Automatica 45 (2009) 1319–1325

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A stability analysis with time-delay of primal–dual power control in optical links ☆

Nem Stefanovic *, Lacra Pavel

University of Toronto, Toronto, Ontario, M5S 3G4, Canada

ARTICLE INFO

Article history:
Received 4 July 2008
Received in revised form
14 October 2008
Accepted 9 January 2009
Available online 9 March 2009

Keywords:
Optical fibre networks
Time-delay
Stability analysis
Nonlinear systems
Singular perturbation method
Singular perturbations

ABSTRACT

Primal-dual power control algorithms in optical networks must be designed properly to remain stable in the presence of time-delays. Control algorithms at the signal transmitters continuously update their channel powers in response to dynamic pricing information from the network links. The control algorithms optimize the optical signal-to-noise ratios (OSNRs) of the channels in a distributive and non-cooperative manner. We consider the case of a single link with multiple channels and a constant time-delay. These results also apply to single sink general network configurations if the time-delay represents the longest round-trip time in the network. We derive sufficient conditions for the stability of the closed loop system based on a tunable parameter in the control algorithm. The work combines singular perturbation theory with Lyapunov–Razumikhin time-delay techniques. We verify our results through simulations based on realistic network parameters.

1. Introduction

The Internet relies on optical communication for high bandwidth transmissions. No other transmission medium provides comparable data throughput rates with no external interference. Optical fibers are laid out over vast surfaces on the Earth spanning oceans and continents.

The high-level operation of optical networks is described as follows. Signal channels are multiplexed together using wavelength division multiplexers. The resultant beam of light is amplified by optical amplifiers every 100 km to compensate for fiber losses. An optical amplifier and a fiber cable constitute an optical fiber span. The trade-off for increased signal powers by optical amplifiers is the introduction of self-generated noise into the channels called amplified spontaneous emission (ASE). The amplifiers also cause channel interference via cross-gain modulation.

The main concern in optical communications is the accurate delivery of information across a network. This is accomplished by maximizing the optical signal-to-noise ratios (OSNRs) of the channels. Typically, OSNR optimization involves static link models with conservative parameter selections to satisfy tolerance margins (Forghieri, Tkach, & Favin, 1998). A heuristic algorithm is derived in Chraplyvy, Nagel, and Tkach (1992) for OSNR equalization without a convergence analysis. A non-cooperative game formulation for the OSNR optimization problem is presented in Pavel (2004). This approach abstracts the channels in the network as players in a game, where an increase in the signal power (increase in the OSNR) of a channel causes increases in noise (decrease in OSNR) in the other channels. The work in Pavel (2004) also devises a network level power control algorithm at the signal sources to ensure the channel powers converge to the unique Nash equilibrium point. Dynamic link pricing is introduced in Pavel (2007) that augments the primal control algorithm of Pavel (2004) to create a primal–dual control algorithm.

Signal propagation delay can not be ignored in optical networks. There is a 1 ms round-trip propagation delay for signals over a 100 km optical span. For such optical spans cascaded in series, the time-delay may be tens of milliseconds long. Stability can not be assumed given a sufficiently large time-delay and an algorithmic update period similar to the duration of the time-delay. The effects of time-delay in primal–dual control strategies have been studied in literature. A series of papers (Paganini, Wang, Doyle, & Low, 2005; Paganini, Wang, & Doyle, 2003; Wang & Paganini, 2002, 2003, 2004) utilize Lyapunov analysis techniques and timescale decoupling to analyze time-delays in nonlinear communication systems. However, the network is restricted to a

☆ A short version of this work appears in [Stefanovic, N., Pavel, L., (2008). Primal-dual power control of optical networks with time-delay, In *Proceedings of the 17th IFAC World Congress*, (pp. 2901–2906). The material in this paper was partially presented at the 17th IFAC World Congress July 6–11, 2008 Seoul, South Korea Control of Networks Session MoC14.3 Monday July 7 "Primal-Dual Power Control of Optical Networks with Time-Delay". This paper was recommended for publication in a revised form by Associate Editor Maria Elena Valcher under the direction of Editor Roberto Tempo.

