Robust Power Control of Optical Links with Time-Delay

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Abstract—We derive robust stability conditions for game-theoretic based power control in optical links with time-delay. We consider a perturbed optical signal-to-noise ratio (OSNR) model in the presence of time-delays. Primal-dual control algorithms adjust the signal powers at the channel sources and receiver to optimize the OSNR values. The link utilization price is regularly updated and fed back to the source algorithms. We apply norm-bounded uncertainty techniques to derive algebraic stability criteria. The main theorem exploits time-decoupling in the primal-dual algorithms by utilizing singular perturbation theory modified to handle Lyapunov-Krasovskii time-delay techniques. We produce a new robust time-delay bound which generalizes the previous nominal time-delay result.

I. INTRODUCTION

Optical networks are the standard for high-speed, long-haul communication. Fiber optic cables have been laid in a global mesh configuration. These networks span thousands of kilometers and traverse oceans and continents.

The objective of optical networks is to transmit signals such that bit errors are minimized. This problem is formulated as the optical signal-to-noise ratio (OSNR) optimization problem. As OSNR values increase, bit error rates decrease. This problem is solved using a game-theoretic framework in [1]. The work in [2] augments the work in [1] by introducing a channel price algorithm located at the network links. These works produce a primal-dual control scheme that solves the OSNR optimization problem. However, propagation delays and uncertainty in the OSNR model are not studied.

Optical networks have non-negligible propagation delays in the tens of milliseconds. The work in [3] applies a frequency domain analysis to study the stability of a time-delayed version of the channel algorithm from [1]. The work in [4] studies the stability of the primal-dual algorithm of [2] with time-delay. The work uses a singular perturbation approach modified to handle Lyapunov-Razumikhin theory. However, these stability results do not account for uncertainties in the OSNR parameters. The gains in the OSNR model are a function of amplified spontaneous emission (ASE) and the channel dependent gains, which are model parameters. The OSNR model assumes the noise gains are constant, when in fact, ASE is time-varying. Also, model parameters change over long periods of time. Thus, we apply a robust framework to the control design to guarantee system stability in the presence of both time-delays and parameter uncertainties in the OSNR model.

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Robust and time-delay analyses are separately studied in [5] for the linear optical signal-noise difference model. The work does not include channel pricing. Here, we study the combined robust and time-delayed stability of a nonlinear OSNR based primal-dual system. In [6], the application of a converse Lyapunov Theorem to time-delayed systems is extremely relevant here. The paper, [7], applies polytopic uncertainty to genetic regulatory networks with time-varying delays. The work utilizes Lyapunov-Krasovskii functionals and linear matrix inequality (LMI) techniques, which are relevant herein. Polytopic uncertainty is also studied in [8] for LTI systems. The work [9] presents an implicit model transformation to solve the guaranteed cost control problem with memoryless state feedback controllers. The paper [10] studies the delay-dependent robust stability problem for linear uncertain systems using a Lyapunov-Krasovskii approach. The paper allows for interval time-varying delay, which can extend the theory presented herein. Finally, the paper [11] applies a Lyapunov-Razumikhin approach to derive sufficient conditions for robust time-delay stability. A study of structured uncertainties for linear systems is found in [12].

In this paper, we derive conditions for the robust stability of primal-dual power control algorithms with time-delay. Using the time-delayed OSNR model from [3], [4], we introduce uncertainty in the system model via a norm-bounded uncertainty framework. We derive new stability conditions as a function of the uncertainty bound $q$, which when set equal to 0, recovers the nominal stability criteria [4]. We utilize singular perturbation theory to exploit the time-scale decoupling in the nonlinear closed-loop system. The singular perturbation theory is modified to handle Lyapunov-Krasovskii techniques and uncertainties.

The paper is organized as follows. Section II reviews the time-delayed OSNR model and control algorithms. Section III presents closed-loop perturbed system. Section IV presents the main theorem. The following section derives the robust time-delay bound. Section VI shows simulations. Section VII gives the conclusions and future work.

II. REVIEW OF TIME-DELAYED OSNR MODEL AND CONTROL ALGORITHMS

We review the time-delayed OSNR model with discrete-time control algorithms and nominal time-delay stability conditions from [3], [4].

