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Brief paper

An analysis of stability with time-delay of link level power control in optical networks[☆]

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ABSTRACT

We study the effects of time-delay on the stability of optical networks. A link level power control scheme adjusts the OSNR value of the signals toward channel OSNR optimization. We utilize the OSNR model from [Pavel, L. (2006). A noncooperative game approach to OSNR optimization in optical networks. *IEEE Transactions on Automatic Control*, *51*(5), 848–852] along with its game-theoretic based control algorithm. Time-delay is incorporated into the closed loop system, for the general network case where every link has a unique time-delay. We derive sufficient conditions for stability under arbitrary time-delays and network configurations. The results are verified via extensive simulations.

1. Introduction

Optical networks comprise the backbone of the Internet. Their reach extends between cities and continents. Signal channels are multiplexed together using wavelength division multiplexing (WDM) (Ramaswami & Sivarajan, 2002). Optical cross connects (OXC) route light signals through the links of the network. Power losses in the signals are compensated by optical amplifiers (OA) every 80–100 km along a link. However, optical amplifiers introduce self-generated noise into the signal channels called amplified spontaneous emission (ASE) (Agrawal, 1997).

Error-free transmission in optical communication networks depends on the optical signal-to-noise ratio (OSNR) at reception (Rx). In Pavel (2006) an OSNR optimization problem is formulated as a non-cooperative game between signal channels (players). As power in one signal increases, thereby increasing its OSNR, the noise in the remaining channels increases, thus decreasing their OSNRs. The work in Pavel (2006) also devises a network level power control algorithm at the sources that converges to a unique Nash equilibrium. Propagation delay in large-scale optical networks is not negligible, yet time-delay is not incorporated in the power control algorithms.

Research on time-delay in networks is an active area. However, there are few results for optical networks themselves (Pavel, 2004) and not for the case of network level control. The work in Paganini, Doyle, and Low (2001) and Paganini, Wang, Doyle, and Low (2005) presents congestion control algorithms based on a frequency domain analysis of time-delays. The main body of work by Paganini is extended in Wang and Paganini (2002, 2004) by utilizing Lyapunov techniques. However, the utility function is decoupled and the network is restricted to the single source single link case. A passivity framework for network flow control is developed in Wen and Arcak (2004) that also utilizes Lyapunov analysis. This work generalizes the work in Paganini et al. (2001) as well as others. A decoupled utility function is assumed. Additional papers by the authors include Fan, Alpcan, Arcak, Wen, and Basar (2006) and Fan, Arcak, and Wen (2004) that study a passivity approach to game-theoretic CDMA power control and the robustness of network flow control with time-delay. A local stability condition with time-delay is derived using frequency domain analysis for the linearized system in Liu, Basar, and Srikant (2003). This paper provides an excellent synopsis of past research. The work in Vinnicombe (2002) utilizes a frequency domain analysis with slight generalizations to the delay models in Paganini et al. (2001). Finally, based on Lyapunov's stability theory, Alpcan and Basar (2003) considers fixed heterogeneous delays.

We are primarily interested in the frequency domain analysis herein. Optical networks have coupled utility functions and

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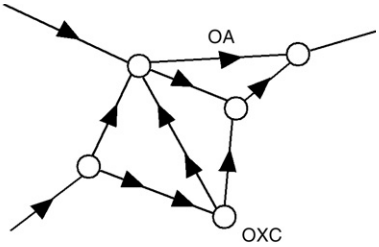

Fig. 1. Mesh optical network.

decentralized control laws. We study the effects of time-delay on the OSNR model and the network level control algorithm presented in Pavel (2006). A time-delay can destabilize the network control algorithm. We derive a theorem that gives sufficient stability conditions for any time-delay. The compensating factors are tunable gain parameters, μ_i. The work herein is an expanded version of the work in Stefanovic and Pavel (2007).

The paper is organized as follows. Section 2 reviews the OSNR model and network level control algorithm (Pavel, 2006). Section 3 presents the closed loop time-delayed system. The next section presents the main theorem and its proof. Section 5 shows simulation results. The final section gives conclusions. The Appendix includes proofs of ancillary results.

2. Review of OSNR model and control algorithms

Consider an optical network that is defined by a set of optical links, $\mathcal{L} = \{1, \ldots, \mathcal{L}\}$, that connect to optical nodes. The optical nodes add, drop, or reroute channels. A link, ℓ, is composed of N_ℓ spans that include one optical amplifier per span. Fig. 1 depicts a mesh optical network. A set of channels, $M = \{1, \ldots, n\}$, (intensity modulated wavelengths) are multiplexed together and transmitted across a link. We denote by M_ℓ the set of channels transmitted over link ℓ, $\ell \in \mathcal{L}$. Also, we denote by R_i, $i \in M$, the set of links from source (Tx) to destination (Rx), that channel i uses in its optical path. We denote by u_i, s_i, and n_i, the optical input power for channel i at Tx, the output signal at Rx, and the output noise at Rx, respectively. The Optical Signal-to-Noise Ratio (OSNR) for any channel, $i \in M$, is defined as $OSNR_i = s_i/n_i$.

