y} \\ u_r \end{smallmatrix}]$ less than or equal to $\tilde{\gamma}$, where $\tilde{\gamma} = \gamma\beta^{-1} + 1$.

Furthermore along the trajectories of $F_l(G^r_{\bar{\mathscr{B}}}, \bar{K}_r)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} W(x^r, \tilde{x}^r) \leqslant (\gamma\beta^{-1} + 1)^2 \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} \bar{y}_r \\ u_r \end{bmatrix} \right\|^2 \forall v \tag{35}$$

where

$$W(x^r, \tilde{x}^r) = (1 + \gamma^{-1}\beta) Z_r(x^r, \tilde{x}^r)$$

and

$$Z_r(x^r, \tilde{x}^r) = \gamma^2 U_{cr}(\tilde{x}^r - x^r) - U_{or}(\tilde{x}^r - x^r) + U_{or}(x^r) \geqslant 0$$

Proof. Appendix D. □

Lemmas 8 and 9 are used now to prove the following theorem which gives sufficient conditions such that K_r robustly stabilizes the full order plant P in Figure 3.

Theorem 3

For any $\beta\varepsilon > 2(\delta_{k+1}(0) + \cdots + \delta_n(0))$, where $\beta = \sqrt{1 - \gamma^{-2}}$, the condition

$$(\gamma + \beta)\varepsilon < 1 \tag{36}$$

ensures that the reduced order controller K_r, (26), asymptotically stabilizes the full order closed loop system in Figure 3 when $w_1 = 0$, $w_2 = 0$.

Proof. Recall that if we show that conditions (S1′, S2′) are satisfied, then, by Theorem 2, the controller $\bar{K}_r$, (28), stabilizes the scaled system $\bar{P}$ in Figure 5. By the equivalence between Figures 3 and 5, this will imply that the controller K_r stabilizes P in Figure 3.

Now, from (i) in Lemma 9, we see that the closed loop system $F_l(G^r_{\mathscr{B}}, \bar{K}_r)$ is asymptotically stable for $v = 0$. This implies that, for $v = 0$, the closed loop system $F_l(G^r_{\mathscr{A}}, \bar{K}_r)$ is asymptotically stable also since the two closed loop systems have the same state trajectories. Therefore, (S1′) holds.

Next, from (ii) in Lemma 9, we see that the closed loop system $F_l(G^r_{\mathscr{B}}, \bar{K}_r)$ has L_2 gain less than or equal to $\tilde{\gamma} = \gamma\beta^{-1} + 1$. Then from (35) we have

$$W(x^r, \tilde{x}^r) - \beta^2 U_{or}(x^r) = (1 + \gamma^{-1}\beta)(\gamma^2 U_{cr}(\tilde{x}^r - x^r) - U_{or}(\tilde{x}^r - x^r) + (1 + \gamma^{-1}\beta - \beta^2) U_{or}(x^r)$$

Since $U_{or}(x^r)$ and $\gamma^2 U_{cr}(\tilde{x}^r - x^r) - U_{or}(\tilde{x}^r - x^r)$ are positive definite and $\beta < 1$ we see that $W(x^r, \tilde{x}^r) - \beta^2 U_{or}(x^r) > 0$. Therefore, by Lemma 8, we see that the L_2 gain of $F_l(G^r_{\mathscr{A}}, \bar{K}_r)$ is less than or equal to $\tilde{\gamma} = \gamma\beta^{-1} + 1$.

Recall now that, by Theorem 1, for any $\beta\varepsilon > 2(\delta_{k+1}(0) + \cdots + \delta_n(0))$, the bound on the L_2 gain of the error system $\left[\begin{smallmatrix}\beta\Delta_N \\ \Delta_M\end{smallmatrix}\right]$, (21), is $\tilde{\varepsilon} = \beta\varepsilon$, (22). Therefore, $\tilde{\gamma}\tilde{\varepsilon} = (\gamma + \beta)\varepsilon$, and by (36) we see that $\tilde{\gamma}\tilde{\varepsilon} < 1$. Hence, condition (S2′) holds also and the theorem is proved. □

4. PERFORMANCE WITH THE REDUCED ORDER CONTROLLER

This section completes the analysis of using the reduced order controller with the full order plant by providing an answer to the second question posed at the end of Section 2.1. This will be done using an input–output approach,[15–17] by developing a number of easily shown results. These are given now in the following lemmas.

Lemma 10

Consider the feedback configuration in Figure 3 with Δ_M, Δ_N null. Then assuming the map

$$\Phi = \begin{bmatrix} I & -K_r \\ -P_r & I \end{bmatrix}^{-1} \tag{37}$$

exists, we see that

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathscr{T} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{38}$$

where

$$\mathscr{T} = \begin{bmatrix} P_r \\ I \end{bmatrix} [I \;\; 0] \; \Phi - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

Let U_r, V_r be any right coprime factors of the controller K_r, i.e., $K_r = U_r V_r^{-1}$. Then the operators Φ, $\mathscr{T}$, (37), (38), can be expressed as

$$\Phi = \begin{bmatrix} M_r & 0 \\ 0 & V_r \end{bmatrix} \mathscr{R} \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} \tag{39}$$

$$\mathscr{T} = \begin{bmatrix} N_r \\ M_r \end{bmatrix} [I \;\; 0] \, \mathscr{R} \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \tag{40}$$

where

$$\mathscr{R} = \begin{bmatrix} M_r & -U_r \\ -\beta N_r & \beta V_r \end{bmatrix}^{-1} \tag{41}$$

Proof. Appendix E. □

Notice that the answer to the second question at the end of Section 2.1 requires the determination of the L_2 gain from $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ in Figure 3 when the reduced order plant $P_r = N_r M_r^{-1}$ is replaced by the full order plant $P = (N_r + \Delta_N)(M_r + \Delta_M)^{-1}$. Therefore if we replace P_r, N_r, M_r, by P, $N_r + \Delta_N$, $M_r + \Delta_M$ in (40) we obtain

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathscr{T}_\Delta \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \mathscr{T}_\Delta = \begin{bmatrix} N_r + \Delta_N \\ M_r + \Delta_M \end{bmatrix} [I \;\; 0]\, \mathscr{R}_\Delta \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \tag{42}$$

where

$$\mathscr{R}_\Delta = \begin{bmatrix} M_r + \Delta_M & -U_r \\ -(\beta N_r + \beta \Delta_N) & \beta V_r \end{bmatrix}^{-1} \tag{43}$$

Now we are going to find an upper bound, $\hat{\gamma}$, on the L_2 gain of the operator $\mathscr{T}_\Delta$, (42), in terms of the upper bound, γ, on the L_2 gain of the operator $\mathscr{T}$, (40). We will do this using Figure 5 and its equivalence to Figure 3. The following lemmas are used to do this.