A. Time-Delayed OSNR Model

The following section presents the time-delayed optical signal-to-noise ratio (OSNR) model for a single optical link.
The link is a cascade of $N$ optical spans that are composed of one optical amplifier and a 100 km optical cable. Consider a set of $M = \{1, ..., n\}$ wavelength division multiplexed signal channels transmitted across the link. We denote by $u_i, s_i,$ and $n_i$, the optical input power for channel $i$ at the transmitter (Tx), the output signal at the receiver (Rx), and the output noise at Rx, respectively. The OSNR value for any channel, $i \in M,$ is defined as $OSNR_i(t) = s_i/n_i$.

An optical span is composed of an optical amplifier (OA) with channel dependent gain, $G_i$ and optical fiber with wavelength independent loss coefficient, $L_k$. The amplifiers have the same spectral shape and are operated in automatic power control (APC) mode with total power targets $P_0$. The OA introduces amplified spontaneous emission (ASE) noise, denoted by $ASE_{k,i}$.

We continuously transmit the OSNR values of each channel from the link output (Rx) back to its source (Tx). Denote by $\tau_f$ the forward propagation delay from Tx to Rx. Denote by $\tau_b$ the backward delay from Rx to Tx. The total round-trip time is denoted by $\tau = \tau_f + \tau_b$.

The following lemma from [4] defines the time-delayed OSNR model for an optical link.

**Lemma 1.** The OSNR for the $i^{th}$ channel returned to the sources (Tx) after a round-trip time-delay $\tau$ is given as

$$OSNR_i(t) = \frac{u_i(t-\tau)}{n_{0,i} + \sum_{j\in M} \Gamma_{i,j} u_j(t-\tau)}$$

where $\Gamma_{i,j}$, elements of the full $(n \times n)$ system matrix $\Gamma$, are defined as

$$\Gamma_{i,j} = \sum_{k=1}^{N} \frac{G^k_i}{G^k_j} ASE_{k,i}$$

and $n_{0,i}$ is the noise power at (Tx) for the $i^{th}$ channel.

**B. Channel Algorithm**

A non-cooperative game between channels was defined in [1] using Lemma 1 without time-delays. Each channel (player) tries to maximize its utility function

$$U_i = \ln \left( 1 + a_i OSNR_i \right)$$

which maximizes its OSNR, and minimizes its bit error rate. Here, $a_i$ is a channel dependent design parameter. Increasing a player’s utility may decrease other players’ utility functions. When the players can not improve their utility functions without adversely affecting the utility functions of the other players, then a Nash equilibrium is reached and optimal OSNR values are attained.

The solution to the Nash game is presented in [1] and [2] without time-delays. A time-delay analysis based on Lyapunov-Razumikhin techniques is produced in [4]. The time-delay control algorithm is

$$u_i(k+1) = u_i(k) - \rho_i u_i(k-\tau) + \rho_i \left( \frac{\beta_i}{\mu_i(k)} \right) - \frac{1}{a_i} \frac{1}{OSNR_{i,in}(k)} \left( 1 - \Gamma_{i,i} \right) u_i(k-\tau)$$

where $\beta_i$ and $a_i$ are design parameters and $k$ is the iterative time step. The parameter $\mu_i(k)$ is the channel price input, which ensures the total link power is below the threshold for nonlinear effects. The notation $OSNR_{i,in}$ represents the OSNR value as an input fed back from the receiver (Rx). The control gains are $\rho_i > 0$ for each channel $i$. The algorithm (2) admits a unique Nash equilibrium $u^*$ such that

$$a_i u_i^* + \sum_{j \neq i} \Gamma_{i,j} u_j^* = \frac{a_i \beta_i}{\mu_i(k)} - n_{0,i}, \forall i$$

if we satisfy the internal condition

$$a_i > \sum_{j \neq i} \Gamma_{i,j} \forall i$$

**C. Link Algorithm**

The channel price, $\mu_i$, is computed every K iterations of the control algorithm (2). We relate the fast time-scale $k$ of the control algorithm to the slow time-scale $\bar{k}$ of the link algorithm by $k = K\bar{k}$, for $K$ large. The link algorithm is

