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An extension of duality to a game-theoretic framework☆

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Abstract

This paper extends some duality results from a standard optimization setup to a noncooperative (Nash) game framework. A Nash game (NG) with coupled constraints is considered. Solving directly such a coupled NG requires coordination among possibly all players. An alternative approach is proposed based on its relation to a special constrained optimization problem for the NG-game cost function, with respect to the second argument that admits a fixed-point solution. Specific separability properties of the NG-game cost are exploited and duality results are developed. This duality extension leads naturally to a hierarchical decomposition into a lower-level NG with no coupled constraints, and a higher-level system optimization problem. In the second part of the paper these theoretical results are applied to a coupled NG with coupled constraints as encountered in optical networks.

Keywords: Game theory; Noncooperative (Nash) games; Coupled constraints; Constrained optimization; Duality; Hierarchical decomposition

1. Introduction

A powerful tool in standard constrained optimization is the duality approach (Bertsekas, 1999). For separable problems, use of duality enables a hierarchical decomposition into a lower-level set of optimization problems, and a higher-level system problem for the Lagrangian multipliers. As the lower-level problems can be analytically tractable or decoupled, this decomposition may offer significant computational advantages. As an example, the duality approach and separability have been used successfully in developing network congestion control algorithms, with the Lagrangian multipliers playing the role of pricing parameters (Kelly, Maulloo, & Tan, 1998; Low & Lapsley, 1999).

Recently, as alternative to the traditional system-wide network optimization (Kandukuri & Boyd, 2002; Yates, 1995; Zander, 1992) game theory approaches have been used for optimization and control of networks (Basar & Olsder, 1999; Nash, 1950). In large-scale networks control decisions are often made independently by users, each according to their own performance objectives (Libman & Orda, 1999). Noncooperative game theory is a suitable framework to model interactions between such self-interested users. Instead of defining a central cost function, a noncooperative game is defined between users (players). Each player maximizes its own utility function (minimizes its cost function) and its action affects the utility of all other players. Game-theoretic approaches have been used for network flow optimization (congestion control), as well as for network power allocation (power control). In network congestion control each user's utility depends only on its own action and is not coupled to the other users' actions (Alpcan & Basar, 2002; Altman, Basar, & Srikant, 2002; Basar & Srikant, 2002; Low & Lapsley, 1999; Shen & Basar, 2004). Coupling appears in the constraints only, specifically in the link capacity constraints. This is contrasted with network power control via game theory, in wireless networks or in optical networks (Alpcan, Basar, Srikant, & Altman, 2001; Koskie & Gajic, 2005; Pavel, 2006b; Saraydar, Mandayam, & Goodman, 2002). In such problems utilities are coupled; each user's utility depends not only on its own action but also on the other users' actions. Of interest are more general game problems that combine the two cases such that both utilities (cost functions) and constraints are coupled.

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In this paper we extend duality results from a standard optimization setup to a game theoretical framework. A short version of this work appeared in Pavel (2005). We consider a noncooperative (Nash) game with coupled cost functions and coupled constraints. Existence conditions for a Nash equilibrium (NE) solution are based on a two-argument system cost function (Basar & Olsder, 1999). This augmented cost function is defined in a Nash game (NG) sense and we call it the NG-game cost function. Solving directly an NG with coupled constraints requires coordination among possibly all players. Alternatively, we first show that such a game is related to a constrained optimization of the NG-game cost function, with respect to its second argument, that admits a fixed-point solution. We exploit the separability property of the NG-cost function with respect to its second argument, and extend standard duality results (Bertsekas, 1999) to a game framework. This enables us to obtain a hierarchical decomposition of the coupled NG into a lower-level modified NG, with no coupled constraints, and a higher-level system optimization problem. In effect this decomposition has the form of a two-level hierarchical game (Basar & Olsder, 1999): the upper-level game is a Stackelberg (leader–follower) game between the system and the players, and the lower-level one is an NG between players. As in standard optimization, this hierarchical decomposition may have computational advantages, for example, if it leads to a simpler or an analytically tractable NG.

In the second part of the paper we discuss such an NG for optical networks. In Pavel (2006b) a noncooperative channel game was formulated towards optical signal-to-noise ratio (OSNR) maximization. This is part of recent efforts aimed at addressing dynamical aspects in optical networks, from stability analysis to optimizing channel OSNR, (Forghieri, Tkach, & Favin, 1998; Pavel, 2004, 2006a). The OSNR NG formulation in (Pavel, 2006b) enabled an explicit closed-form NG solution. However, the link total power constraint was not considered. Such a constraint needs to be considered as optical fiber is shared by all wavelength-multiplexed channels in a link and total power needs to be below the nonlinearity threshold (Agrawal, 2002; Mecozzi, 1998). This coupled constraint can be regarded as a link power capacity constraint.

We formulate an OSNR NG with coupled utilities (as in wireless networks) and coupled constraints (as in congestion control). Solving directly such an NG with coupled constraints requires coordination among possibly all channels and is impractical (Pan & Pavel, 2005). Instead we use a duality approach. We show how to hierarchically decompose the coupled NG into a lower-level NG with no coupled constraints between channels, and a higher-level network optimization problem for pricing. Moreover, the lower-level NG is analytically tractable; its solution can be found decentralized based on its relation to the game in Pavel (2006b). The higher-level problem induces players to cooperate towards satisfying the coupled constraint by adjusting the price.

The paper is organized as follows. In Section 2 we review NG with uncoupled constraints. In Section 3 we consider NG with coupled constraints. We show that such an NG is related to a constrained optimization for the NG-game cost function, with respect to its second argument. We develop extension of Lagrange multiplier results and express them in a two-argument form, with fixed-point solution. In Section 4 we develop duality results and hierarchical decomposition in an NG-game sense. In Section 5–7 we apply these results to an OSNR game in optical networks. In Section 5 we review the OSNR model in Pavel (2006b) and we formulate a coupled OSNR NG with total power constraints. Applying duality results, in Section 6 we decompose it into a lower-level NG with no coupled constraints, and a higher-level link problem for pricing. We propose a recursive hierarchical algorithm, based on the explicit solution of the lower-level NG and on price adjustment. For a given price from the link, at the lower-level the algorithm is decentralized with respect to channels. The pricing parameter is adjusted at the link level, based on the measured total power. This adjustment acts as a coordination so that channels satisfy the total power constraint. An example is given in Section 7 and conclusions in Section 8.

In the following we present some preliminary notation. For a twice continuously differentiable function $f : \mathbb{R}^m \to \mathbb{R}$, we denote its gradient by ∇f, defined as the row vector whose ith component is $\partial f(\mathbf{x})/\partial x_i$, $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$. We denote its Hessian by $\nabla^2 f(\mathbf{x})$, defined as the symmetric matrix whose (i, j)th entry is equal to $(\partial^2 f/\partial x_i \partial x_j)(\mathbf{x})$. For a two-argument function, $f(\mathbf{u}; \mathbf{x})$, $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, we denote its gradient by $\nabla f(\mathbf{u}; \mathbf{x}) = [\nabla_u f(\mathbf{u}; \mathbf{x}) \;\; \nabla_x f(\mathbf{u}; \mathbf{x})]$, where $\nabla_u f(\mathbf{u}; \mathbf{x})$, $\nabla_x f(\mathbf{u}; \mathbf{x})$ are the gradients with respect to the first argument $\mathbf{u}=(u_1, \dots, u_m) \in \mathbb{R}^m$ and second argument $\mathbf{x}=(x_1, \dots, x_m) \in \mathbb{R}^m$, respectively.

2. NGs with uncoupled constraints

In this section we review some definitions and results for noncooperative (Nash) games with uncoupled constraints, i.e., with decoupled actions sets (Basar & Olsder, 1999; Nash, 1950). We give definitions with respect to both the individual cost functions of the players and the NG-game cost function.

Consider an m-player NG, each player i minimizing the individual cost function $J_i : \Omega \to \mathbb{R}$, $i \in \mathscr{M}$, $\mathscr{M} = \{1, \dots, m\}$. We typically assume J_i to be continuously differentiable. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$, $\Omega_i = [m_i, M_i]$, be the action set, so that Ω is a separable set. Let $\mathbf{u} = [u_1 \dots u_i \dots u_m]^{\mathrm{T}} \in \Omega$, with $u_i \in \Omega_i$, denote the vector of player actions. Let $\mathbf{u}_{-i} = [u_1 \dots u_{i-1}, u_{i+1} \dots u_m]^{\mathrm{T}}$ denote the vector obtained by deleting the ith element from $\mathbf{u}$. We then write $\mathbf{u} = (\mathbf{u}_{-i}, u_i) \in \Omega$, or $\mathbf{u} = [u_i] \in \Omega$. An NE solution of the game is defined as follows.