Lemma 11

Consider the feedback configuration shown in Figure 5 with Δ_M, Δ_N null. Let U_r, V_r be any right coprime factors of K_r, i.e., $\bar{K}_r = K_r \beta^{-1} = U_r V_r^{-1} \beta^{-1}$, and assume Φ, (37), exists. Then

$$\bar{\Phi} = \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} \Phi \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} \tag{44}$$

and

$$\xi = \bar{\mathscr{A}}\left(\begin{bmatrix} w_1 \\ \bar{w}_2 \end{bmatrix} \right) \tag{45}$$

where

$$\bar{\Phi} = \begin{bmatrix} I & -\bar{K}_r \\ -\bar{P}_r & I \end{bmatrix}^{-1} \tag{46}$$

$$\begin{aligned} \bar{\mathscr{A}} &= M_r^{-1} \; [I \;\; 0] \bar{\Phi} \\ &= [I \;\; 0]\, \mathscr{R} \end{aligned} \tag{47}$$

with $\mathscr{R}$ defined in (41).

Proof. Appendix E. □

It can be easily shown that when $\bar{P}_r$, (20), M_r^{-1}, (30), and $G^r_{\bar{\mathscr{A}}}$, (31), each have the same initial conditions, the input–output behaviour of the closed loop system from $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ to ξ, with I/O map $\bar{\mathscr{A}}$, (45), (47), is the same as the input–output behaviour of the closed loop system $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)$, when $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ is set equal to $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$. Recalling that $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)$ is asymptotically stable for $v = 0$, with the L_2 gain from v to ξ less than or equal to $\tilde{\gamma} = \gamma\beta^{-1} + 1$, provided the conditions in Theorem 3 are satisfied, we see that

$$\| \bar{\mathscr{A}}(\omega) \|_2 \leqslant (\gamma\beta^{-1} + 1) \| \omega \|_2 \quad \forall \omega \in L_2[0, T] \tag{48}$$

Another lemma which will be needed in the proof of the main result in this section is given now as follows.

Lemma 12

$\mathscr{R}_\Delta$ and $\mathscr{R}$, (43), (41), are related as

$$\mathscr{R}_\Delta = \mathscr{R}\Omega$$

where

$$\Omega = \left(I - \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \bar{\mathscr{A}} \right)^{-1} \tag{49}$$

and, if $\varepsilon(\gamma + \beta) < 1$, then

$$\| \Omega(e) \|_2 \leqslant \frac{1}{1 - \varepsilon(\gamma + \beta)} \| e \|_2 \quad \forall e \in L_2[0, T] \tag{50}$$

Proof. Appendix E. □

Using the foregoing results we are now able to prove the following theorem which gives an upper bound on the L_2 gain of $\mathscr{T}_\Delta$, (42).

Theorem 4

An upper bound on the L_2 gain of $\mathscr{T}_\Delta$,(42), given by

$$\hat{\gamma} = \frac{\gamma\beta^{-1} + \varepsilon(\gamma\beta^{-1} + 1)}{1 - \varepsilon(\gamma + \beta)} \quad \text{with} \quad \varepsilon(\gamma + \beta) < 1$$

Proof. Expanding the relation for $\mathscr{T}_\Delta$, (42), gives

$$\mathscr{T}_\Delta = \begin{bmatrix} N_r \\ M_r \end{bmatrix} [I \quad 0]\, \mathscr{R}_\Delta \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} [I \quad 0]\, \mathscr{R}_\Delta \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

and substituting for $\mathscr{R}_\Delta$ from Lemma 12 gives

$$\mathscr{T}_\Delta = \begin{bmatrix} N_r \\ M_r \end{bmatrix} [I \quad 0]\, \mathscr{R}\Omega \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} [I \quad 0]\, \mathscr{R}\Omega \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \tag{51}$$

Next from (40) we see that

$$\begin{bmatrix} N_r \\ M_r \end{bmatrix} [I \quad 0]\, \mathscr{R} = \mathscr{T} \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} \tag{52}$$

and using (47) we see that

$$\mathscr{T}_\Delta = \left(\mathscr{T} \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} \right) \Omega \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \bar{\mathscr{A}}\Omega \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

$$= \mathscr{T}\begin{bmatrix} I & 0 \\ 0 & \beta^{-1}I \end{bmatrix}\Omega\begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}\bar{\mathscr{A}}\Omega\begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \beta^{-1}I \end{bmatrix}(I - \Omega^{-1})\Omega\begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix}$$

Next notice from Lemma 12 that

$$I - \Omega^{-1} = \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix}\bar{\mathscr{A}}$$

which enables $\mathscr{T}_\Delta$ to be rewritten as

$$\mathscr{T}_\Delta = \mathscr{T}\Psi_1 + \begin{bmatrix} \Delta_N \\ 0 \end{bmatrix}\Psi_2 \tag{53}$$

where

$$\Psi_1 = \begin{bmatrix} I & 0 \\ 0 & \beta^{-1}I \end{bmatrix}\Omega\begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} \quad \text{and} \quad \Psi_2 = \bar{\mathscr{A}}\Omega\begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix}$$

Thus, from (53), use of the triangle inequality yields

$$\| \mathscr{T}_\Delta(w) \|_2 \leqslant \| \mathscr{T}\Psi_1(w) \|_2 + \left\| \begin{bmatrix} \Delta_N \\ 0 \end{bmatrix}\Psi_2(w) \right\|_2 \tag{54}$$

Now from (48), (50) and $\beta < 1$ we see that

$$\| \Psi_1(w) \|_2 \leqslant \frac{\beta^{-1}}{1 - \varepsilon(\gamma + \beta)} \| w \|_2 \quad \text{and} \quad \| \Psi_2(w) \|_2 \leqslant \frac{\gamma\beta^{-1} + 1}{1 - \varepsilon(\gamma + \beta)} \| w \|_2 \quad \forall w \in L_2[0, T] \tag{55}$$

Moreover from the discussion following Lemma 8 we see that

$$\| \mathscr{T}(w) \|_2 \leqslant \gamma \| w \|_2 \forall w \in L_2[0, T] \tag{56}$$

In addition from (22) we see that

$$\left\| \begin{bmatrix} \Delta_N \\ 0 \end{bmatrix}(v) \right\|_2 \leqslant \varepsilon \| v \|_2 \quad \forall v \in L_2[0, T] \tag{57}$$

Therefore use of (55)–(57) in (54) yields

$$\| \mathscr{T}_\Delta(w) \|_2 \leqslant \frac{\gamma\beta^{-1} + \varepsilon(\gamma\beta^{-1} + 1)}{1 - \varepsilon(\gamma + \beta)} \| w_2 \| \quad \forall w \in L_2[0, T]$$

and the theorem is proved. □

Although the performance bound just obtained for input affine nonlinear plants exceeds the performance bound obtained in Reference 2 for linear plants, it is shown in Appendix F that our result reduces to the result in Reference 2 when the plant is restricted to the linear case.