$$\mu(\bar{k} + 1) = \mu(\bar{k}) + \eta \sum_{j=1}^{u_{j,in}(K)}(K) - P_0$$

where $\mu(\bar{k}) > 0$ is the channel price, $\eta > 0$ is the step-size, and $k$ is the iterative time step for (5). Denote by $u_{j,in}$ the channel powers as inputs to the link algorithm. The total power of the signal channels, $\sum_{j=1}^{u_{j,in}(K)}$, at Rx is easily measured in real-time. The equilibrium point of (5) is

$$\sum_{j} u_{j,in} = P_0$$

**D. Channel-Link Stability Conditions**

A continuous-time stability analysis of the combined channel-link algorithm, (2) and (5), is presented in [4]. The combined channel-link algorithms, (2) and (5), converge to the optimal NE channel powers and price $(u^*, \mu^*)$ if

$$a_i > \frac{1}{2} \sum_{j \neq i} (\Gamma_{i,j} + \Gamma_{j,i})$$

and we satisfy the time-delay bound on the fast time-scale

$$\tilde{\tau} < \frac{\rho(\mu + \tilde{\Gamma}) \tilde{\Gamma}}{2 \sqrt{\rho (\mu^2 + \tilde{\Gamma}^2) \tilde{\Gamma}^2}}$$

where $\tilde{\tau} = K\tau, \rho = \text{diag}(\rho_i)$ and $\tilde{\Gamma}$ is defined as

$$\tilde{\Gamma}_{i,j} = \begin{cases} \frac{1}{\Gamma_{i,i}} & i = j \\ \frac{1}{\Gamma_{i,j}} & i \neq j \end{cases}$$

Figure 1 depicts the control algorithm (2) and the link algorithm (5) acting on the OSNR model (1).

**III. UNCERTAIN CONTINUOUS-TIME CLOSED-LOOP SYSTEM**

We apply additive uncertainty to the $\Gamma_{i,j}$ terms in the OSNR model (1). We then present the continuous-time closed-loop system with control algorithms (5) and (2).

The OSNR model (1) has uncertainty in the gains, $\Gamma_{i,j}$. The uncertainty of $\Gamma_{i,j}$ is due to the time-varying nature of ASE and the slow variation of the link parameters over long periods of time. We model the uncertainty of the $\Gamma_{i,j}$
where the equilibrium shifted variables are denoted by $z_i(t) = u_i(t) - u_i^*$ and $x(t) = \mu(t) - \mu^*$, such that $z$ is the column vector with elements $z_i$. We substituted $\mu(t) = \mu(t - \tau_b)$, the backward time-delayed signal power, and $u_{j,in} = u_j(t - \tau_f)$, the forward time-delayed time-delayed signal powers. In addition, $\beta$ is a column matrix with elements $\beta_i$, $I_{row}$ is a row vector with elements equal to 1, and $\rho = \text{diag}(\rho_i)$.

Equation (15) is on a “fast” time-scale versus (14) which is on a “slow” time-scale. Here, $\epsilon = 1/K$, where $K$ is large and defined in section 2.3. Finally, we re-write (14) and (15),

$$
\dot{x} = f(z(t - \tau_f))
$$
\hspace{1cm} (16)

$$
\epsilon \dot{z} = g(x(t - \tau_b), z(t - \tau))
$$
\hspace{1cm} (17)

where

$$
f(z(t - \tau_f)) = \eta_1 \rho(\tau_f)z(t - \tau_f)
$$
\hspace{1cm} (18)

$$
g(x(t - \tau_b), z(t - \tau)) = \rho \left\{ - \beta \frac{x(t - \tau_b)}{\mu^*(x(t - \tau) + \mu^*)} - \tilde{\Gamma}_o z(t - \tau) \right\}
$$
\hspace{1cm} (19)

We introduce the following co-ordinate shift to simplify the time-delays in the closed-loop system. Let

$$
y(t) = z(t - \tau_f) - h(x(t))
$$
\hspace{1cm} (20)

where

$$
h(x) = \tilde{\Gamma}_o^{-1} \beta \left( \frac{-x(t)}{\mu^*(x(t) + \mu^*)} \right)
$$
\hspace{1cm} (21)

is the isolated root of the RHS of (17). Note that we are assuming that $\tilde{\Gamma}_o$ is invertible. In fact, we impose this condition later in Lemma 3. Substitute (20) into (16),

$$
\dot{x} = f(y(t) + h(x(t)))
$$
\hspace{1cm} (22)