Defintion 1. Consider an m-player game, each player minimizing the cost function $J_i : \Omega \to \mathbb{R}$. A vector $\mathbf{u}^* = [u_i^*]$ or $\mathbf{u}^* = (\mathbf{u}_{-i}^*, u_i^*) \in \Omega$ is called an NE solution of this game if for every given $\mathbf{u}_{-i}^*$,

$$J_i(\mathbf{u}_{-i}^*, u_i^*) \leqslant J_i(\mathbf{u}_{-i}^*, x_i) \quad \forall x_i \in \Omega_i \;\; \forall i \in \mathscr{M}.$$

Such an NG has coupled cost functions but decoupled constraints (action spaces Ω_i are decoupled); any two players i and

j can choose independently their actions from separate action sets, Ω_j and Ω_j, $j \neq i$.

By Definition 1 $\mathbf{u}^*$ is an NE solution when u_i^* is solution to the individual optimization problem for player i, given all other players have equilibrium actions $\mathbf{u}_{-i}^*$. In this sense, each cost function J_i (parameterized by $\mathbf{u}_{-i}^*$) is minimized individually, but the NE solution has to satisfy simultaneously the set of m inequalities in Definition 1. Equivalently this definition can be formulated using an augmented "system-like" cost function (Basar & Olsder, 1999, p. 176) that we will call NG-game cost function. The NG-game cost function $\widetilde{J} : \Omega \times \Omega \to \mathbb{R}$ is defined as the two-argument function

$$\widetilde{J}(\mathbf{u}; \mathbf{x}) := \sum_{i=1}^{m} J_i(\mathbf{u}_{-i}, x_i) \quad \forall \mathbf{x} \in \Omega, \quad \mathbf{u} \in \Omega \tag{1}$$

with $\mathbf{x} = [x_1 \ldots x_i \ldots x_m]^{\mathrm{T}}$, and $\mathbf{u}$ $\mathbf{u}_{-i}$ defined as before. An NE solution can be defined equivalently with respect to the NG-game cost function $\widetilde{J}$, (1), as follows.

Defintion 2. Consider an m-player game, with each player minimizing the cost function $J_i : \Omega \to \mathbb{R}$. A vector $\mathbf{u}^* \in \Omega$ is called an NE solution of this game if its NG-game cost function $\widetilde{J}$ defined by (1) satisfies

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*; \mathbf{x}) \quad \forall \mathbf{x} \in \Omega$$

This is equivalent to

$$\sum_{i=1}^{m} J_i(\mathbf{u}_{-i}^*, u_i^*) \leqslant \sum_{i=1}^{m} J_i(\mathbf{u}_{-i}^*, x_i) \quad \forall \mathbf{x} \in \Omega$$

for every given $\mathbf{u}_{-i}^*$. If $\mathbf{u}^*$ is an NE in the sense of Definition 1, it follows immediately that $\mathbf{u}^*$ satisfies also Definition 2. Conversely, it can be shown that if $\mathbf{u}^*$ is an NE in the sense of Definition 2, it satisfies component-wise all inequalities in Definition 1 (Basar & Olsder, 1999, p. 177). The NG-cost function $\widetilde{J}$ defined by (1) is separable in the second argument $\mathbf{x}$, for any given first argument $\mathbf{u}$. We call this property separability in an NG-game sense. The NG-cost function and its separability property will be instrumental in our developments.

An NE solution exists if well-defined reaction curves of all players exist and these have a common intersection point. Since in what follows we will be using the NG-cost function equivalence, we reformulate results for NE solution (existence and necessary conditions) with respect to the NG-game cost function $\widetilde{J}$. Without loss of generality we assume that assumption (A.1) holds such that an NE solution is inner.

(A.1). $u_i = m_i$, $u_i = M_i$ are not minimizing J_i, i.e., for any given $\mathbf{u}_{-i}$, $J_i(\mathbf{u}_{-i}, m_i) > J_i(\mathbf{u}_{-i}, u_i)$, $\forall u_i \neq m_i$; $J_i(\mathbf{u}_{-i}, M_i) > J_i(\mathbf{u}_{-i}, u_i)$, $\forall u_i \neq M_i$.

In general, player parameters can be selected to satisfy (A.1) (Alpcan et al., 2001; Pavel, 2006b), or alternatively projection methods can be used (Bertsekas, 1999).

Proposition 1. *Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$, with Ω_i a closed, bounded and convex subset of $\mathbb{R}$. For each $i \in \mathscr{M}$ the cost function $J_i : \Omega \to \mathbb{R}$ is continuously differentiable on Ω and strictly convex in u_i for every $u_j \in \Omega_j$, $j \neq i$. Then the associated NG with NG-game cost function $\widetilde{J}$ defined by* (1) *admits an NE solution.*

Under (A.1), *an NE solution* $\mathbf{u}$ *satisfies the following necessary conditions with respect to the NG-game cost $\widetilde{J}$, $\nabla_x \widetilde{J}(\mathbf{u}; \mathbf{u}) = 0$, i.e.,*

$$\nabla_x \widetilde{J}(\mathbf{u}; \mathbf{x})|_{\mathbf{x}=\mathbf{u}} = 0, \tag{2}$$

where the notation used denotes a fixed-point solution.

Proof. Under the conditions of convexity of J_i and compactness of Ω_i, by Theorem 4.3 in Basar and Olsder (1999, p. 173), it follows that there exists an NE solution to the game with individual costs J_i. With respect to the NG-game cost $\widetilde{J}$, (1) , we use arguments similar to those used in Theorem 4.4 of Basar and Olsder (1999, p. 176). Since $\widetilde{J}$ is separable in the second argument $\mathbf{x}$, using (1) we see that

$$\nabla_x \widetilde{J}(\mathbf{u}; \mathbf{x}) = \left[\frac{\partial J_1(\mathbf{u}_{-1}, x_1)}{\partial x_1} \cdots \frac{\partial J_m(\mathbf{u}_{-m}, x_m)}{\partial x_m} \right]. \tag{3}$$

Hence , the Hessian of $\widetilde{J}$ with respect to $\mathbf{x}$, $\nabla_{xx}^2 \widetilde{J}(\mathbf{u}; \mathbf{x})$, is a diagonal matrix and its ith diagonal element is

$$[\nabla_{xx}^2 \widetilde{J}(\mathbf{u}; \mathbf{x})]_{(i,i)} = \frac{\partial^2 J_i(\mathbf{u}_{-i}, x_i)}{\partial x_i^2}$$

By strict convexity of J_i with respect to its argument it follows that $\widetilde{J}$ itself is strictly convex with respect to its second argument $\mathbf{x}$, for every given $\mathbf{u}$. Thus, there exists an $\mathbf{x}^*$ minimizing $\widetilde{J}(\mathbf{u}; \mathbf{x})$ over $\mathbf{x}$, for every given $\mathbf{u}$. We can introduce the reaction set of the game

$$\Psi(\mathbf{u}) = \{\mathbf{v} \in \Omega | \widetilde{J}(\mathbf{u}; \mathbf{v}) \leqslant \widetilde{J}(\mathbf{u}; \mathbf{x}), \forall \mathbf{x} \in \Omega\} \tag{4}$$

so that $\mathbf{x}^* \in \Psi(\mathbf{u})$. The properties of $\widetilde{J}$ imply that Ψ is an upper semicontinuous mapping that maps each point $\mathbf{u}$ in the convex and compact set Ω into a closed convex subset of Ω. By a fixed-point theorem (Basar & Olsder, 1999, Theorem C.2, p. 483), it follows that $\Psi(\mathbf{u})$ has a fixed point $\mathbf{u}^*$, i.e., $\mathbf{u}^* \in \Psi(\mathbf{u}^*)$, so that, by (4), $\mathbf{u}^*$ satisfies

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*; \mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \tag{5}$$

i.e., in effect $\mathbf{x} = \mathbf{u}^*$ minimizes $\widetilde{J}(\mathbf{u}; \mathbf{x})$, over $\mathbf{x} \in \Omega$, and is a fixed-point of $\Psi(\mathbf{u})$. By Definition 2, $\mathbf{u}^*$ is an NE solution to the m-player game. The individual components of $\mathbf{u}^*$ constitute an NE solution in the sense of Definition 1 and by (A.1) it is inner.