5. CONCLUSIONS

This paper has solved the problem of reducing the order of the $H_\infty(L_2)$ controller for nonlinear input affine plants, by following the approach in Reference 2 for linear plants.

The controller reduction procedure is based on nonlinear H_∞ balancing.[6, 7] The development began by showing that the energy functions needed to achieve a balanced realization could be obtained by solving a special type of control problem referred to as the normalized $H_\infty(L_2)$ control problem. This was followed by the development of a normalized right coprime factorization of a scaled version of the given input affine plant, viz., $\bar{P} = \beta N M^{-1}$. This scaling matched the HJEs for the normalized $H_\infty(L_2)$ control problem for the unscaled plant to the HJES needed to obtain a normalized coprime factorization of the scaled plant. The development continued by obtaining a reduced order model for the scaled plant by balanced truncation of $Q = [\begin{smallmatrix}\beta N \\ M\end{smallmatrix}]$. The reduced order controller was then obtained by solving the normalized $H_\infty(L_2)$ control problem for the reduced order unscaled model.

The usefulness of the controller reduction procedure has been justified by characterizing the properties of the reduced order controller when connected to the full order plant. Conditions were derived which ensure that the reduced order controller stabilizes not only the reduced order plant but also the full order plant. This was done by relating the order reduction problem to a type of unstructured robust stabilization problem. The original full order plant is one of the elements of the set of plants generated by perturbing the factors of a normalized coprime factorization of the scaled reduced order plant. The use of ideas from unstructured robust stabilization yields an additional beneficial property for the control system. More specifically the reduced order controller obtained here stabilizes a family of full order plants having the actual full order plant as a member. Thus the present reduced order control system has a degree of robustness. As in the linear case,[2] the stability conditions are sufficient. The main reason for this is that only an upper bound on the H_∞ norm (L_2 gain) of the error system is available a priori, i.e., Glover's result,[18] which leads to sufficient conditions.

In addition, the degree to which the performance of the closed loop system decreases, with respect to the L_2 gain, which using the reduced order controller has been considered. It was shown that, when specialized to linear systems, our result recovers the result in Reference 2.

APPENDIX A. BALANCING PROCEDURE

Suppose that in some co-ordinates with the state denoted by ξ, the plant P, (1), is characterized by a pair of energy functions, denoted $\hat{E}_c(\xi)$ and $\hat{E}_o(\xi)$, satisfying the following assumptions

1. $\hat{E}_c$ and $\hat{E}_o$ exist and are smooth on $\mathscr{W} \subset R^n$
2. $\frac{\partial^2 \hat{E}_c}{\partial \xi^2}(0) > 0$ and $\frac{\partial^2 \hat{E}_o}{\partial \xi^2}(0) > 0$

A balancing procedure which uses these energy functions was given in References 5 and 6 and is summarized here as follows.

Step 1

There exists (Morse's Lemma, (Reference 19), a co-ordinate transformation $\xi = \psi(\tilde{\xi})$, $\psi(0) = 0$, such that in the new coordinates we have

$$\hat{E}_c(\psi(\tilde{\xi})) = \tilde{\xi}^{\mathrm{T}} \tilde{\xi}$$
$$\hat{E}_o(\psi(\tilde{\xi})) = \tilde{\xi}^{\mathrm{T}} M(\tilde{\xi})\, \tilde{\xi}$$

where $M(\tilde{\xi})$ is an $n \times n$ symmetric matrix with entries which are smooth functions of $\tilde{\xi}$, and $M(0) = \frac{\partial^2 \hat{E}_o}{\partial \xi^2}(0)$

Step 2

On a neighbourhood $\mathscr{W}$ of 0 where the number of distinct eigenvalues of $M(\tilde{\xi})$, $\tilde{\xi} \in \mathscr{W}$, is constant, there exists a co-ordinate transformation $\bar{x} = \varphi(\tilde{\xi})$, $\varphi(0) = 0$, such that in the new co-ordinates the two functions have the form

$$\hat{E}_c(\psi(\varphi^{-1}(\bar{x}))) = \bar{x}^{\mathrm{T}}\bar{x}$$

$$\hat{E}_o(\psi(\varphi^{-1}(\bar{x}))) = \bar{x}^{\mathrm{T}}\,\mathrm{Diag}[\omega_1(\bar{x}), \ldots, \omega_n(\bar{x})]\,\bar{x}$$

where $\omega_1(\bar{x}) \geqslant \cdots \geqslant \omega_n(\bar{x})$ are smooth functions of $\bar{x}$.

Step 3

Define another smooth co-ordinate transformation $x = \eta(\bar{x})$, such that

$$\eta(\bar{x}) = [\eta_1(\bar{x}_1) \cdots \eta_n(\bar{x}_n)]^{\mathrm{T}}$$

$$\eta_i(\bar{x}_i) = \omega_i^{1/4}(0, \ldots, 0, \bar{x}_i, 0, \ldots, 0)\,\bar{x}_i$$

and in the new co-ordinates, called balanced co-ordinates, the two functions have the form

$$E_c(x) = x^{\mathrm{T}}\,\mathrm{Diag}[\alpha_c^1(x_1), \ldots, \alpha_c^n(x_n)]\,x$$

$$E_o(x) = x^{\mathrm{T}}\,\mathrm{Diag}[\beta_o^1(x_1), \ldots, \beta_o^n(x)]\,x$$

where $E_c(x) = \hat{E}_c(\psi \circ \varphi^{-1} \circ \eta^{-1}(x))$, $E_o(x) = \hat{E}_o(\psi \circ \varphi^{-1} \circ \eta^{-1}(x))$ and

$$\alpha_c^i(x_i) = \omega_i^{-1/2}(0, \ldots, 0, \eta_i^{-1}(x_i), 0, \ldots, 0)$$

$$\beta_o^i(x) = \alpha_c^i(x_i)\,\omega_i(\eta^{-1}(x)) \quad i = 1, \ldots, n$$

It follows that, for $i = 1, \ldots, n$,

$$E_c(0, \ldots, 0, x_i, 0, \ldots, 0) = \tfrac{1}{2}\,x_i^2\,\alpha_c^i(x_i)$$

$$E_o(0, \ldots, 0, x_i, 0, \ldots, 0) = \tfrac{1}{2}\,x_i^2\,[\alpha_c^i(x_i)]^{-1}$$