* Corresponding author. Tel.: +1 416 274 8265; fax: +1 416 978 0804.
E-mail addresses: nem@control.toronto.edu (N. Stefanovic), pavel@control.toronto.edu (L. Pavel).

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.01.005

single source single link case, and the utility function is decoupled with respect to the state. A paper by Wen and Arcak (2004) uses a passivity approach and Lyapunov techniques to present a unifying framework for several flow control schemes. The control schemes are similar to optical network control schemes due to the gradient-like positive projection dynamics and the use of primal–dual algorithms. However, the utility function is decoupled. The survey paper (Liu, Basar, & Srikant, 2003) analyzes a positive projection gradient algorithm. The work in Ait-Hellal and Altman (2003) studies the stability of ABR congestion control. The work studies nonlinear time-delay equations. Heuristic approximations and the boundary-layer approach for time-scale decoupling are used. Also, the slowly varying variable is assumed to be fixed. We apply a more rigorous theoretical analysis. The work of Mazenc and Niculescu (2003) is a good reference for the application of Lyapunov analysis to time-delayed systems. Finally, Alpcan and Basar (2003) considers fixed heterogeneous delays in the network using Lyapunov stability theory. The work herein uses the Lyapunov–Razumikhin approach, as in Gu, Kharitonov, and Chen (2003), to study stability in the presence of time-delays. We couple this technique with the singular-perturbation approach (Khalil, 2002).

We study the conditions under which stability is ensured for power control in optical networks with time-delay. The analysis is based on the OSNR model and the network level control algorithms presented in Pavel (2004, 2007), which do not consider time-delay. Stability conditions for the primal algorithm of Pavel (2004) in the presence of time-delays are derived in Stefanovic and Pavel (2009) using frequency domain techniques. The work herein augments the primal control algorithm by including dynamic pricing (Pavel, 2007) at the links with time-delays. The pricing term, μ, is adjusted to produce a total link power below P_0, the threshold for limiting nonlinear effects (Agrawal, 1997). Due to the nonlinear nature of the primal–dual algorithm, we can no longer apply a frequency based stability analysis. We simplify the network model by assuming a single link with multiple signal channels. However, the work also applies to single sink general network topologies if we assume the time-delay represents the worst case round-trip time in the network. We apply singular perturbation theory (Khalil, 2002) to exploit the time-decoupling between the source algorithms and the channel pricing algorithm. Since singular perturbation theory does not inherently handle time-delays, we augment the theory using Lyapunov–Razumikhin techniques (Gu et al., 2003). Our design parameter is a tunable gain at the sources. A short version of the work here appears in Stefanovic and Pavel (2008) which uses added assumptions that we relax herein. We also expand on the work in Stefanovic and Pavel (2008) to include a complete proof for the main theorem as well as the proofs of the ancillary lemmas that were omitted.

The paper is organized as follows. Section 2 reviews the OSNR model and control algorithms derived in Pavel (2004) and introduces the link algorithm (Pavel, 2007). Section 3 introduces the tuning parameter and presents a continuous-time closed loop system. The following section introduces time-delays into the closed loop system. Section 5 presents the main stability result and its corresponding proof. Section 6 presents the simulations. The last section gives the conclusions and future work. Appendix A gives the Lemma proofs.

2. Review of OSNR model and control algorithms

We begin by reviewing the OSNR model. The discrete-time control algorithm from Pavel (2004) is presented next. We then introduce the link control law from Pavel (2007).

2.1. OSNR model

Consider a link composed of N spans that include one optical amplifier per span. A set of channels, $M = \{1, \ldots, m\}$, (intensity modulated wavelengths) are multiplexed together and transmitted across the link. We denote by u_i, s_i, and n_i, the optical input power for channel i at the transmitter (Tx), the output signal at the receiver (Rx), and the output noise at Rx, respectively. The Optical Signal-to-Noise Ratio (OSNR) for any channel, $i \in M$, is defined as $\text{OSNR}_i = s_i/n_i$.

The following provides the framework for modeling OSNR in a single link. An optical span is composed of an optical amplifier (OA) with channel dependent gain, G_i and an optical fiber with wavelength independent loss coefficient, L_k. The amplifiers have the same spectral shape and are operated in automatic power control (APC) mode with total power targets P_0. The OA introduces ASE noise power, denoted by $ASE_{k,i}$. The following lemma, adjusted from Pavel (2004), describes the OSNR model for an optical link.