Next we consider the necessary conditions to find an NE solution, with respect to J_i first and then to $\widetilde{J}$. In order to find u_i^* we solve the necessary condition

$$\frac{\partial J_i}{\partial u_i}(\mathbf{u}_{-i}, u_i) = 0 \quad \forall i \in \mathscr{M} \tag{6}$$

which defines the reaction curve R_i of the ith player. The optimal u_i^* is parameterized in $\mathbf{u}_{-i}$, $u_i^* = R_i(\mathbf{u}_{-i})$. An NE solution $\mathbf{u}^*$ is a fixed-point vector solution to the set of m

equations, (6), or $u_i^* = R_i(\mathbf{u}_{-i}^*)$. We will then denote $J_i^* = J_i(\mathbf{u}_{-i}^*, u_i^*)$ the Nash individual optimal values.

Now with respect to $\widetilde{J}$, in order to find an $\mathbf{x}^*$ minimizing $\widetilde{J}(\mathbf{u};\mathbf{x})$, over $\mathbf{x}$ as in (4), we need to solve first

$$\nabla_x \widetilde{J}(\mathbf{u};\mathbf{x}) = 0 \tag{7}$$

for every given $\mathbf{u}$. This gives $\mathbf{x}^*$ parameterized by $\mathbf{u}$, $\mathbf{x}^* \in \Psi(\mathbf{u})$. Since an NE solution $\mathbf{u}$ is a fixed point of $\Psi(\mathbf{u})$, we need to solve (7) for a fixed-point solution: we set $\mathbf{x} = \mathbf{u}$ and solve $\nabla_x \widetilde{J}(\mathbf{u};\mathbf{u}) = 0$. This is written compactly as in (2), which using (3) yields the component-wise form

$$\left.\frac{\partial J_i}{\partial x_i}(\mathbf{u}_{-i}, x_i)\right|_{x_i=u_i} = 0 \quad \forall i \in \mathscr{M}$$

In the foregoing we set $x_i = u_i$ to look for a fixed-point solution yielding $(\partial J_i/\partial u_i)(\mathbf{u}_{-i}, u_i) = 0$, $\forall i \in \mathscr{M}$, which is identical to (6). Hence the two-argument form (2) is equivalent to the component-wise form (6). □

Remark 1. As described above, the procedure to find an NE solution is as follows: find a solution to each individual minimization with cost function J_i, stack the resulting set of m parameterized equations in vector form and find a fixed-point solution. With respect to $\widetilde{J}$, this procedure involves minimizing $\widetilde{J}$ with respect to the second argument $\mathbf{x}$ as in (4), followed by solving for a fixed-point solution as in (5). This last step is realized by setting $\mathbf{x} = \mathbf{u}$ and solving (7), or $\mathbf{u} = \Psi(\mathbf{u})$, for a solution $\mathbf{u}^*$. For convenience we use the compact notation in (2) to denote this fixed-point minimization of $\widetilde{J}$. Specifically, from (5) we write

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) = \left.\min_{\mathbf{x}\in\Omega} \widetilde{J}(\mathbf{u};\mathbf{x})\right|_{\mathbf{x}=\mathbf{u}} \tag{8}$$

and

$$\mathbf{u}^* = \arg\left\{ \left.\min_{\mathbf{x}\in\Omega} \widetilde{J}(\mathbf{u};\mathbf{x})\right|_{\mathbf{x}=\mathbf{u}} \right\}. \tag{9}$$

3. NGs with coupled constraints: Lagrangian extension

In this section we consider NGs with coupled constraints, i.e., with coupled action spaces. Without loss of generality we consider only inequality constraints, since equality constraints can be treated similarly. Solving directly such an NG requires coordination among possibly all players. Alternatively, we exploit the fact that an NG with coupled constraints is related to a constrained minimization of the NG-game cost function, with respect to the second argument and fixed-point solution (Lemma 1). Then as in standard optimization, we develop Lagrange multiplier results and express them in a two-argument form, with fixed-point solution (Lemma 2). In the next section we use this extension to obtain duality results and hierarchical decomposition of NGs with coupled constraints.

Consider an m-player NG, each player minimizing the individual cost function $J_i\colon \Omega \to \mathbb{R}$, $\Omega = \Omega_1 \times \cdots \times \Omega_m$, subject to R coupled inequality constraints

$$g_r(\mathbf{u}) \leqslant 0, \quad r = 1, \ldots, R,$$

or $\mathbf{g}(\mathbf{u}) \leqslant 0$ in vector form, with $\mathbf{g}(\mathbf{u}) = [g_1(\mathbf{u}) \ldots g_R(\mathbf{u})]^{\mathrm{T}}$. The overall action set $\overline{\Omega} \subset \mathbb{R}^m$ is coupled in this case,

$$\overline{\Omega} = \{\mathbf{u} \in \Omega \mid \mathbf{g}(\mathbf{u}) \leqslant 0\}. \tag{10}$$

A vector $\mathbf{u} = (\mathbf{u}_{-i}, u_i)$ is called feasible if $\mathbf{u} \in \overline{\Omega}$. As in Definition 1, $\mathbf{u}^*$ is an NE solution if

$$J_i(\mathbf{u}_{-i}^*, u_i^*) \leqslant J_i(\mathbf{u}_{-i}^*, x_i) \quad \forall x_i \in \overline{\Omega}_i(\mathbf{u}_{-i}^*) \;\; \forall i \in \mathscr{M}, \tag{11}$$

where now $\overline{\Omega}_i(\mathbf{u}_{-i}^*)$ is the projection set defined as

$$\overline{\Omega}_i(\mathbf{u}_{-i}^*) := \{x_i \in \Omega_i \mid (\mathbf{u}_{-i}^*, x_i) \in \overline{\Omega}\}. \tag{12}$$

Unlike games with uncoupled action sets (Section 2), in this case there are no separate sets from which players can choose independently their actions. For such coupled action sets, the following theorem gives sufficient conditions for existence of an NE solution (Basar & Olsder, 1999, Theorem 4.4, p. 176). We restate this result here as Theorem 1.

Theorem 1. *Let $\overline{\Omega}$ be a closed, bounded and convex subset of $\mathbb{R}^m$, and for each $i \in \mathscr{M}$ let the cost function $J_i\colon \overline{\Omega} \to \mathbb{R}$ be continuous on $\overline{\Omega}$ and convex in u_i, for every $u_j \in \overline{\Omega}_j$, $j \in \mathscr{M}$, $j \neq i$. Then the associated NG admits an NE solution.*

We show next that an NG with coupled constraints can be related to a constrained minimization of the NG-cost function, with respect to the second argument, that has a fixed-point solution. This result will be followed by Lagrangian extension in a game theoretical setup.

Lemma 1. *Consider an NG for m players, each player minimizing the individual cost function $J_i(\mathbf{u}_{-i}, u_i)$, $\mathbf{u} \in \Omega \subset \mathbb{R}^m$, subject to coupled constraints $g_r(\mathbf{u}) \leqslant 0$, $r = 1, \ldots, R$. Assume that Theorem 1 is satisfied so that an NE solution $\mathbf{u}^*$ exists, and denote by $J_i^* = J_i(\mathbf{u}_{-i}^*, u_i^*)$ the Nash individual optimal values.*

(a) *Then with respect to $\widetilde{J}$ defined by (1), an NE solution $\mathbf{u}^*$ satisfies*

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*;\mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \quad \mathbf{g}(\mathbf{u}_{-i}^*, x_i) \leqslant 0 \quad \forall i, \tag{13}$$

where $\mathbf{g}(\mathbf{u}_{-i}, x_i) = [g_1(\mathbf{u}_{-i}, x_i) \ldots g_R(\mathbf{u}_{-i}, x_i)]^{\mathrm{T}}$.

(b) *Let $\mathbf{u}^*$ be a solution for the constrained minimization of NG-game cost function $\widetilde{J}$ defined by (1) such that*

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*;\mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \;\; \widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{x}) \leqslant 0 \tag{14}$$

with $\widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{u}^*) \leqslant 0$, where the two-argument augmented constraint $\widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{x}) : \Omega \times \Omega \to \mathbb{R}^R$ is defined as

$$\widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{x}) = \sum_{i=1}^{m} \mathbf{g}(\mathbf{u}_{-i}^*, x_i). \tag{15}$$

Then $\mathbf{u}^*$ is an NE solution and the optimal NG cost is

$$\widetilde{J}^* = \widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) = \sum_i J_i^*. \tag{16}$$

Proof. We give the proof for a single coupled constraint. Extension to the multiple constraint case can be done by appropriately using vector inequalities.

(a) Using (10), we see that the projection set (12) is

$$\overline{\Omega}_i(\mathbf{u}^*_{-i}) = \{x_i \in \Omega_i | g(\mathbf{u}^*_{-i}, x_i) \leqslant 0\}.$$

Using also (11), we see that $\mathbf{u}^*$ as an NE solution satisfies

$$J_i(\mathbf{u}^*_{-i}, u^*_i) \leqslant J_i(\mathbf{u}^*_{-i}, x_i) \quad \forall x_i \in \Omega_i, \quad g(\mathbf{u}^*_{-i}, x_i) \leqslant 0 \tag{17}$$

for all $i \in \mathscr{M}$, with $g(\mathbf{u}^*_{-i}, u^*_i) \leqslant 0$. Using (1), (17) it follows immediately that (13) holds.