APPENDIX B. PROOF OF THEOREM 1

The first part of the theorem is proved using the same arguments as in Reference 6. Since Q, (15), is in balanced co-ordinates, it follows that $L_{o,Q}$ and $L_{c,Q}$ are as in (6), i.e.,

$$L_{c,Q}(x) = x^{\mathrm{T}}\,\mathrm{Diag}[\sigma_c^1(x_1), \ldots, \sigma_c^n(x_n)]x \quad L_{o,Q}(x) = x^{\mathrm{T}}\,\mathrm{Diag}[\mu_o^1(x), \ldots, \mu_o^n(x)]x$$

for some $\sigma_c^i(x_i)$ and $\mu_o^i(x)$. Since $\sigma_c^1(x_1), \ldots, \sigma_c^k(x_k)$ do not depend on x^q, we see that

$$\frac{\partial L_{c,Q}}{\partial x^q}(x^r, 0) = 0$$

Using the foregoing relation together with (18) and the conditions stated in the theorem it can be seen that

$$\frac{\partial U_c}{\partial x^q}(x^r, 0) f^q(x^r, 0) = 0 \quad \frac{\partial U_c}{\partial x^q}(x^r, 0)\, g^q(x^r, 0) = 0 \tag{58}$$

$$\frac{\partial U_o}{\partial x^q}(x^r, 0) f^q(x^r, 0) = 0 \quad \frac{\partial U_o}{\partial x^q}(x^r, 0)\, g^q(x^r, 0) = 0 \tag{59}$$

Next, setting $x^q = 0$ in the HJB equation (8) and using (59) as well as the notations in (20) yields

$$\frac{\partial U_o}{\partial x^r}(x^r, 0) \hat{f}(x^r) - \frac{1}{4}\beta^2 \frac{\partial U_o}{\partial x^r}(x^r, 0)\, \hat{g}(x^r) \hat{g}^{\mathrm{T}}(x^r) \frac{\partial U_o^{\mathrm{T}}}{\partial x^r}(x^r, 0) + \hat{h}^{\mathrm{T}}(x^r) \hat{h}(x^r) = 0$$

so that $U_{or}(x^r) = U_o(x^r, 0)$ and $\beta^2 U_{or}(x^r)$ is solution to (23). Using (16) and the second relation in (59), we see from the definition of $F_r(x^r)$ (20) that (24) holds. It can be immediately checked that all the conditions in Lemma 3 hold so that Q_r, (19) is inner. Furthermore, under these conditions $(\beta N_r, M_r)$, (19) is an NRCF of the reduced order scaled plant βP_r, (20). The reduced order approximation of the original plant P, i.e., P_r, (25), is obtained by multiplying the output equation of βP_r by β^{-1}, so that (N_r, M_r) is a right coprime factorization of P_r although not a normalized one.

Similarly, setting $x^q = 0$ in the HJB equation (9) and using (58), (20) yields

$$\frac{\partial U_c}{\partial x^r}(x^r,0)\hat{f}(x^r) + \frac{1}{4}\frac{\partial U_c}{\partial x^r}(x^r,0)\,\hat{g}(x^r)\hat{g}^{\mathrm{T}}(x^r)\frac{\partial U_c^{\mathrm{T}}}{\partial x^r}(x^r,0) - \beta^2\hat{h}^{\mathrm{T}}(x^r)\hat{h}(x^r) = 0$$

so that $U_{cr}(x^r) = U_c(x^r, 0)$. It follows immediately, using Lemma 1, that the controller which solves the normalized $H_\infty(L_2)$ control problem for the reduced model P_r, (25), is K_r, (26).

For the last part, recall that (15) is already in balanced form with singular value functions $\sigma_1(x) \geqslant \cdots \geqslant \sigma_n(x)$. Under the condition stated in the theorem and from Theorem 4.6 in Reference 4 it follows that the linearized system of (15) is asymptotically stable and in balanced form with the Hankel singular values $\delta_1(0) \geqslant \cdots \geqslant \delta_n(0)$, where $\delta_i(0) = \sigma_i^{1/2}(0)$, $i = 1, \ldots, n$. Under the assumptions stated in the theorem, the reduced order nonlinear model (19) is also in balanced form with singular values $\sigma_1(x^r, 0) \geqslant \cdots \geqslant \sigma_k(x^r, 0)$, so that its linearization has the Hankel singular values $\delta_1(0) \geqslant \cdots \geqslant \delta_k(0)$. Also, the linear subsystem thus obtained is asymptotically stable.

It can be easily shown that the linearization of the nonlinear error system (21), denoted G_e, is identical to the linear error system between the linearization of the full order plant, (15), and the linearization of the reduced order system (19). Therefore, the linearization of the nonlinear error system (21), i.e., G_e, is asymptotically stable,[20] so that the nonlinear error system (21), is locally asymptotic stable.

Also, from Theorem 9.6 in Reference 18, we see that the linearized error system G_e, has the H_∞-norm satisfying $\|G_e\|_\infty \leqslant 2(\delta_{k+1}(0) + \cdots + \delta_n(0))$, so that $\|G_e\|_\infty < \tilde{\varepsilon}$ for any $\tilde{\varepsilon} > 2(\delta_{k+1}(0) + \cdots + \delta_n(0))$. Applying Theorem 8 in Reference 21 it follows that the nonlinear error system (21) has also locally L_2 gain less than or equal to $\tilde{\tilde{\xi}} := \beta\varepsilon$ for any

$$\tilde{\varepsilon} > 2(\delta_{k+1}(0) + \cdots + \delta_n(0))$$

and the conclusion follows. □

APPENDIX C. PROOF OF THEOREM 2

The proof of Theorem 2 uses the following result.

Lemma 13[22]

Consider the system

$$\dot{x} = A(x) + B(x)s(x) \tag{60}$$

where $A(\cdot)$, $B(\cdot)$ and $s(\cdot)$ are sufficiently smooth functions with $A(0) = 0$ and $s(0) = 0$. Suppose that the following conditions hold for all $x \in \mathscr{B}_r$, for some $r > 0$, where $\mathscr{B}_r = \{x \in R^n \mid \|x\| < r\}$:

(a) There exists a positive semi-definite function $\Phi(\cdot)\colon R^n \to R$ on $\mathscr{B}_r$ such that, for some $\rho > 0$

$$\frac{\partial \Phi}{\partial x}(x)(A(x) + B(x)s(x)) \leqslant -\rho\,\|s(x)\|^2$$

(b) The system $\dot{x} = A(x)$ has an asymptotically stable equilibrium at the origin on $\mathscr{B}_r$.

Then the system (60) is asymptotically stable on $\mathscr{B}_r$.