Next, take the derivative of (20) with respect to time, $t$, and substitute in (20) for $z$, (17) for $\dot{z}$, and substitute the RHS of (22) for $\dot{x}$ to obtain

$$
\epsilon \frac{dy(t)}{dt} = g(x(t - \tau_f), y(t - \tau) + h(x(t - \tau)))
$$
\hspace{1cm} (23)

$$
- \epsilon \frac{dh(x(t))}{dx}f(y(t) + h(x(t)))
$$
\hspace{1cm} (23)

Thus, (22) and (23) are the co-ordinate shifted system (16) and (17) using (20). We now have only one time-delay, $\tau$.

IV. MAIN THEOREM

We exploit the time-scale decoupling in (22) and (23) to derive the main theorem via singular perturbation theory modified to handle Lyapunov-Krasovskii techniques.

We first define $C([t - r, 0], \mathbb{R}^n)$ as the set of all continuous functions mapping $[t - r, 0]$ to $\mathbb{R}^n$. Let $C = C([t - r, 0], \mathbb{R}^n)$.

**Definition 1:** The continuous norm is defined as

$$
||y||_C = \max_{-r \leq \theta \leq 0} ||y(t + \theta)||_2
$$

where $y(t) \in C$, is the set of values, $y(q)$ for all $q \in [t - r, t]$. The following stability definition for time-delayed systems is presented from [13].

**Definition 2:** Consider the general system

$$
\dot{\bar{x}} = F(t, \bar{x}_t)
$$
\hspace{1cm} (24)
Thus, evaluating in (23) in the fast time-scale, \( t \) has faster dynamics than (22). To write the dynamics of \( y \) produces a large gain in the dynamics of \( \tilde{\epsilon} \) in the slow time-scales, respectively. Notice that a small \( \tilde{\epsilon} \) so we get (22) and (23).

Let \( \tilde{\epsilon} \) be defined as (22) with \( y \) dynamics converges to zero instantaneously, and denote by \( \tilde{x}(t) \) its solution, to get

\[
\dot{x}(t) = f(h(\tilde{x}(t)))
\]

Let the boundary system be defined as (23) in the \( t_f \) time-scale and setting \( \epsilon = 0 \), and denote by \( \tilde{y}(t_f) \) its solution

\[
\frac{d\tilde{y}}{dt_f}(t_f) = g(x, \tilde{y}(t_f - \tilde{\tau}) + h(x(t)))
\]

where \( x = x(0) \) is a frozen parameter, \( t_f = \frac{t}{\tau} \) and \( h(x) \) is defined in (21).

Using Definition 3, the reduced and boundary-layer systems of (22) and (23) are

\[
\dot{\tilde{x}}(t) = \eta_1 \omega \tilde{\epsilon}(\tilde{\epsilon}(t) + \mu^* \tilde{\epsilon})
\]

\[
\frac{d\tilde{y}}{dt_f} = -\rho \tilde{\epsilon}(t_f - \tilde{\tau})
\]

The reduced system (26) is a scalar, nonlinear system with no time-delay. The boundary-layer system (27) is linear, with one time-delay. The systems are decoupled.

The following lemma presents a robust time-delay bound. A detailed proof is provided in the following section.

**Lemma 2: Robust Time-Delay Bound**

Consider the boundary-layer system (27), where \( -\rho \tilde{\epsilon} \Delta \in \Omega \) with \( \Omega \) defined in (12). Let

\[
a = \frac{c_3}{\alpha} - c_1 \delta + \delta^2 q + q
\]

\[
b = \delta q^2 c_1 - 2\delta^2 q^3 - q^3 + q \delta^2 c_2
\]

\[
c = \delta^2 q^3 (c_2 - c_2)
\]

where

\[
c_1 = \sigma(A + A^T)
\]

\[
c_2 = \sigma(A^T A) = \sigma(A A^T)
\]

\[
c_3 = \sigma(\bar{\delta} A + q \bar{I}) A S_1^{-1} A^T (\bar{\delta} A^T + q I)
\]