(b) We show the next part by contradiction. Since $\widetilde{g}$, (15), has a special form with the same constraint repeated for each term in the sum, note that $\widetilde{g}(\mathbf{u}^*; \mathbf{u}^*) = mg(\mathbf{u}^*)$. Hence $\widetilde{g}(\mathbf{u}^*; \mathbf{u}^*) \leqslant 0$ is equivalent to $g(\mathbf{u}^*_{-i}, u^*_i) \leqslant 0$ (feasibility).

Suppose that $\mathbf{u}^*$ is a solution of (14), i.e.,

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*; \mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \quad \widetilde{g}(\mathbf{u}^*; \mathbf{x}) \leqslant 0$$

with $\widetilde{g}(\mathbf{u}^*; \mathbf{u}^*) \leqslant 0$, but is not an NE in the sense of (11) or (17). Then, it follows that there exists an $i_0 \in \mathscr{M}$ and some $\bar{x}_{i_0} \in \Omega_{i_0}$ with

$$g(\mathbf{u}^*_{-i_0}, \bar{x}_{i_0}) \leqslant 0 \tag{18}$$

such that

$$J_{i_0}(\mathbf{u}^*_{-i_0}, \bar{x}_{i_0}) < J_{i_0}(\mathbf{u}^*_{-i_0}, u^*_{i_0}).$$

Then adding $\sum_{i \neq i_0} J_i(\mathbf{u}^*)$ on both sides yields

$$\widetilde{J}(\mathbf{u}^*; \bar{\mathbf{x}}_0) < \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*), \quad \bar{\mathbf{x}}_0 = (\mathbf{u}^*_{-i_0}, \bar{x}_{i_0}) \in \Omega.$$

Now from (18) and $g(\mathbf{u}^*_{-i}, u^*_i) \leqslant 0$, it follows that

$$\widetilde{g}(\mathbf{u}^*; \bar{\mathbf{x}}_0) = \sum_{i \neq i_0} g(\mathbf{u}^*_{-i}, u^*_i) + g(\mathbf{u}^*_{-i_0}, \bar{x}_{i_0}) \leqslant 0, \quad \forall i \neq i_0$$

so that $\bar{\mathbf{x}}_0$ is feasible. The two foregoing inequalities imply that $\mathbf{u}^*$ is not a solution of (14), which is false. Hence (14) implies (17) or (11), and (16) follows. □

This relation to a special constrained optimization problem is exploited in the following. Optimality conditions for standard constrained optimization involve a set of Lagrange multipliers (Bertsekas, 1999, chap. 3). The next result gives necessary and sufficient conditions for an NE game optimal solution, in terms of Lagrange multipliers.

Lemma 2. *Consider an NG with individual cost functions* $J_i(\mathbf{u}_{-i}, u_i)$, $\mathbf{u} \in \Omega \subset \mathbb{R}^m$, *subject to the constraints* $g_r(\mathbf{u}) \leqslant 0$, $r = 1, \ldots, R$. *Assume* J_i *and* g_r *are continuously differentiable convex functions, and let* $\mathscr{A}(\mathbf{u}) = \{r | g_r(\mathbf{u}) = 0\}$ *denote the set of active constraints.*

(a) (*Necessity*): *Let* $\mathbf{u}$ *be an NE solution. Then there exist unique* μ^*, $\mu^*_r \geqslant 0$ *such that*

$$\nabla_x \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu^*)\big|_{\mathbf{x}=\mathbf{u}} = 0 \tag{19}$$

and $\mu^*_r = 0$, $\forall r \notin \mathscr{A}(\mathbf{u})$, *i.e.*, $\mu^*_r g_r(\mathbf{u}) = 0$, $r = 1, \ldots, R$, *where* $\widetilde{L}$ *is the two-argument Lagrangian function defined by*

$$\widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) = \widetilde{J}(\mathbf{u}; \mathbf{x}) + \mu^{\mathrm{T}} \widetilde{\mathbf{g}}(\mathbf{u}; \mathbf{x}). \tag{20}$$

(b) (*Sufficiency*): *Let* $\mathbf{u}^*$ *be a feasible point and* μ *be such that* $\mu_r \geqslant 0$, $r = 1, \ldots, R$, *with* $\mu_r = 0$, $\forall r \notin \mathscr{A}(\mathbf{u})$, *or*

$$\mu_r g_r(\mathbf{u}) = 0, \quad r = 1, \ldots, R. \tag{21}$$

Assume that $\mathbf{u}^*$ *minimizes the Lagrangian* $\widetilde{L}$ (20), *over* $\mathbf{x} \in \Omega$, *as a fixed point, i.e.*,

$$\mathbf{u}^* = \arg\left\{\min_{x \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu)|_{\mathbf{x}=\mathbf{u}}\right\} \tag{22}$$

in the sense of (9). *Then* $(\mathbf{u}; \mathbf{x}) = (\mathbf{u}^*; \mathbf{u}^*)$ *is an NE game solution in the sense of* (14), *i.e.*,

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*; \mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \quad \widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{x}) \leqslant 0,$$

and hence is an NE solution in the sense of (11).

Proof. (a) (Necessity): The proof is based on a penalty approach as in standard constrained optimization (Bertsekas, 1999, Section 3.1.1, p. 281). We consider only the set of active constraints, treated as equality constraints. For the inactive constraints a zero penalty can be taken. Around an NE solution $\mathbf{u} = (\mathbf{u}_{-i}, u_i)$, the original game with cost functions $J_i(\mathbf{u})$ and constraints $\mathbf{g}(\mathbf{u})$ can be approximated locally by an unconstrained game with penalized cost functions

$$F^k_i(\mathbf{u}^k) = J_i(\mathbf{u}^k) + \frac{k}{2}\|\mathbf{g}(\mathbf{u}^k)\|^2 + \frac{\alpha}{2}|u^k_i - u_i|^2 \quad \forall i$$

with $\mathbf{u}^k = (\mathbf{u}^k_{-i}, u^k_i)$, for some $\alpha > 0$ and $k = 1, 2, \ldots$. The same penalty term is added to all cost functions J_i to penalize violation of the constraints. For each i, it can be shown that $\{u^k_i\}$ converges to u_i as $k \to \infty$ (as in Bertsekas, 1999, Propositions 3.1.1 & 3.3.1, p. 278). Thus, component-wise the sequence $\{\mathbf{u}^k\}$ (NE solutions of the unconstrained game) converges to $\mathbf{u}$. Now for an NE solution $\mathbf{u}^k$ of the unconstrained game with cost functions $F^k_i(\mathbf{u}^k)$ necessary conditions as in (6), Proposition 1 hold. Thus the set of m equations $(\partial F^k_i / \partial u^k_i)(\mathbf{u}^k) = 0$, $\forall i$, needs to be solved for a fixed-point solution. Equivalently, for $\forall i$,

$$\frac{\partial J_i}{\partial u^k_i}(\mathbf{u}^k_{-i}, u^k_i) + k\mathbf{g}(\mathbf{u}^k)\frac{\partial \mathbf{g}}{\partial u^k_i}(\mathbf{u}^k_{-i}, u^k_i) + \alpha|u^k_i - u_i| = 0.$$

By an argument similar to that in Propositions 3.1.1 and 3.3.1 of Bertsekas (1999), it can be shown that $\lim_{k\to\infty} k\mathbf{g}(\mathbf{u}^k) = \mu^* \geqslant 0$. Taking $k \to \infty$ and using $\{u^k_i\} \to u_i$ in the foregoing set yields that μ^* satisfies

$$\frac{\partial J_i}{\partial u_i}(\mathbf{u}_{-i}, u_i) + \mu^{*\mathrm{T}} \frac{\partial \mathbf{g}}{\partial u_i}(\mathbf{u}_{-i}, u_i) = 0 \quad \forall i. \tag{23}$$

Thus, (23) are the necessary conditions for an NE solution $\mathbf{u}$ of the original game, given in a component-wise form. As in (2), we can write these conditions in a compact two-argument form with fixed-point notation, based on $\widetilde{L}$ defined in (20). Using the separability property of both $\widetilde{J}$, (1), and $\widetilde{\mathbf{g}}$, (15), it can be seen that $\widetilde{L}$, (20), is separable and satisfies a relation similar to (3). Moreover component-wise, relation (19), i.e.,

$$\nabla_x \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu^*)\big|_{\mathbf{x}=\mathbf{u}} = 0,$$

is identical to the set of m equations (23). So, the necessary conditions for an NE solution are given in a compact two-argument form as in (19).