Lemma 1. *The OSNR for the ith channel is given as*

$$\text{OSNR}_i = \frac{u_i}{n_{0,i} + \sum_{j \in M} \Gamma_{i,j} u_j} \tag{1}$$

where the elements $\Gamma_{i,j}$ of the full $(n \times n)$ system matrix Γ, are defined as

$$\Gamma_{i,j} = \sum_{k=1}^{N} \frac{G_j^k}{G_i^k} \frac{ASE_{k,i}}{P_0}$$

and $n_{0,i}$ is the noise optical power at transmitter (Tx) for the ith channel.

With Lemma 1 presented, we can next introduce the channel and link control laws presented in Pavel (2004, 2007).

2.2. Channel algorithm

Given the OSNR model in Lemma 1, a non-cooperative game between channels was defined in Pavel (2004). The objective of each channel (player) is to maximize its utility

$$U_i = ln \left(1 + a_i \frac{\text{OSNR}_i}{1 - \Gamma_{i,i}\text{OSNR}_i} \right)$$

where a_i is a channel dependent design parameter. A greater utility implies greater OSNR values which further implies a lower bit error rate in the optical link. Each channel adjusts its power towards this goal in the presence of other channels. The game settles at an equilibrium when no channel can improve its utility unilaterally; the equilibrium of the game being a Nash equilibrium. By converging to the Nash equilibrium the distributed system will achieve optimal OSNR values for its channels without centralized control. A full Nash game solution was presented in Pavel (2004). This work was then modified in Pavel (2007) to include an adjustable cost parameter, μ, that reflects the network channel utilization. The parameter μ is the channel price, which is adjusted to ensure the total link power is below the threshold for nonlinear effects, P_0 (Agrawal, 1997). The OSNR game admits a unique Nash equilibrium, u^*, which is the solution of

$$a_i u_i^* + \sum_{j \neq i} \Gamma_{i,j} u_j^* = \frac{a_i \beta_i}{\mu(\bar{k})} - n_{0,i} \quad \forall i. \tag{2}$$

Based on this solution an iterative network level control algorithm was proposed in Pavel (2004) to control the network (1) at the sources

$$u_i(k+1) = \frac{\beta_i}{\mu(\bar{k})} - \frac{1}{a_i} \left(\frac{1}{\text{OSNR}_i(k)} - \Gamma_{i,i} \right) u_i(k) \tag{3}$$

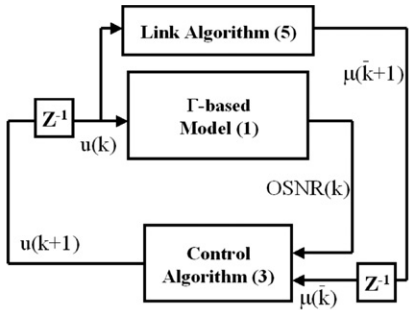

Fig. 1. Discrete-time control algorithm (3) and link algorithm (5) applied to OSNR system (1).

where β_i and a_i are design parameters and k is the iteration time step. The control algorithm (3) converges to the Nash equilibrium (2) if

$$a_i > \sum_{j\neq i} \Gamma_{i,j}. \tag{4}$$

2.3. Link algorithm

The link algorithm is computed every K iterations of the control algorithm (3). The value K is a large number such that the link algorithm is updated infrequently with respect to the control algorithm (ie. $K = 200$). The link algorithm (Pavel, 2007), without the $[\cdot]^+$ notation as done in Liu et al. (2003), is:

$$\mu(\bar{k}+1) = \mu(\bar{k}) + \eta\left(\sum_{j=1} u_j(K) - P_0\right) \tag{5}$$

where $\mu > 0$ is the channel price and $\eta > 0$ is the step-size. The variable $\bar{k}$ represents the discrete-time variable which is on a different time-scale than the control algorithm (3). Notice that $\sum_{j=1} u_j(K)$ is the received total power for all channels in the link. This is easily measured in practice as a real-time value. Moreover, under the assumption of stationary channel powers, which can be made given that the link algorithm updates very slowly, (5) corresponds to a gradient descent algorithm. The combined channel-link algorithm converges to the optimal NE channel power and price (u^*, μ^*) (Pavel, 2007). A slow time-scale for the link price update allows the fast time-scale variables, u_j, to settle at constant values before a new channel price, μ, is computed. Thus, the channel algorithm and the link algorithm are decoupled in time.