and \( A = -\rho \tilde{\epsilon} \), \( \delta > 0 \) and \( \alpha > 0 \) are scalar design parameters, \( S_1 \) is a positive definite matrix design parameter and \( q \geq 0 \), defined in (13), is a scalar that reflects the degree of uncertainty within \( -\rho \tilde{\epsilon} \Delta \). Then, the uncertain boundary-layer system (27) is exponentially stable if we select \( \alpha, \rho, S_1, \delta, \alpha, \lambda \) which is also a design parameter, such that

\[
c_1 > q (\delta + \frac{1}{\delta})
\]

\[
c_2 \geq q^2
\]

\[
\alpha > c_1 \delta - q \delta^2
\]

\[
\lambda > -b - \sqrt{b^2 - 4ac}
\]

and we satisfy the time-delay bound

\[
\hat{\tau} < \frac{-a \lambda^2 - b \lambda - c}{\lambda q (\lambda - q^2)^2 + \lambda q (\lambda - q^2) c_2 + a \sigma(S_1)(\lambda - q^2)^2}
\]

There exists a solution to (34)-(38) such that the uncertain boundary-layer system (27) is exponentially stable.

**Remark:** The time-delay bound (38) maps directly to (8) if we select \( S_1 = \alpha P = \alpha \frac{1}{2} \tilde{x} \). We then impose \( q = 0 \) to eliminate all uncertainty. After some manipulation, we exactly recover (8). Note that \( \delta \) and \( \lambda \) cancel out.

Lemma 3 is modified from [4] to have uncertainty \( q \). The proof is omitted.

**Lemma 3:** The uncertain reduced system (26) is exponentially stable if

\[
a_i > \frac{1}{2} \left( 1 - \frac{q}{\rho_i} \right) \sum_{j \neq i}(\Gamma_{i,j} + \Gamma_{j,i}) \quad \forall i
\]

\[
\rho_i > q \quad \forall i
\]

**Remark:** If we increase \( a_i \), we decrease \( q \), to satisfy (40). If \( q = 0 \), then (40) is satisfied, and (39) reduces to (7).

We state the main stability result using lemmas 2 and 3.

**Theorem 2:** Consider the uncertain system (14) and (15), with uncertainty set (12). There exists an \( \epsilon^* > 0 \) such that for \( 0 < \epsilon < \epsilon^* \) the origin is asymptotically stable if (34)-(40) are satisfied.

**Proof:** We prove semiglobal asymptotic stability for (22) and (23) using a composite Lyapunov functional based on the reduced and boundary layer system functionals.
For the reduced system (26), we select the Lyapunov function $V(\bar{x}) = \frac{1}{2} \bar{x}^2$. If (39) and (40) hold, then

$$\frac{dV}{dt} f(h(\bar{x})) \leq -\eta \frac{\nu}{\mu + \nu} \bar{x}^2 \leq -k \bar{x}^2$$

where $-\bar{F}_{x} \in \Omega$, $\bar{x} \leq r_1$, where $r_1 > -\mu^*$, and $k > 0$ is a constant. We can make $r_1$ arbitrarily large, so (41) is satisfied semiglobally for $\bar{x}$.

By Lemma 2, the boundary-layer system (27) is exponentially stable. Thus, by the converse Lyapunov Theorem [14][Lemma 33.1] and [6], there exists a Lyapunov function $W(\bar{y}_t)$ such that

$$\bar{V}_i || \bar{y}_t ||^2 \leq W(\bar{y}_t) \leq \bar{V}_2 || \bar{y}_t ||^2$$

$$W'(\bar{y}_t) g_t (x, \bar{y}(t - \tau)) \leq -k || \bar{y}_t ||^2$$

where we used $W(\bar{y}_t) = \frac{dW(\bar{y}_t)}{dt} = W'(\bar{y}_t) \frac{d\bar{y}_t}{dt}$ with $W'(\bar{y}_t)$ denoting the Fréchet derivative of $W$ (see remark after Theorem 1), and $\frac{d\bar{y}_t}{dt} = q_t (\bar{y}(t - \tau))$ is obtained from $q(\bar{y}(t - \tau))$ as in [15](Theorem 2.4.6, pg. 60). Here, $\bar{V}_i > 0$ and $\bar{V}_2 > 0$.