(b) (Sufficiency): Since $\mathbf{u}^*$ is a feasible point satisfying (21), using (15) gives $\mu^{\mathrm{T}}\widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{u}^*) = 0$. Then (20) yields

$$\widetilde{L}(\mathbf{u}^*; \mathbf{u}^*; \mu) = \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*).$$

From (22), $\mathbf{u}^*$ minimizes the augmented Lagrangian $\widetilde{L}(\mathbf{u}; \mathbf{x}; \mu)$, with respect to the second argument and is a fixed-point solution. Then similar to (5) we have

$$\widetilde{L}(\mathbf{u}^*; \mathbf{u}^*; \mu) \leqslant \widetilde{L}(\mathbf{u}^*; \mathbf{x}; \mu) \quad \forall \mathbf{x} \in \Omega.$$

Using (20) and $\mu \geqslant 0$ on the right-hand side yields

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*; \mathbf{x}) \quad \forall \mathbf{x} \in \Omega \quad \widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{x}) \leqslant 0$$

for $\forall \mathbf{x}$ such that $\widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{x}) \leqslant 0$, so (14) holds. By Lemma 1 it follows that $\mathbf{u}^*$ is an NE solution as in (11). □

Note that as J_i and g_r are differentiable convex functions and $\Omega = \mathbb{R}^m$, the Lagrangian function $\widetilde{L}(\mathbf{u}; \mathbf{x}; \mu)$ is convex with respect to $\mathbf{x}$, so the Lagrangian minimization is equivalent to the first order necessary condition. Thus in the presence of convexity the first order optimality conditions are also sufficient.

Remark 2. By Lemma 2 an NE solution $\mathbf{u}^*$ can be obtained by minimizing the augmented Lagrangian $\widetilde{L}(\mathbf{u}; \mathbf{x}; \mu)$, with respect to the second argument and finding a fixed-point solution, (22), via (19). This $\mathbf{u}^*$ is a solution to the constrained minimization of the NG-cost function $\widetilde{J}$, with respect to the second argument, in the sense of (14). We can interpret $\widetilde{L}$, (20), as a Lagrangian for $\widetilde{J}$ and $\widetilde{\mathbf{g}}$. As in (8), we can write in a compact notation,

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) = \left[\min_{\mathbf{x} \in \Omega, \widetilde{\mathbf{g}}(\mathbf{u}; \mathbf{x}) \leqslant 0} \widetilde{J}(\mathbf{u}; \mathbf{x}) \right] \Bigg|_{\mathbf{x}=\mathbf{u}},$$

where in fact $\mathbf{u}^*$ is obtained via (19).

4. Duality and hierarchical decomposition

In this section we develop duality results for NGs with coupled constraints, based on the Lagrangian extension. We introduce a dual cost function related to the minimization of the associated Lagrangian function (cf. Lemma 2), similar to standard optimization (see Bertsekas, 1999, p. 359). For NGs with convex coupled constraints, we show that duality enables decomposition into a lower-level NG with no coupled constraints, and a higher-level optimization problem.

Consider an NG with coupled constraints as in Section 3. Recall the associated Lagrangian function $\widetilde{L}$, (20), and its minimization in a fixed-point sense, as in (22), Lemma 2. The resulting fixed-point solution is a function of μ, $\mathbf{u}^* = \mathbf{u}^*(\mu)$. Consider the *dual cost* $D(\mu)$ as a function of μ, defined as

$$D(\mu) = \widetilde{L}(\mathbf{u}^*; \mathbf{u}^*; \mu),$$

or in a fixed-point notation as in (8),

$$D(\mu) := \left[\min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) \right] \Bigg|_{\mathbf{x}=\mathbf{u}}, \tag{24}$$

where $\widetilde{\mathbf{g}}(\mathbf{u}; \mathbf{u}) \leqslant 0$. The *dual NG problem* can be defined as *maximizing* $D(\mu)$ subject to $\mu \geqslant 0$, with the dual optimal value defined as

$$D^* = \max_{\mu \geqslant 0} D(\mu). \tag{25}$$

The following result characterizes the primal and dual optimal solution pairs.

Theorem 2. $(\mathbf{u}^*; \mu^*)$ *is optimal NE solution–Lagrange multiplier pair in the sense of* (25) *if and only if*

(*NG-feasibility*):

$$\mathbf{u}^* \in \Omega, \quad \widetilde{g}_r(\mathbf{u}^*; \mathbf{u}^*) \leqslant 0, \quad r = 1, \ldots, R.$$

(*Dual feasibility*):

$$\mu^* \geqslant 0.$$

(*Lagrangian optimality*):

$$\mathbf{u}^* = \arg \left\{ \left[\min_{x \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu^*) \right] \Bigg|_{\mathbf{x}=\mathbf{u}} \right\}.$$

(*Complementary slackness*):

$$\mu_r^* \widetilde{g}_r(\mathbf{u}^*; \mathbf{u}^*) = 0, \quad r = 1, \ldots, R.$$

Proof. If $(\mathbf{u}^*; \mu^*)$ is an optimal NE solution–Lagrange multiplier pair, then $\mathbf{u}^*$ is feasible and μ^* is dual feasible and the first two relations follow directly. The last two relations follow from Lemma 2.

For sufficiency, using Lagrangian optimality we obtain

$$\widetilde{L}(\mathbf{u}^*; \mathbf{u}^*, \mu^*) = \left[\min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu^*) \right] \Bigg|_{\mathbf{x}=\mathbf{u}}$$

so that

$$\widetilde{L}(\mathbf{u}^*; \mathbf{u}^*, \mu^*) \leqslant \widetilde{L}(\mathbf{u}^*; \mathbf{x}; \mu^*) \quad \forall \mathbf{x} \in \Omega.$$

Using (20) and complementary slackness yields

$$\widetilde{L}(\mathbf{u}^*; \mathbf{u}^*, \mu^*) = \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*).$$

Then from the foregoing inequality we can write

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leqslant \widetilde{J}(\mathbf{u}^*; \mathbf{x}) \quad \forall x \in \Omega, \quad \widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{x}) \leqslant 0.$$

Therefore (14) holds and, by Lemma 1, $\mathbf{u}^*$ is an optimal NE game solution with $\widetilde{J}^* = \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*)$. Using (24), evaluated at μ^*, and the foregoing relations yields

$$D(\mu^*) = \min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu^*) \Bigg|_{\mathbf{x}=\mathbf{u}} = \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*),$$

and for the optimal dual cost D^*, (25),

$$D^* \geqslant D(\mu^*) = \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) = \widetilde{J}^* \qquad \square$$

If a Lagrange multiplier μ is known then all optimal NE solutions $(\mathbf{u}^*, \mathbf{u}^*)$ can be found by minimizing the Lagrangian $\widetilde{L}$ over $\mathbf{x} \in \Omega$, in the fixed-point sense as in (22). However, among those solutions $\mathbf{u}^*(\mu)$, there may be vectors that do not satisfy the coupled NG-feasibility condition $\mathbf{g}(\mathbf{u}^*) \leqslant 0$, so this has to be checked.

In the following we exploit the fact that both the NG-game cost and the constraints are separable in the second argument.We show that the dual NG-cost function $D(\mu)$ can be decomposed and, equivalently, found by solving a modified NG with no coupled constraints.