Proof of Theorem 2. Let the state space descriptions of G and K be given as

$$G\colon \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h_1(x) + d_{11}(x)w + d_{12}(x)u \\ y = h_2(x) + d_{21}(x)w + d_{22}(x)u \end{cases} \quad K\colon \begin{cases} \dot{\tilde{x}} = F_K(\tilde{x}) + G_K(\tilde{x})y \\ u = H_K(\tilde{x}) \end{cases} \tag{61}$$

and let the perturbational plant Δ be described by

$$\Delta\colon \begin{cases} \dot{x}_\Delta = a(x_\Delta) + b(x_\Delta)z \\ w = c(x_\Delta) + k(x_\Delta)z \end{cases} \tag{62}$$

Denoting the composite state as

$$x_c = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

and using the above state space descriptions we see that the closed loop system $F_l(G, K)$, obtained from Figure 4 by disconnecting Δ, is described by

$$F_l(G,K)\colon \begin{cases} \dot{x}_c = f_c(x_c) + g_c(x_c)w \\ z = h_c(x_c) + d_c(x_c)w \end{cases} \tag{63}$$

where

$$f_c(x_c) = \begin{bmatrix} f(x) + g_2(x)H_K(\tilde{x}) \\ F_K(\tilde{x}) + G_K(\tilde{x})(h_2(x) + d_{22}(x)H_K(\tilde{x})) \end{bmatrix} \quad g_c(x_c) = \begin{bmatrix} g_1(x) \\ G_K(\tilde{x})d_{21}(x) \end{bmatrix}$$

$$h_c(x_c) = h_1(x) + d_{12}(x)H_K(\tilde{x}) \qquad d_c(x_c) = d_{11}(x)$$

Under assumption (AS2) and condition (S2) of the theorem it follows[10, 21, 23] that there exists a function $W(x_c) \geqslant 0$, $W(0) = 0$ such that

$$\dot{W}(x_c) = \frac{\partial W}{\partial x_c}(f_c(x_c) + g_c(x_c)w) \leqslant \tilde{\gamma}^2 \| w \|^2 - \| z \|^2 \tag{64}$$

Again under assumption (AS3) and the assumptions on the uncertain system Δ, i.e., $\Delta \in \mathscr{F}\mathscr{G}(\tilde{\varepsilon})$, there exists a function $Y(x_\Delta) \geqslant 0$, $Y(0) = 0$, such that

$$\dot{Y}(x_\Delta) = \frac{\partial Y}{\partial x_\Delta}(a(x_\Delta) + b(x_\Delta)z) \leqslant \tilde{\varepsilon}^2 \| z \|^2 - \| w \|^2 \tag{65}$$

Now consider the following function

$$\Phi(x_c, x_\Delta) = W(x_c) + \eta^2 Y(x_\Delta)$$

where η is restricted as $\tilde{\gamma} < \eta < 1/\tilde{\varepsilon}$. Notice that condition (S2) of the theorem guarantees the existence of such an η.

Then from the state description of Δ, (62), and $F_l(G, K)$, (63), we see that the closed loop system shown in Figure 4 has state equation

$$\mathscr{S}: \dot{\bar{x}} = A(\bar{x}) + B(\bar{x})s(\bar{x})$$

where $\bar{x}^{\mathrm{T}} = [x_c^{\mathrm{T}} \;\; x_\Delta^{\mathrm{T}}]$ and

$$A(\bar{x}) = \begin{bmatrix} f_c(x_c) \\ a(x_\Delta) \end{bmatrix} \quad B(\bar{x}) = \begin{bmatrix} g_c(x_c) & 0 \\ 0 & b(x_\Delta) \end{bmatrix} \quad s(\bar{x}) = \begin{bmatrix} \omega(\bar{x}) \\ h_c(x_c) + d_c(x_c)\omega(\bar{x}) \end{bmatrix}$$

with

$$\omega(\bar{x}) = S_\omega^{-1}[c(x_\Delta) + k(x_\Delta)h_c(x_\Delta)]$$

$$S_\omega = I - k(x_\Delta)d_{11}(x)$$

where the inverse of S_ω is assured by assumption (AS1).

Now from (64), (65) we see that the following relation holds along solutions of $\mathscr{S}$

$$\begin{aligned} \dot{\Phi}(x_c, x_\Delta) &= \dot{W}(x_c) + \eta^2 \dot{Y}(x_\Delta) \\ &\leqslant -(\eta^2 - \tilde{\gamma}^2)\| w \|^2 - (1 - \eta^2\tilde{\varepsilon}^2)\| z \|^2 \end{aligned}$$

Denote by

$$\rho = \min\{1 - \eta^2\tilde{\varepsilon}^2, \quad \eta^2 - \tilde{\gamma}^2\}$$

Since $\tilde{\gamma} < \eta < 1/\tilde{\varepsilon}$, it follows that $\rho > 0$, which implies that $\dot{\Phi}(x_c, x_\Delta) \leqslant 0$ along all state trajectories of $\mathscr{S}$. Also, since Δ is connected to $F_l(G, K)$ the previous inequality can be written as

$$\dot{\Phi}(x_c, x_\Delta) \leqslant -\rho(\|\omega(\bar{x})\|^2 + \|h_c(x_c) + d_c(x_c)\omega(\bar{x})\|^2) \tag{66}$$

To prove that $\mathscr{S}$ is asymptotically stable we note that the state space description of $\mathscr{S}$ has the same form as (60). Moreover, from the previous inequality, we have

$$\frac{\partial \Phi}{\partial \bar{x}}(\bar{x})\,(A(\bar{x}) + B(\bar{x})s(\bar{x})) \leqslant -\rho\|s(\bar{x})\|^2$$

which is condition (a) in Lemma 13.

Now $\dot{x}_c = f_c(x_c)$ and $\dot{x}_\Delta = a(x_\Delta)$ are each being assumed asymptotically stable. Therefore $\dot{\bar{x}} = A(\bar{x})$ is asymptotically stable and condition (b) in Lemma 13 is satisfied. Thus we see, by Lemma 13, that the closed loop system $\mathscr{S}$ is asymptotically stable. □