We define the composite Lyapunov function as $\bar{V}(x, y_t) = V(\bar{x}) + W(\bar{y}_t)$. We apply $\bar{V}(x, y_t)$ to (22) and (23). By exploiting the inequalities (41)-(44), we prove asymptotic stability by Theorem 1 for $\epsilon$ small enough. The proof follows as in Theorem 11.4 in [16].

V. ROBUST TIME-DELAY BOUND DERIVATION

The following section proves Lemma 2, which replaces the time-delay bound (8). We map the uncertainty in (27) into a feedback $u = F y$ [13], with uncertainty set (12). Let $A = -\rho \Gamma$. The system (27) is stable if the following LMI from [13][Proposition 6.17, pg.222] is negative definite

$$\begin{pmatrix}
M_n & 0 & -PA^T & \sqrt{\eta} & \frac{\eta}{\lambda} I & -PA \sqrt{\eta} \\
0 & S_0 & 0 & 0 & 0 & 0 \\
-(A^T)^T P & 0 & S_1 + \lambda q I & -\sqrt{\eta} A^T & 0 & 0 \\
-\sqrt{\eta} I & 0 & 0 & -\lambda I & -I & -q I \\
-\sqrt{\eta} A^T P & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

where $M_n = \frac{1}{\alpha} [PA + AT^T] + S_0 + S_1$, and $S_0$, $S_1$ and $P$ are symmetric and positive definite. Rewrite (45) as

$$\begin{pmatrix}
\bar{A} & \bar{B} & \bar{C} \\
\end{pmatrix} < 0$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are the corresponding 3×3 block matrices of (45). Apply the Schur complement to obtain

$$\lambda > q^2$$

$$\bar{A} - \bar{B} C^{-1} \bar{B}^T < 0$$

Let $||S_0|| \rightarrow 0^+$ be very small. Also, pick

$$S_1 = \lambda q I + \frac{\lambda q}{\lambda - q^2} A^T A + \alpha \bar{S}_1$$

where $\bar{S}_1 > 0$ is a design matrix. Then (48) evaluates to

$$\begin{pmatrix}
M_n & N_n \\
N_n^T & -\alpha \bar{S}_1 \\
\end{pmatrix} < 0.$$
the uncertainty. However, the convergence time increased and the OSNR values decreased compared to figure 3. We derived a new robust time-delay bound using singular perturbation theory and Lyapunov-Krasovskii scheme from [4]. We derived a new robust time-delay bound using singular perturbation theory and Lyapunov-Krasovskii scheme from [4].

The size of the system matrix, which implies $||F||_2 \leq q$. For the uncertainty $\bar{q} = 1.823 \times 10^{-4}$, or $q = 0.116$, we get the response in figure 4. We see instability since one channel gets dropped. The uncertainty $\bar{q}$ is significant at approximately four times the maximum system gain. Figure 5 shows the robust compensated, perturbed system where we increase $\alpha_1$ by a factor of 2.9. The maximum upper bound on the time-delay is found using (38), where the optimal parameters, $\delta = 1.34$, $\alpha = 0.021$ and $\lambda = 0.102$, are computed as constraints to a nonlinear optimization problem. The time-delay bound is $\tilde{\tau} < 10.5 \times \bar{q}$. Compared to figure 4, we have stability. However, the convergence time increased and the OSNR values decreased compared to figure 3.

VII. CONCLUSIONS AND FUTURE WORK

We applied norm-bounded uncertainty to the power control scheme from [4]. We derived a new robust time-delay bound using singular perturbation theory and Lyapunov-Krasovskii time-delay analysis. For no uncertainty, i.e. $q = 0$, we recovered the nominal stability conditions. Simulations showed

stability is ensured for perturbed systems at the expense of convergence time and OSNR values. Future work will extend the robust stability results to multi-link networks.

REFERENCES


Fig. 3. The base case $\bar{q} = 0$, no uncertainty in the system with nominal control laws (2) and (5).

Fig. 4. The case $\bar{q} = 1.823 \times 10^{-4}$ for the unstable perturbed system with nominal control laws (2) and (5).

Fig. 5. The case $\bar{q} = 1.823 \times 10^{-4}$ for the perturbed system with robust compensation with $\alpha_1$ increased by a factor of 2.9.