Theorem 3. *Consider an NG with cost functions $J_i(\mathbf{u}_{-i}, u_i)$, $\mathbf{u} \in \Omega \subset \mathbb{R}^m$, subject to the coupled constraints $g_r(\mathbf{u}) \leqslant 0$, where J_i and g are continuously differentiable and convex functions. The dual cost function $D(\mu)$ (24) can be decomposed as*

$$\begin{aligned} D(\mu) &= \sum_{i=1}^{m} \left[\min_{x_i \in \Omega_i} L_i(\mathbf{u}_{-i}, x_i, \mu) \right] \Big|_{x_i = u_i} \\ &= \sum_{i=1}^{m} L_i(\mathbf{u}^*_{-i}(\mu), u_i^*(\mu), \mu), \end{aligned} \tag{26}$$

where

$$L_i(\mathbf{u}_{-i}, x_i, \mu) = J_i(\mathbf{u}_{-i}, x_i) + \mu^{\mathrm{T}} \mathbf{g}(\mathbf{u}_{-i}, x_i) \tag{27}$$

and $\mathbf{u}^*(\mu) = [u_i^*(\mu)]$ is a fixed-point solution to the set of m minimizations in (26). *Alternatively, $D(\mu)$ can be obtained by solving a modified NG with cost functions L_i,* (27) *and no coupled constraints.*

Proof. By Lemma 2, the necessary conditions for NE optimality with respect to the Lagrangian $\widetilde{L}$, (20), require to solve

$$\nabla_x \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu)\big|_{\mathbf{x}=\mathbf{u}} = 0, \tag{28}$$

or, equivalently component-wise, $(\partial \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu)/\partial x_i)|_{x_i = u_i} = 0$, $i = 1, m$. Using (1) and (15) we can write $\widetilde{L}$, (20), as

$$\widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) = \sum_{i=1}^{m} L_i(\mathbf{u}_{-i}, x_i, \mu) \tag{29}$$

with L_i as in (27). Using (29) into the foregoing yields

$$\sum_{j=1}^{m} \frac{\partial}{\partial x_i} L_j(\mathbf{u}_{-j}, x_j; \mu) = 0, \quad i = 1, m.$$

Due to separability with respect to $\mathbf{x}$, this yields

$$\frac{\partial}{\partial x_i} L_i(\mathbf{u}_{-i}, x_i; \mu) = 0, \quad i = 1, m. \tag{30}$$

Therefore, component-wise (28) is the same as (30). For either (28) or (30) we need to find a fixed-point solution. Now (30) are the first order necessary conditions for minimizing L_i (27) with respect to x_i. Since J_i and g_r are convex, they are also sufficient. For each given $\mathbf{u}_{-i}$, from (30) we obtain $x_i^* = x_i(\mathbf{u}_{-i})$, so that

$$x_i(\mathbf{u}_{-i}) = \arg \min_{x_i \in \Omega_i} [L_i(\mathbf{u}_{-i}, x_i, \mu)], \quad i = 1, m. \tag{31}$$

Moreover, because we look for a fixed-point solution we set $\mathbf{x} = \mathbf{u}$, i.e., component-wise we need to solve

$$x_i(\mathbf{u}_{-i}) = u_i \quad \forall i = 1, \ldots, m$$

for a fixed-point vector denoted by $\mathbf{u}^* = [u_i^*]$ and $\mathbf{x} = [u_i^*]$, which depends on μ. With this $\mathbf{u}^*$ we return now to the value functional in (29). The first step taken in order to obtain $\mathbf{u}^*$ was minimization with respect to $\mathbf{x}$, so that from (29) we have

$$\min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) = \min_{\mathbf{x} \in \Omega} \sum_{i=1}^{m} L_i(\mathbf{u}_{-i}, x_i, \mu), \quad \mathbf{x} \in \Omega$$

for any given $\mathbf{u}$, with L_i as in (27). Since $\Omega = \Omega_1 \times \cdots \times \Omega_m$ and the right-hand side is separable with respect to $\mathbf{x} = [x_i]$, $x_i \in \Omega_i$, it follows that

$$\min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) = \sum_{i=1}^{m} \min_{x_i \in \Omega_i} L_i(\mathbf{u}_{-i}, x_i, \mu) \tag{32}$$

for any given $\mathbf{u}$. Now evaluating (32) at the fixed-point $\mathbf{u}^* = [u_i^*]$, $\mathbf{x} = [u_i^*]$ obtained as above, we can write

$$\begin{aligned} &\left[\min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) \right] \Big|_{\mathbf{u}=\mathbf{u}^*, \mathbf{x}=\mathbf{u}^*} \\ &\quad = \sum_{i=1}^{m} \left[\min_{x_i \in \Omega_i} L_i(\mathbf{u}_{-i}, x_i, \mu) \right] \Big|_{u_i = u_i^*, x_i = u_i^*} \\ &\quad = \sum_{i=1}^{m} L_i(\mathbf{u}^*_{-i}(\mu), u_i^*(\mu), \mu). \end{aligned}$$

We write this in a compact fixed-point notation, as in (8),

$$\begin{aligned} &\left[\min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) \right] \Big|_{\mathbf{x}=\mathbf{u}} \\ &\quad = \sum_{i=1}^{m} \left[\min_{x_i \in \Omega_i} L_i(\mathbf{u}_{-i}, x_i, \mu) \right] \Big|_{x_i = u_i} \end{aligned}$$

which using (24) gives (26). The proof is completed by recalling that $\mathbf{u}^*$ is a fixed-point solution to the set of m optimizations (31), i.e., equivalently $\mathbf{u}^*$ is an NE solution to the NG with cost functions L_i, (27). $\square$

Theorem 3 yields a decomposition into a lower-level modified NG with cost functions L_i, (27), with no coupled constraints, and a higher-level optimization problem. In general, $\mathbf{u}^*(\mu)$ may not be NE optimal for the given μ, in the sense of attaining the minimum NG cost such that $L_i^* = J_i^*$. However, by Theorem 2 there exists a dual optimal price $\mu^* \geqslant 0$ such that $\mathbf{u}(\mu^*) = [u_i(\mu^*)]$ is NE optimal. Hence, μ^* can be found as the maximizer in (25), with $D(\mu)$ as in (26). A sufficient condition is that the dual cost $D(\mu)$ is strictly concave in μ, for $\mathbf{u}^*(\mu)$

as obtained from the lower-level game, (27). Alternatively, the price μ can be adjusted until the slackness conditions in Theorem 2 are satisfied indicating that the dual optimal price μ^* is found.

The formulation in Theorem 3 has a hierarchical game interpretation (Basar & Olsder, 1999). At the upper level is a Stackelberg game (Basar & Olsder, 1999, p. 179): the system is the leader that sets "prices" (Lagrange multipliers) and the m players are the followers. Given prices as set by the leader, an NG is played at the lower level between m players, with cost functions L_i, (27). Each player reacts to given "prices" and the price acts as a coordination signal.

As in standard optimization, the hierarchical game decomposition in Theorem 3 may offer computational advantages. For example, the lower-level game may admit a closed-form explicit solution, or the higher-level problem may have a reduced dimension. One such application of these results is presented in the following sections.

5. OSNR game in optical networks

In this section we consider an NG with coupled constraints in the context of maximizing OSNR in optical networks, (Pavel, 2006b).

Consider an optical link composed of N cascaded optical amplifiers and optical fiber spans. A set $\mathscr{M} = \{1, \ldots, m\}$ of channels are transmitted across the same optical fiber by wavelength-multiplexing. Every few tens of km of fiber an optical amplifier is used to compensate the loss of the previous optical span. Such an amplifier simultaneously boosts the optical power of all channels, each with gain G_i, at the expense of introducing amplified spontaneous emission (ASE) noise, with power ASE_i, $i \in \mathscr{M}$. Optical amplifiers are operated to maintain a constant total power at their output. This compensates for variations in optical span loss across a link (Forghieri et al., 1998). The total power target needs to be selected below the threshold for nonlinear effects (Mecozzi, 1998). Then at the input of each intermediary optical span the following condition holds:

$$\sum_{j=1}^{m} p_{k,j} = P_0 \quad \forall k = 1, \ldots, N, \tag{33}$$

where $p_{k,j}$ is the ith channel power at the output of the kth span, or input of the $(k+1)$th span.

Let u_i and $n_{0,i}$ denote the ith channel signal and noise optical power at the input (at Tx), respectively. Similarly, let $p_{N,i}$ and $n_{N,i}^{\text{out}}$ be the ith channel signal and noise optical power at the output (at Rx). The channel OSNR at the receiver (Rx), defined as $OSNR_i = p_{N,i}/n_{N,i}^{\text{out}}$, is given as (see Pavel, 2006b, Lemma 2)

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j \in \mathscr{M}} \Gamma_{i,j} u_j}, \tag{34}$$

where $\mathbf{\Gamma} = [\Gamma_{i,j}]$ is system matrix with

$$\Gamma_{i,j} = \sum_{v=1}^{N} \frac{G_j^v}{G_i^v} \frac{ASE_i}{P_0} \quad \forall i, j \in \mathscr{M}.$$

Channel OSNR (34) is affected by noise accumulation in optical amplifiers and depends on all channel powers at the input (Tx). Based on this model, an NG can be formulated towards OSNR optimization.

The same notation as in the previous sections is used. Here $\mathbf{u} = [u_i]$, or $\mathbf{u} = (\mathbf{u_{-i}}, u_i)$, is the vector of channel powers at the Tx, with $u_i \in \Omega_i$, $\Omega_i = [0, u_{\max}]$, $\mathbf{u} \in \Omega$.