Fig. 1 depicts the control algorithm (3) and the link algorithm (5) acting on the OSNR system (1).

3. Continuous-time closed loop system

We generalize the control algorithm (3) by introducing a control gain, $\rho_i > 0$, for each channel i. We convert the control and link algorithm (3) and (5) into the continuous-time domain. We then formulate the closed loop system by applying the primal–dual controller to the OSNR model (1).

We introduce a tuning parameter at each source, ρ_i, into (3) to act as a weighting between the RHS of (3) and $u_i(k)$

$$u_i(k+1) - u_i(k) = \rho_i\left\{\frac{\beta_i}{\mu(\bar{k})} - \frac{1}{a_i}\left(\frac{1}{\mathrm{OSNR}_i(k)} - \Gamma_{i,i} + a_i\right)u_i(k)\right\}. \tag{6}$$

Note that $\rho_i = 1$ gives (3). Approximating the LHS of (6) by $\frac{\mathrm{d}u_i(t)}{\mathrm{d}t_f}$, where t_f denotes the "fast" time variable, we write the continuous-time version of the control algorithm (6) as

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t_f} = \rho_i\left\{\frac{\beta_i}{\mu(\bar{k})} - \frac{1}{a_i}\left(\frac{1}{\mathrm{OSNR}_i(t)} - \Gamma_{i,i} + a_i\right)u_i(t)\right\}. \tag{7}$$

Substituting (1) into (7), yields the closed loop system

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t_f} = \rho_i\left\{\frac{\beta_i}{\mu(\bar{k})} - \frac{n_{0,i} + \sum_{j\in M}\Gamma_{i,j}u_j}{a_i} + \left(\frac{\Gamma_{i,i}}{a_i} - 1\right)u_i\right\}. \tag{8}$$

Similarly, the dual control law (5) is rewritten in the continuous-time domain as

$$\dot{\mu}(t) = \eta\left(\sum_{j=1} u_j(t) - P_0\right). \tag{9}$$

We relate the link algorithm time variable, t, to t_f according to the relation $t_f = tK$. Let $\varepsilon = \frac{1}{K}$, such that $\varepsilon t_f = t$. The role of ε is crucial, since we will see later that stability of the closed-loop system is assured if ε is sufficiently small. A smaller ε value produces a greater time-scale decoupling between the control and link algorithms. From (8) and (9), along with

$$\frac{\mathrm{d}u_i}{\mathrm{d}t_f} = \frac{\mathrm{d}u_i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t_f} = \varepsilon\frac{\mathrm{d}u_i}{\mathrm{d}t} \tag{10}$$

the closed loop, time-scale decoupled system is given in the standard singular perturbation form as

$$\dot{\mu}(t) = \eta\left(\sum_{j=1} u_j(t) - P_0\right) \tag{11}$$

$$\varepsilon\dot{u}_i = \rho_i\left\{\frac{\beta_i}{\mu(t)} - \frac{n_{0,i} + \sum_{j}\Gamma_{i,j}u_j}{a_i} + \left(\frac{\Gamma_{i,i}}{a_i} - 1\right)u_i\right\}. \tag{12}$$