APPENDIX D. PROOF OF LEMMAS 8 AND 9

Proof of Lemma 8. Since $G^r_{\mathscr{T}}$, (31), $G^r_{\mathscr{B}}$, (32) have the same state and measured output equations, use of the same controller $\bar{K}_r$ with each of these systems produces closed loop systems $F_l(G^r_{\mathscr{T}}, \bar{K}_r)$, $F_l(G^r_{\mathscr{B}}, \bar{K}_r)$, which have the same state trajectories. Therefore, assuming $X_r(x^r) = \beta^2 U_{or}(x^r)$ solves $\mathscr{H}_{GC}(X_r, x^r) = 0$, (23), and $F_r(x^r)$ satisfies (24) (Theorem 1), the time derivative of $X_r(x^r)$ along the state trajectories of $G^r_{\mathscr{T}}$ or $G^r_{\mathscr{B}}$ for any u, v becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}X_r(x^r) &= \frac{\partial X_r}{\partial x^r}(x^r)(\hat{f}(x^r) + \hat{g}(x^r)v_1 + \hat{g}(x^r)u)\\
&= \frac{1}{4}\frac{\partial X_r}{\partial x^r}(x^r)\hat{g}(x^r)\mathrm{g}^{\mathrm{T}}(x^r)\frac{\partial \mathrm{X}_r^{\mathrm{T}}}{\partial x^r}(x^r) - \beta^2\hat{h}^{\mathrm{T}}(x^r)\hat{h}(x^r) + \frac{\partial X_r}{\partial x^r}(x^r)(\hat{g}(x^r)v_1 + \hat{g}(x^r)u)\\
&= F_r^{\mathrm{T}}(x^r)F_r(x^r) - \beta^2\hat{h}^{\mathrm{T}}(x^r)\hat{h}(x^r) - 2F_r^{\mathrm{T}}(x^r)(v_1 + u) + \|v_1 + u\|^2 - \|v_1 + u\|^2\\
&= \|-F_r(x^r) + v_1 + u\|^2 - \left\|\begin{bmatrix}\beta\hat{h}(x^r)\\ v_1 + u\end{bmatrix}\right\|^2
\end{aligned}$$

Using (31), (32) in the foregoing equality gives

$$\frac{\mathrm{d}}{\mathrm{d}t}X_r(x^r) = \|\xi\|^2 - \left\|\begin{bmatrix}\bar{y}_r\\ u_r\end{bmatrix}\right\|^2 \tag{67}$$

Consider the following function

$$\mathscr{Y}(x^r, \tilde{x}^r) = W_r(x^r, \tilde{x}^r) - X_r(x^r)$$

where $X_r(x^r) = \beta^2 U_{or}(x^r)$. By the assumption stated in the lemma $\mathscr{Y}(x^r, \tilde{x}^r) \geqslant 0$. Compute the time-derivative of $\mathscr{Y}(x^r, \tilde{x}^r)$ along $F_l(G^r_{\mathscr{B}}, \bar{K}_r)$, or equivalently, along $F_l(G^r_{\mathscr{T}}, \bar{K}_r)$, using (33), (67)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Y}(x^r, \tilde{x}^r) = \frac{\mathrm{d}}{\mathrm{d}t}W_r(x^r, \tilde{x}^r) - \frac{\mathrm{d}}{\mathrm{d}t}X_r(x^r) \leqslant \tilde{\gamma}^2\left\|\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right\|^2 - \left\|\begin{bmatrix}\bar{y}_r\\ u_r\end{bmatrix}\right\|^2 - \|\xi\|^2 + \left\|\begin{bmatrix}\bar{y}_r\\ u_r\end{bmatrix}\right\|^2$$

Therefore,

$$\dot{\mathscr{Y}}(x^r, \tilde{x}^r) \leqslant \tilde{\gamma}^2\left\|\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right\|^2 - \|\xi\|^2$$

Integrating on both sides from $t = 0$ to $t = T$, with $(x^r(0), \tilde{x}^r(0)) = (0, 0)$ we see that

$$0 \leqslant \mathscr{Y}(x^r(T), \tilde{x}^r(T)) \leqslant \tilde{\gamma}^2\left\|\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right\|_2^2 - \|\xi\|_2^2$$

and the lemma is proved. □

Proof of Lemma 9

(i) Using (34), (26) we see that $F_l(G_r, K_r)$ has state space description given by

$$F_l(G_r, K_r): \quad \begin{cases} \begin{bmatrix} \dot{x}^r \\ \dot{\tilde{x}}^r \end{bmatrix} = \begin{bmatrix} \hat{f}(x^r) + \hat{g}(x^r)C_K(\tilde{x}^r) \\ \hat{A}_K(\tilde{x}^r) + \hat{B}_K(\tilde{x}^r)\hat{h}(x^r) \end{bmatrix} + \begin{bmatrix} \hat{g}(x^r) & 0 \\ 0 & \hat{B}_K(\tilde{x}^r) \end{bmatrix} w \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \hat{h}(x^r) \\ C_K(\tilde{x}^r) \end{bmatrix} \end{cases} \tag{68}$$

From (32), (28) we see that the state space description of $F_l(G^r_{\mathscr{B}}, \bar{K}_r)$ is

$$F_l(G^r_{\mathscr{B}}, \bar{K}_r): \quad \begin{cases} \begin{bmatrix} \dot{x}^r \\ \tilde{x}^r \end{bmatrix} = \begin{bmatrix} \hat{f}(x^r) + \hat{g}(x^r)C_K(\tilde{x}^r) \\ \hat{A}_K(\tilde{x}^r) + \hat{B}_K(\tilde{x}^r)\hat{h}(x^r) \end{bmatrix} + \begin{bmatrix} \hat{g}(x^r) & 0 \\ 0 & \hat{B}_K(\tilde{x}^r)\beta^{-1} \end{bmatrix} v \\ \begin{bmatrix} \bar{y}_r \\ u_r \end{bmatrix} = \begin{bmatrix} \beta\hat{h}(x^r) \\ C_K(\tilde{x}^r) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} v \end{cases} \tag{69}$$

Comparing with the previous relations and using (C1′) it follows that $F_l(G^r_{\mathscr{B}}, \bar{K}_r)$ is asymptotically stable for $v = 0$.

(ii) Recalling (12) in Lemma 1, we see that the corresponding inequality which holds along the trajectories of $F_l(G_r, K_r)$, (68), for any w is

$$\dot{Z}_r(x^r, \tilde{x}^r) \leqslant \gamma^2 \left\| \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\|^2 \tag{70}$$

where

$$Z_r(x^r, \tilde{x}^r) = \gamma^2 U_{cr}(\tilde{x}^r - x^r) - U_{or}(\tilde{x}^r - x^r) + U_{or}(x^r) \geqslant 0$$

From (68) and (69) we see that if $w = \left[\begin{smallmatrix} v_1 \\ \beta^{-1} v_2 \end{smallmatrix}\right]$ then the two systems have the same state equations, i.e., the same state trajectories, and the outputs are related by

$$\begin{aligned} \bar{y}_r &= \beta z_1 \\ u_r &= z_2 + v_1 \end{aligned} \tag{71}$$