An NG can be defined where each channel (player) minimizes an individual cost function J_i, by adjusting its own transmission power, in response to the other channels' actions. Consider the following cost J_i:

$$J_i(\mathbf{u_{-i}}, u_i) = -\beta_i U_i(\mathbf{u_{-i}}, u_i), \tag{35}$$

where U_i is the channel utility function related to OSNR maximization defined as

$$U_i(\mathbf{u_{-i}}, u_i) = \ln\left(1 + \frac{a_i}{1/OSNR_i - \Gamma_{i,i}}\right) \tag{36}$$

and a_i is a channel parameter. Note that this logarithmic utility function is monotone in OSNR and can be expressed as

$$U_i(\mathbf{u_{-i}}, u_i) = \ln\left(1 + a_i \frac{u_i}{X_{-\mathbf{i}}}\right), \quad a_i > 0, \tag{37}$$

where $X_{-\mathbf{i}} = \sum_{j \neq i} \Gamma_{i,j} u_j + n_{0,i}$. Thus, U_i is twice continuously differentiable, monotone increasing and strictly concave in u_i. In the above, β_i is a parameter indicating channel's desire to maximize its OSNR, selected such that an NE solution is inner.

Remark 3. The cost function (35) is the simplest one, as it has only an utility term. A similar utility function, that had an additional ad hoc pricing term, was used in the game formulation in Pavel (2006b),

$$\overline{J}_i(\mathbf{u_{-i}}, u_i) = \alpha_i u_i - \beta_i U_i(\mathbf{u_{-i}}, u_i). \tag{38}$$

However no coupled constraints were considered in Pavel (2006b). For a game with cost functions $\overline{J}_i$ (38), and no coupled constraints, Theorem 3 of Pavel (2006b) gives conditions for existence and uniqueness of the NE solution. We restate here this result which will be used in the following.

Proposition 2. *The m-player NG with individual cost functions* $\overline{J}_i$, (38), *admits a unique NE solution* $\mathbf{u}^*$ *if* a_i *are selected such that*

$$\sum_{j \neq i} \Gamma_{i,j} < a_i \qquad \forall i \in \mathscr{M}.$$

The optimal NE solution $\mathbf{u}^*$ *is inner and* $\mathbf{u}^* = \widetilde{\mathbf{\Gamma}}^{-1} \tilde{\mathbf{b}}$, *where* $\widetilde{\mathbf{\Gamma}} = [\widetilde{\Gamma}_{i,j}]$ *and* $\tilde{\mathbf{b}} = [\tilde{b}_i]$ *are defined as*

$$\widetilde{\Gamma}_{i,j} = \begin{cases} a_i, & j = i, \\ \Gamma_{i,j}, & j \neq i, \end{cases} \quad \tilde{b}_i = \frac{a_i \beta_i}{\alpha_i} - n_{0i}$$

and $\Gamma_{i,j}$ *being defined in* (34).

In the following we formulate a game that considers explicitly such a coupled constraint that corresponds to link power capacity. Such a constraint needs to be considered because all wavelength-multiplexed channels in a link share the optical fiber, and because total power needs to be below the threshold for nonlinear effects (Agrawal, 2002). As in (33), the total power at Tx has to be constrained also, $\sum_{j=1}^{m} u_j \leqslant P_0$, so that nonlinearity is limited. Hence, a general OSNR game in optical networks has utilities (37), and constraints being coupled, and hence has a coupled action set

$$\overline{\Omega} = \left\{ \mathbf{u} \in \Omega \,\middle|\, \sum_{j=1}^{m} u_j \leqslant P_0 \right\}. \tag{39}$$

As in (11), (12), an NE solution $\mathbf{u}^* \in \Omega$ satisfies

$$J_i(\mathbf{u}_{-i}^*, u_i^*) \leqslant J_i(\mathbf{u}_{-i}^*, x_i) \quad \forall x_i \in \overline{\Omega}_i(\mathbf{u}_{-i}^*), \ \ \forall i \tag{40}$$

where $\overline{\Omega}_i(\mathbf{u}_{-i}^*)$ is the projection set from $\overline{\Omega}$, (39),

$$\overline{\Omega}_i(\mathbf{u}_{-i}^*) = \left\{ x_i \in \Omega_i \,\middle|\, \sum_{j \neq i}^{m} u_j^* + x_i \leqslant P_0 \right\}. \tag{41}$$

6. OSNR game: a hierarchical decomposition

In this section we consider the coupled OSNR NG with cost functions J_i, (35), and coupled constraint (39). The existence of an optimum NE solution is guaranteed by Theorem 1, since the cost functions J_i, (35), are strictly convex, on a compact and convex action set $\overline{\Omega}$. Each channel's cost function J_i (35), (37) is coupled to all other channels' actions due to the presence of X_{-i}. Moreover, due to the coupled constraint (39), solving directly for an NE solution of this game requires coordination among possibly all channels and is impractical. We apply the results in Section 4 as a natural way to obtain a hierarchical decomposition.

Based on Theorem 3 applied to the OSNR game, with separable NG-cost function and linear constraints, we have the following decomposition result.

Corollary 1. *Consider the coupled OSNR NG with cost functions* $J_i(\mathbf{u}_{-i}, u_i)$, (35), (37), *subject to the linear constraint* (39). *Then the dual cost function* $D(\mu)$,

$$D(\mu) := \left[\min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) \right] \Bigg|_{\mathbf{x}=\mathbf{u}},$$

can be decomposed as

$$D(\mu) = \sum_{i=1}^{m} \overline{L}_i(\mathbf{u}_{-i}^*, u_i^*(\mu), \mu) + \sum_{i=1}^{m} \mu(\mathbf{e}^{\mathrm{T}} \mathbf{u}_{-i}^* - P_0), \tag{42}$$

where $\mathbf{u}^*(\mu) = [u_i^*(\mu)]$ *is an NE solution to the NG with cost functions* $\overline{L}_i$ (43) *and no coupled constraints.*

$$\overline{L}_i(\mathbf{u}_{-i}, x_i, \mu) = J_i(\mathbf{u}_{-i}, x_i) + \mu x_i, \quad i \in \mathscr{M}. \tag{43}$$

Proof. Since each cost function J_i, (35), is convex in u_i and the constraints are linear, Theorem 3 and (26), (27) hold. We rewrite the linear constraint (39) in a two-argument form:

$$g(\mathbf{u}_{-i}, x_i) = \mathbf{e}^{\mathrm{T}} \mathbf{u}_{-i} + x_i - P_0, \tag{44}$$

where $\mathbf{e} = [1, \ldots, 1]^{\mathrm{T}}$ is the $(m-1) \times 1$ all ones vector. Using (44) we see that L_i, (27), is given here as

$$L_i(\mathbf{u}_{-i}, x_i, \mu) = J_i(\mathbf{u}_{-i}, x_i) + \mu x_i + \mu(\mathbf{e}^{\mathrm{T}} \mathbf{u}_{-i} - P_0) \tag{45}$$

and (26) holds for L_i, (45). Recall that in (26) we minimize first with respect to x_i on the right-hand side, and then solve for a fixed-point solution. From (45) we see that only the first two terms depend on x_i. Hence, substituting for $L_i(\mathbf{u}_{-i}, x_i, \mu)$, (45), on the right-hand side of (26) and isolating the terms that are independent of x_i yields

$$D(\mu) = \sum_{i=1}^{m} \min_{x_i \in \Omega_i} \overline{L}_i(\mathbf{u}_{-i}, x_i, \mu) \Bigg|_{x_i = u_i} + \sum_{i=1}^{m} \mu(\mathbf{e}^{\mathrm{T}} \mathbf{u}_{-i} - P_0)$$

with $\overline{L}_i$ defined as in (43). A fixed-point solution $\mathbf{u}^* = [u_i^*]$ to the set of m optimizations on the right-hand side of the foregoing is an NE solution to the NG with cost functions, $\overline{L}_i$, (43), and the last part of the claim follows. □

Corollary 1 leads to a hierarchical decomposition into a lower-level modified NG with cost functions $\overline{L}_i$ (43), and a higher-level optimization problem used for coordination. This decomposition is computationally simpler as shown below. For a given price μ, the lower-level game admits a closed-form explicit solution. Specifically using (43), (35) we see that $\overline{L}_i$ satisfies

$$\overline{L}_i(\mathbf{u}_{-i}, x_i, \mu) = \overline{J}_i(\mathbf{u}_{-i}, x_i)$$

for $\alpha_i = \mu$, $\forall i$, where $\overline{J}_i$ is defined in (38). Therefore, for each given μ, the NE solution $\mathbf{u}^*(\mu)$ to the lower-level game with cost $\overline{L}_i$ is unique and can be obtained from Proposition 2 as

$$\mathbf{u}^*(\mu) = \widetilde{\mathbf{\Gamma}}^{-1} \left(\frac{1}{\mu} \mathbf{b}_0 - \mathbf{n}_0 \right), \tag{46}$$

where $\mathbf{b}_0 = [a_i \beta_i]$, $\mathbf{n}_0 = [n_{0i}]$. Next we propose a recursive hierarchical algorithm based on the explicit solution (46) and on price coordination at the higher level. By Theorem 2 applied to the coupled OSNR game J_i, (35), (39), $(\mathbf{u}^*, \mu^*)$ is an optimal NE solution–Lagrange multiplier pair if and only if $\mathbf{u}^*$ is NG feasible,

$$\sum_{i=1}^{m} u_i^*(\mu) \leqslant P_0, \quad u_i^* \in \Omega_i, \ \ i \in \mathscr{M}, \tag{47}$$