The equilibrium point, (u^*, μ^*), of the closed loop system (12) and (11) is obtained by setting the dynamics equal to 0. Eq. (12) gives the unique Nash equilibrium u^* with components u_i^* as in (2), where $\mu(\bar{k}) = \mu^*$. From (11) we get

$$\sum_j u_j^* = P_0. \tag{13}$$

Shifting (12) and (11) around the equilibrium point (2) and (13), and using the change of variables $\tilde{u}_i = u_i - u_i^*$ and $\tilde{\mu} = \mu - \mu^*$, we obtain the closed loop system

$$\dot{\mu}(t) = \eta\sum_j u_j(t)$$

$$\varepsilon\dot{u}_i(t) = \rho_i\left\{\frac{-\mu\beta_i}{\mu^*(\mu+\mu^*)} - \sum_j\frac{\tilde{\Gamma}_{i,j}}{a_i}u_j(t)\right\} \tag{14}$$

where for simplicity we denote by (u_i, μ) the shifted variables $(\tilde{u}_i, \tilde{\mu})$. The value $\tilde{\Gamma}_{i,j}$ is defined as

$$\tilde{\Gamma}_{i,j} = \begin{cases} a_i, & i = j \\ \Gamma_{i,j}, & i \neq j. \end{cases} \tag{15}$$

4. Time-delay system

The closed loop system (14) does not take time-delay into account. Forward time-delay, $\tau_f \geq 0$, occurs from the channel sources u_j to the outputs OSNR_i at the end of the link. Similarly, the backward time-delay, $\tau_b \geq 0$, occurs from the outputs OSNR_i back to channel sources u_i. We denote by $\tau = \tau_f + \tau_b$ the total round

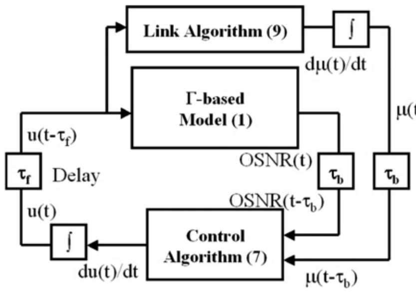

Fig. 2. Continuous-time control algorithm (7) and link algorithm (9) applied to OSNR system (1) with time-delay.

trip delay. We explicitly introduce round-trip time-delay into (1) as

$$\mathrm{OSNR}_i(t) = \frac{u_i(t-\tau)}{n_{0,i} + \sum_{j\in M} \Gamma_{i,j} u_j(t-\tau)}. \tag{16}$$

Substituting (16) into (7), and using (10), gives

$$\varepsilon \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = \rho_i \left\{ \frac{\beta_i}{\mu(t)} - \frac{1}{a_i}\left(\frac{n_{0,i} + \sum_j \Gamma_{i,j} u_j(t-\tau)}{u_i(t-\tau)} - \Gamma_{i,i} + a_i \right) u_i(t) \right\}. \tag{17}$$

Since (17) derives from our control algorithm, we can modify (17) to eliminate $u_i(t-\tau)$ in the denominator by design. If we keep a record of past power inputs, $u_i(t-\tau)$, we can then appropriately replace $u_i(t)$ with $u_i(t-\tau)$ to obtain

$$\varepsilon \dot{u}_i(t) = \rho_i \left\{ \frac{\beta_i}{\mu(t-\tau_b)} - \frac{1}{a_i}\left(n_{0,i} + \sum_j \tilde{\Gamma}_{i,j} u_j(t-\tau) \right) \right\} \tag{18}$$

where $\tilde{\Gamma}_{i,j}$ is defined in (15). The backward propagation delay for μ is included in (18). We next add the link dynamics (9) along with a forward time-delay on the input powers

$$\dot{\mu}(t) = \eta \left(\sum_{j=1} u_j(t-\tau_f) - P_0 \right). \tag{19}$$

Thus, (18) and (19) represent the closed loop, time-delayed model. Fig. 2 shows the block diagram. Shifting (19) and (18) about the equilibrium point (u^*, μ^*) as defined in (13) and (2), and rewriting in matrix form gives

$$\dot{x}(t) = \eta 1_{row} z(t-\tau_f) \tag{20}$$

$$\varepsilon \dot{z}(t) = \rho \left\{ -\beta \frac{x(t-\tau_b)}{\mu^*(x(t-\tau_b)+\mu^*)} - \tilde{\Gamma}_a z(t-\tau) \right\} \tag{21}$$

where the shifted variables, $z_i = u_i - u_i^*$ and $x = \mu - \mu^*$, in vector form are denoted by z and x, $\tilde{\Gamma}_a$ is a matrix with elements $\frac{\tilde{\Gamma}_{i,j}}{a_i}$, $\tilde{\Gamma}_{i,j}$ defined as (15), β is a column matrix with elements β_i, 1_{row} is a row vector with all elements equal to 1, $\rho = \mathrm{diag}(\rho_i)$. Note that $\tilde{\Gamma}_a$ is invertible because of the diagonal dominance condition (4).