Therefore, taking $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} v_1 \\ \beta^{-1} v_2 \end{smallmatrix}\right]$ in (70) and using (71), we see that along the trajectories of $F_l(G^r_{\mathscr{B}}, \bar{K}_r)$ the following relation holds for any v

$$\dot{Z}_r(x^r, \tilde{x}^r) \leqslant \gamma^2 \left\| \begin{bmatrix} v_1 \\ \beta^{-1} v_2 \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} \beta^{-1}\bar{y}_r \\ u_r - v_1 \end{bmatrix} \right\|^2$$

Since $\beta = \sqrt{1 - \gamma^{-2}} < 1$ this implies that

$$\dot{Z}_r(x^r, \tilde{x}^r) \leqslant \gamma^2 \beta^{-2} \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 - \| \bar{y}_r \|^2 - \| u_r \|^2 - \| v_1 \|^2 + 2u_r^{\mathrm{T}} v_1 \tag{72}$$

Now expanding the following inequality

$$\left(\sqrt{\theta} v_1 - \frac{1}{\sqrt{\theta}} u_r\right)^{\mathrm{T}} \left(\sqrt{\theta} v_1 - \frac{1}{\sqrt{\theta}} u_r\right) \geqslant 0$$

which holds for any $\theta > 0$, gives

$$2u_r^{\mathrm{T}} v_1 \leqslant \frac{1}{\theta} \| u_r \|^2 + \theta \| v_1 \|^2 \quad \forall \theta > 0$$

Thus substituting the foregoing inequality into (72) yields

$$\dot{Z}_r(x^r, \tilde{x}^r) \leqslant \gamma^2\beta^{-2}\left\|\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\|^2 - \|\bar{y}_r\|^2 - \left(1-\frac{1}{\theta}\right)\|u_r\|^2 + (\theta-1)\|v_1\|^2 \tag{73}$$

Next, for $\theta > 1$, (73) implies that

$$\dot{Z}_r(x^r, \tilde{x}^r) \leqslant (\gamma^2\beta^{-2} + \theta - 1)\left\|\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\|^2 - \left(1-\frac{1}{\theta}\right)\left\|\begin{bmatrix} \bar{y}_r \\ u_r \end{bmatrix}\right\|^2$$

and multiplying throughout by $\theta/(\theta-1)$ yields

$$\frac{\theta}{\theta-1}\dot{Z}_r(x^r, \tilde{x}^r) \leqslant \tilde{\gamma}^2\left\|\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\|^2 - \left\|\begin{bmatrix} \bar{y}_r \\ u_r \end{bmatrix}\right\|^2 \quad \text{with} \quad \tilde{\gamma}^2 = \theta\left(\frac{\gamma^2\beta^{-2}}{\theta-1} + 1\right)$$

Now the positive value of θ which reduces $\tilde{\gamma}^2$ to its smallest value is $\theta = \gamma\beta^{-1} + 1$. Therefore using this value of θ in the foregoing inequality yields

$$(1 + \gamma^{-1}\beta)\,\dot{Z}_r(x^r, \tilde{x}^r) \leqslant (\gamma\beta^{-1} + 1)^2\left\|\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\|^2 - \left\|\begin{bmatrix} \bar{y}_r \\ u_r \end{bmatrix}\right\|^2$$

which gives (ii) in the lemma. □

APPENDIX E. PROOF OF LEMMAS 10–12

Proof of Lemma 10. Since the perturbations Δ_M, Δ_N are assumed null we see from Figure 3 that ϑ_1 and ϑ_2 are each null and

$$u_r = w_1 + K_r(y)$$

$$y = w_2 + P_r(u_r)$$

which gives

$$\begin{bmatrix} u_r \\ y \end{bmatrix} = \Phi\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Thus expanding the right hand side of (38) yields

$$\begin{bmatrix} P_r & 0 \\ I & 0 \end{bmatrix}\Phi\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} P_r & 0 \\ I & 0 \end{bmatrix}\begin{bmatrix} u_r \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ w_1 \end{bmatrix}$$

which equals the left hand side since $z_1 = P_r(u_r)$, $z_2 = u_r - w_1$.

Next using the coprime factorizations for P_r and K_r, viz., $P_r = N_r M_r^{-1}$ and $K_r = U_r V_r^{-1}$ in the definition of Φ^{-1}, (37), yields

$$\Phi^{-1} = \begin{bmatrix} I & 0 \\ 0 & \beta^{-1}I \end{bmatrix}\begin{bmatrix} M_r & -U_r \\ -\beta N_r & \beta V_r \end{bmatrix}\begin{bmatrix} M_r^{-1} & 0 \\ 0 & V_r^{-1} \end{bmatrix}$$

which gives (39) and the expression for $\mathscr{T}$, (40), follows. □

Proof of Lemma 11. Recall from (37) that

$$\Phi^{-1} = \begin{bmatrix} I & -K_r \\ -P_r & I \end{bmatrix}$$

Therefore, using $\bar{P}_r = \beta P_r$ and $\bar{K}_r = K_r\beta^{-1}$, we see that $\bar{\Phi}^{-1}$, (46), can be written as

$$\bar{\Phi}^{-1} = \begin{bmatrix} I & -K_r \\ -\beta P_r & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} I & -K_r \\ -P_r & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix}$$

which shows that $\bar{\Phi}$ is as given in the lemma. As a remark notice that, in the nonlinear case, $K_r\beta^{-1} \neq \beta^{-1}K_r$ and $\beta P_r \neq P_r\beta$. Recall that in the case when $\bar{P}_r = \beta P_r$ and $\tilde{P}_r = P_r\beta$ have linear state descriptions, with state vector x and output $\bar{y}_r$ and, respectively, $\tilde{x}$ and $\tilde{y}_r$, we have $\tilde{y}_r(t) = \bar{y}_r(t)$ for any input $u(t)$ and any initial conditions such that $\tilde{x}(0) = \beta x(0)$. However, in nonlinear case, in general one cannot find, for any $x(0)$, an initial condition $\tilde{x}(0)$ such that the input–output behaviour of $\tilde{P}_r = P_r\beta$ is the same as the input–output behaviour of $\bar{P}_r = \beta P_r$.

Next proceeding as in the proof of Lemma 10 it can be shown that

$$\begin{bmatrix} u_r \\ \bar{y} \end{bmatrix} = \bar{\Phi} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

and from Figure 5 we see that

$$\xi = M_r^{-1}(u_r) = [M_r^{-1} \;\; 0]\, \bar{\Phi} \begin{bmatrix} w_1 \\ \bar{w}_2 \end{bmatrix}$$

Finally using (44), (39) we obtain the expression for $\bar{\mathscr{A}}$ given in the lemma. □

Proof of Lemma 12. Using the expressions for $\bar{\mathscr{A}}$, (47), and $\mathscr{R}$, (41), to expand $\Omega^{-1}\mathscr{R}^{-1}$ yields

$$\Omega^{-1}\mathscr{R}^{-1} = \mathscr{R}_\Delta^{-1}$$

and the relation between $\mathscr{R}$ and $\mathscr{R}_\Delta$ is proved.