$\mu^* \geqslant 0$, $\mu^*(\sum_{i=1}^{m} \mathbf{u}_i^* - P_0) = 0$ (slackness condition) and the Lagrangian optimality condition

$$\mathbf{u}^* = \left[\arg \min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu^*) \right] \Bigg|_{\mathbf{x}=\mathbf{u}} \tag{48}$$

holds. By Corollary 1 and (42), we see that $\mathbf{u}^*(\mu)$ solving (48) can be found as an NE solution to the modified NG $\overline{L}_i$ (43), with no coupled constraints. For every given price μ, this NE solution $\mathbf{u}^*(\mu)$ is unique as in (46). Furthermore, from (46) it is seen that all components of $\mathbf{u}^*(\mu)$ decrease with μ. We can exploit the linear constraint and adjust the price μ to satisfy the slackness condition. Instead of maximizing $D(\mu)$, the optimal price μ^* can be obtained such that the slackness condition holds, i.e., as the point of interception between the curve representing total power, $u_{\mathrm{T}}^*(\mu)=\sum_{i=1}^{m} u_i^*(\mu)$, with the level P_0 (Fig. 1). This method has the interpretation of a coordination mechanism. The link as the coordinator sets the price at the optimal value μ^*. The channels respond by adjusting their power levels to $u_i^*(\mu^*)$ that minimizes their own cost. A hierarchical adjustment algorithm is proposed for both coordinating link price (higher level) and channel powers (lower level).

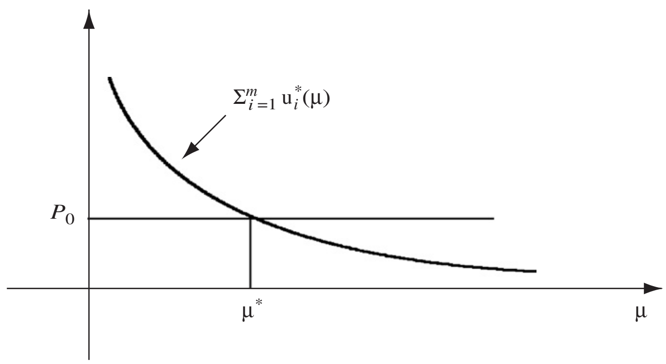

Fig. 1. Graphical plot of total power versus price.

6.1. Link algorithm

For every K iterations of the channel algorithm, the new link price μ is computed based on the received total power for all channels in the link $u_{\mathrm{T}}(K) = \sum_{j=1} u_j(K)$ as

$$\mu(k+1) = [\mu(k) + \eta(u_{\mathrm{T}}(K) - P_0)]^+, \tag{49}$$

where η is the step size and $[z]^+ = \max\{z, 0\}$.

This simple price update based on Fig. 1 requires only measurement of total power. Moreover, it corresponds to a gradient descent technique if link price is adjusted slower than channel powers. At the higher level, $\mu(k)$ acts as a coordination signal that aligns individual optimality with the system constraint, (39) or (47). This is the new price given to the channels, who repeat K iterations of the following algorithm.

6.2. Channel algorithm

Based on a price $\mu(k)$ from the link, the optimal channel power vector $\mathbf{u}^*(\mu(k))$ can be found explicitly as in (46). This requires global centralized information. However, the following iterative update algorithm can be used:

$$u_i(n+1) = \frac{\beta_i}{\mu(k)} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(n)} - \Gamma_{i,i}\right) u_i(n). \tag{50}$$

This is a decentralized algorithm, since the only information fed back is the individual channel $OSNR_i$, that can be measured in

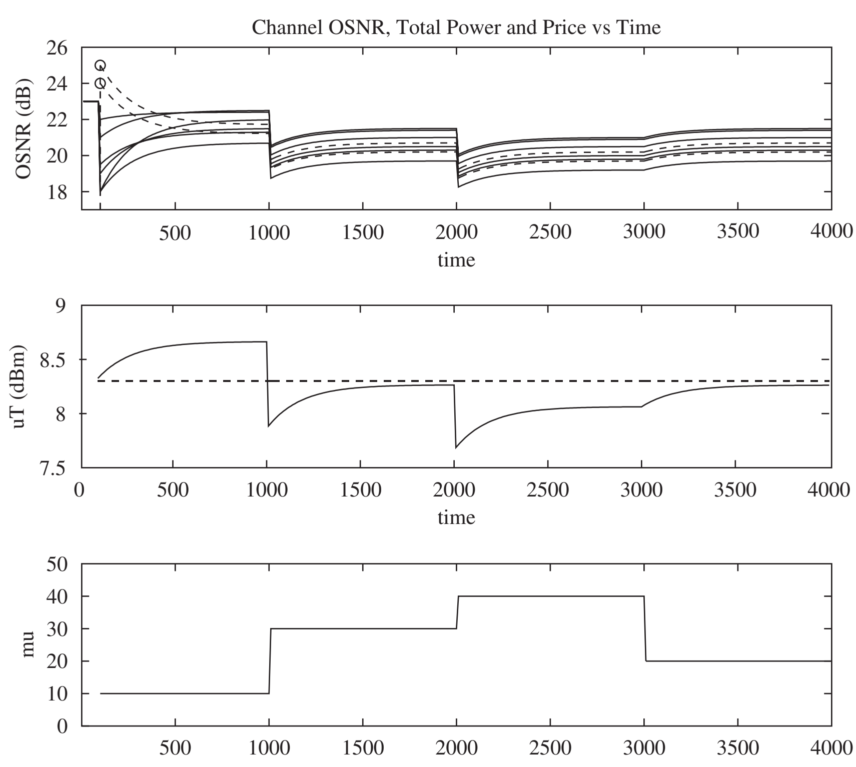

Fig. 2. OSNR, total power and price evolution.

real time, and the channel "gain", $\Gamma_{i,i}$. In Pavel (2006b) it was proved that for fixed μ this algorithm converges to the optimal NE solution (46).

Even if the optimal solution is coupled as in (46), it can be iteratively computed by using the decentralized algorithm (50). Thus, individual channels do not have to coordinate with other channels at the lower-level game. We note that the approach in this paper is different from the very recent results in Pan and Pavel (2005) where the capacity constraint was considered indirectly by modifying each channel cost function. The results in Pan and Pavel (2005) do not allow for an analytically tractable NE solution and developing decentralized algorithms is not immediate.

Our decomposition approach is similar to the one used for optimization flow control that relies on utilities being decoupled and system cost function being separable (Low & Lapsley, 1999). The problem in our case is not separable in a standard sense, but is a genuine coupled NG with coupled constraints. However, we exploited the fact that this coupled OSNR game is separable in an NG sense.

7. Numerical example

A MATLAB simulation was used for a link with 10 amplified spans and eight channels. Each optical amplifier has a parabolic gain profile and the link has a total power constraint of 8.3 dBm. Assume initially that only the first six channels were present with powers set for equal OSNR. At $t = 100$ two new channels 7 and 8 are added. The OSNR for existing channels has a sudden drop due to the extra two channels in the link (see Fig. 2). If the channel algorithm is used to adjust all powers ($\beta_i = 1$, $a_i = \Gamma_{i,i}$ and $\mu = 10$), OSNR converges to new steady-state values, but total power limit is exceeded (Fig. 2). For every $K = 100$ iterations (1000 time steps in Fig. 2), the link adjusts the price μ via the link algorithm and channels readjust their powers. After a few link iterations, the total power satisfies the constraint, while the OSNR levels are slightly below than that during the first interval.

8. Conclusions

We extended duality results from a standard optimization to a noncooperative (Nash) game framework. We showed that an NG with coupled constraints can be solved by solving a constrained optimization of the NG-game cost function with respect to the second argument, and fixed-point solution. We exploited the separability of the NG-cost function with respect to the second argument, and extended duality and hierarchical decomposition in an NG sense. In the second part of the paper we applied these theoretical results to an NG with coupled constraints in optical networks. The duality approach offered a natural way to hierarchically decompose the coupled OSNR NG into a lower-level NG, with no coupled constraints, and a higher-level link problem. Moreover the lower-level NG is analytically tractable and its solution can be found iteratively, decentralized with respect to channels.

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