The closed loop system model (20) and (21) is more realistic and less assumptive than the model in Stefanovic and Pavel (2008). Here, we keep the price algorithm at the links rather than the sources. We also keep the natural time delay for the x variable in (21) rather than artificially increasing it.

5. Main result

We use the Lyapunov–Razumikhin theory and the singular perturbation approach to study the stability of the delayed closed loop system (20) and (21). The main result gives sufficient conditions for stability.

We first define $\mathcal{C}([-r,0],\mathfrak{R}^n)$ as the set of all continuous functions mapping $[-r,0]$ to $\mathfrak{R}^n$. We simplify the notation by letting $\mathcal{C} = \mathcal{C}([-r,0],\mathfrak{R}^n)$. We also define the continuous norm as $\|y_t\|_c = \max_{-r\le\theta\le 0}\|y(t+\theta)\|_2$, where $y_t \in \mathcal{C}$, represents the set of values, $y(q)$, for all $q \in [t-r,t]$. We use the following definition of stability from Gu et al. (2003),

Definition 1. Consider the general system

$$\dot{\bar{x}} = F(t, \bar{x}_t) \tag{22}$$

where $\bar{x}_t \in \mathcal{C}$ represents the set of values $\bar{x}(q)$ for all $q \in [t-r,t]$ and $F : \mathfrak{R}\times\mathcal{C} \to \mathfrak{R}^n$. We say (22) is stable if for any $t_0 \in \mathfrak{R}$ and any $\varepsilon > 0$, there exists a $\delta = \delta(t_0,\varepsilon) > 0$ such that $\|\bar{x}_{t_0}\|_c < \delta$ implies $\|\bar{x}(t)\|_2 < \varepsilon$ for $t \ge t_0$. System (22) is asymptotically stable if it is stable and $\lim_{t\to\infty}\bar{x}(t) = 0$.

We state the restricted Razumikhin theorem (Gu et al., 2003) (Proposition 5.1, pg. 149) that gives general conditions for asymptotic stability for time-delay systems based on Definition 1.

Theorem 1. *A time-delay system with maximum time-delay r is asymptotically stable if there exists a quadratic Lyapunov function V such that for some* $\psi > 0$ *and* $K > 0$*, it satisfies*

$$\psi\|y\|_2^2 \le V(y) \le K\|y\|_2^2$$

and its derivative along the system trajectory $\dot{V}(y(t))$ *satisfies*

$$\dot{V}(y(t)) \le -\psi\|y(t)\|_2^2$$

whenever

$$V(y(t+\xi)) \le pV(y(t)), \quad -r \le \xi \le 0$$

for some constant $p > 1$.

Theorem 1 presents a sufficient condition, namely the existence of a Lyapunov function endowed with special properties, ensuring the stability of a time-delay system. Theorem 1 is used to prove Lemma 2.

Re-write (20) and (21) as

$$\dot{x} = f(z(t-\tau_f)) \tag{23}$$

$$\varepsilon\dot{z} = g(x(t-\tau_b), z(t-\tau)) \tag{24}$$

where

$$f(z(t-\tau_f)) = \eta 1_{row} z(t-\tau_f) \tag{25}$$

$$g(x(t-\tau_b), z(t-\tau)) = \rho\left\{ -\beta \frac{x(t-\tau_b)}{\mu^*(x(t-\tau_b)+\mu^*)} - \tilde{\Gamma}_a z(t-\tau) \right\}. \tag{26}$$

We rewrite (23) and (24) into a more convenient analytical form using the co-ordinate shift

$$y(t) = z(t-\tau_f) - h(x(t)) \tag{27}$$

where

$$h(x) = \tilde{\Gamma}_a^{-1}\beta\left(\frac{-x}{\mu^*(x+\mu^*)} \right) \tag{28}$$

is the isolated root of (26). Substitute (27) into (23) to obtain

$$\dot{x} = f(y(t) + h(x(t))). \tag{29}$$