Next recall that the L_2 gain of $F_l(G^r_{\bar{\mathscr{A}}}, \bar{K}_r)$, or equivalently, $\bar{\mathscr{A}}$ is less than or equal to $\tilde{\gamma} = \gamma\beta^{-1} + 1$, (48), i.e.,

$$\| \bar{\mathscr{A}}(w) \|_2 \leqslant (\gamma\beta^{-1} + 1) \| w \|_2 \quad \forall w \in L_2[0, T]$$

Moreover from the upper bound on the L_2 gain of the uncertainty, (22), we see that

$$\left\| \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} (v) \right\|_2 \leqslant \beta\varepsilon \| v \|_2 \quad \forall v \in L_2[0, T]$$

Therefore it follows from these inequalities that

$$\left\| \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \bar{\mathscr{A}}(w) \right\|_2 \leqslant \beta\varepsilon \| \bar{\mathscr{A}}(w) \|_2 \leqslant \varepsilon(\gamma + \beta) \| w \|_2 = \gamma_1 \| w \|_2 \quad \forall w \in L_2[0, T] \tag{74}$$

where it is assumed that the sufficient condition for stability, (36), of Theorem 3 is satisfied, i.e.,

$$\gamma_1 = \varepsilon(\gamma + \beta) < 1$$

To complete the proof of lemma consider the following feedback configuration (Figure 6), where $\mathscr{Y} = \left[\begin{smallmatrix} -\Delta_M \\ \beta\Delta_N \end{smallmatrix}\right] \bar{\mathscr{A}}$.

Now applying the triangle inequality to the relation between signals implied by this configuration and using (74) gives

$$\| w \|_2 \leqslant \| e \|_2 + \gamma_1 \| w \|_2$$

which implies

$$\| w \|_2 \leqslant \frac{1}{1 - \gamma_1} \| e \|_2 \quad \forall e \in L_2[0, T]$$

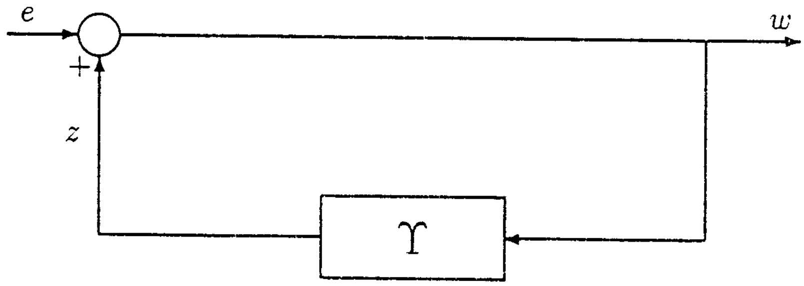

Figure 6. Feedback configuration: proof of Lemma 12

However, the configuration implies

$$w = (I - \Upsilon)^{-1}(e)$$

and since $\Omega = (I - \mathscr{Y})^{-1}$ we see that the inequality (50) in the lemma follows. □

APPENDIX F. L_2 BOUND: LINEAR CASE

In the case when all the maps are linear, applying the inversion lemma,[24] to (49) gives

$$\Omega = I + \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \Pi \bar{\mathscr{A}} \quad \text{where} \quad \Pi = \left(I - \bar{\mathscr{A}} \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \right)^{-1} \tag{75}$$

Thus using (75) and the linearity of $\mathscr{R}$, we see that $\mathscr{T}_\Delta$, (15), can be expressed as

$$\mathscr{T}_\Delta = \begin{bmatrix} N_r \\ M_r \end{bmatrix} [I \ 0] \ \mathscr{R} \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} N_r \\ M_r \end{bmatrix} [I \ 0] \ \mathscr{R} \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \Pi \bar{\mathscr{A}} \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix}$$
$$+ \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} [I \ 0] \ \mathscr{R} \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} [I \ 0] \ \mathscr{R} \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \Pi \bar{\mathscr{A}} \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix}$$

Next, using (40), (47) and (52), gives

$$\mathscr{T}_\Delta = \mathscr{T} + \left(\mathscr{T} \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} \right) \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \Pi \mathscr{O}$$
$$+ \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \mathscr{O} + \begin{bmatrix} \Delta_N \\ \beta\Delta_M \end{bmatrix} \bar{\mathscr{A}} \begin{bmatrix} -\Delta_M \\ \beta\Delta_N \end{bmatrix} \Pi \mathscr{O} \tag{76}$$

where

$$\mathscr{O} = \bar{\mathscr{A}} \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix}$$

Recalling the feedback configuration in Figure 5 with Δ_M, Δ_N null, and lemma 11, we see that

$$\xi = \mathscr{O} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Recall also that $\bar{\mathscr{A}}$, has, by (48), L_2 gain less than or equal to $\tilde{\gamma} = \gamma\beta^{-1} + 1$. Using the same arguments it can be easily shown that $\mathscr{O}$ has a smaller upper bound on the L_2 gain, viz.,

$$\| \mathscr{O}(w) \|_2 \leqslant (\gamma + 1) \| w \|_2 \quad \forall w \in L_2[0, T]$$

Finally, a relation similar to (50), Lemma 12, can be easily proved for the operator Π, (75), viz.,

$$\| \Pi(w) \|_2 \leqslant \frac{1}{1-\varepsilon(\gamma+\beta)} \| w \|_2 \quad \forall w \in L_2[0, T]$$

Using the two foregoing relations, together with (48), (56) and (22), in (76) gives

$$\| T_\Delta(w) \|_2 \leqslant \left(\gamma + (\gamma\beta^{-1}+1) \frac{\beta\varepsilon(\gamma+1)}{1-\varepsilon(\gamma+\beta)} + \varepsilon(\gamma+1) + \varepsilon(\gamma\beta^{-1}+1) \frac{\beta\varepsilon(\gamma+1)}{1-\varepsilon(\gamma+\beta)} \right) \| w \|_2$$

from which we see that

$$\| \mathscr{T}_\Delta(w) \|_2 \leqslant \tilde{\gamma}_L \| w \|_2 \quad \forall w \in L_2[0, T]$$

with

$$\hat{\gamma}_L = \gamma + \frac{\varepsilon(\gamma+1)(\gamma+\beta+1)}{1-\varepsilon(\gamma+\beta)}$$

Therefore, the linear results in Reference 2 are recovered. Notice that in order to obtain the tighter upper bound (77) we made use of the inversion lemma which does not hold for nonlinear maps.

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