The following provides the framework for modeling OSNR in a general optical network. An optical span is composed of an optical amplifier (OA) with channel dependent gain, $G_{\ell,k,i}$ and an optical fiber with wavelength independent loss coefficient, $L_{\ell,k}$. The OA introduces ASE noise power, denoted by $ASE_{\ell,k,i}$. The optical span transmission, for the kth span on the ℓth link, and the ith channel, is $h_{\ell,k,i} = G_{\ell,k,i}L_{\ell,k}$, $\forall k = 1, \ldots, N_\ell$. The ℓth link transmission is $T_{\ell,i} = \prod_{q=1}^{N_\ell} h_{\ell,q,i}$, $\forall \ell$.

Assumptions: (i) ASE noise power is small compared with the signal power and it does not contribute to gain saturation (Agrawal, 1997, Chapter 8.5.3). (ii) All spans in a link have equal length. (iii) All amplifiers in a link have the same spectral shape and are operated in automatic power control (APC) mode, with the same total power target $P_{0,\ell}$. Thus, $G_{\ell,k,i} = G_{\ell,i}$ $\forall k$.

Assumptions (i)–(iii) are all reasonable with respect to physical optical networks. Assumption (i) is justified because ASE is on the order of 20 dB smaller than the signal power (Agrawal, 1997, Chapter 8.6.4). Assumption (ii) models the OA spacing of 100 km along uniformly designed fiber optic lines (Agrawal, 1997). The operation of the OAs in APC mode is common. This ensures a common power output for each optical span and compensates for variations in fiber-span loss across a link (Agrawal, 1997, Chapter 8.5.3). Assumption (iii) is justified if assumption (ii) holds. The output power is chosen to be below the threshold for nonlinear effects (Agrawal, 1997, Chapter 2.6). The following lemma (Pavel, 2006) describes the OSNR model for optical networks.

Lemma 1. *Under assumptions* (i)–(iii), *the OSNR for the ith channel is given as*

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j\in M} \Gamma_{i,j} u_j} \tag{1}$$

where $\Gamma_{i,j}$, elements of the full $(n \times n)$ system matrix Γ, are

$$\Gamma_{i,j} = \sum_{\ell\in R_i\cap R_j} \sum_{k=1}^{N_\ell} \frac{G^k_{\ell,j}}{G^k_{\ell,i}} \left(\prod_{q=1}^{\ell-1} \frac{T_{q,j}}{T_{q,i}}\right) \frac{ASE_{\ell,k,i}}{P_{0,\ell}}$$

and $n_{0,i}$ is the noise optical power (Tx) for the ith channel.

Each channel (player) attempts to maximize its utility (Pavel, 2006)

$$U_i = \ln\left(1 + a_i \frac{OSNR_i}{1 - \Gamma_{i,i} OSNR_i}\right)$$

where a_i is a channel dependent design parameter. The game settles at an equilibrium when no channel can improve its utility unilaterally; the equilibrium of the game being a Nash equilibrium. A full Nash game solution was presented in Pavel (2006). Provided that the following condition holds

$$a_i > \sum_{j\neq i} \Gamma_{i,j} \tag{2}$$

it is shown that the OSNR game admits a unique Nash equilibrium, u^*, which is the solution of (3)

$$a_i u_i^* + \sum_{j\neq i} \Gamma_{i,j} u_j^* = \frac{a_i \beta_i}{\alpha_i} - n_{0,i} \quad \forall i. \tag{3}$$

Based on this solution an iterative network level control was proposed. We state the discrete-time algorithm to control the network (1) at the sources as outlined in Pavel (2006)

$$u_i(k+1) = \frac{\beta_i}{\alpha_i} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(k)} - \Gamma_{i,i}\right) u_i(k) \tag{4}$$

where α_i and β_i are design parameters. The control algorithm (4) converges to the Nash equilibrium (3) if (2) holds.

3. OSNR model and control algorithm with time-delay

In this section, we introduce time-delay into the OSNR model. First, we generalize the control algorithm (4) by subtracting $u_i(k)$ from each side of the equation and introducing a control gain, μ_i, $0 < \mu_i \leq 1$, for each channel i, as a multiple of the RHS. Note that $\mu_i = 1$ recovers (4). Approximating the LHS, $u_i(k+1) - u_i(k)$, by $\frac{\mathrm{d}u_i(t)}{\mathrm{d}t}$, we obtain

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = \mu_i \left\{\frac{\beta_i}{\alpha_i} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(t)} - \Gamma_{i,i} + a_i\right) u_i(t)\right\}. \tag{5}$$

Substituting (1) into (5), and using a change of variable to shift the equilibrium point u^* from (3) to the origin yields the closed loop system

$$\dot{u}_i(t) = -\frac{\mu_i}{a_i}\left\{\sum_{j\in M} \Gamma_{i,j} u_j(t) + (-\Gamma_{i,i} + a_i) u_i(t)\right\}. \tag{6}$$

The closed loop system (6) does not take time-delay into account. Forward time-delay occurs from the channel sources u_j to the OSNR outputs $OSNR_i$. Let $\tau_{i,j} \geq 0$ represent a time-delay from source j to OSNR output i. Let $\tau_{i,B}$ represent the backward time-delay from

$OSNR_i$ back to its associated source u_i. We introduce forward and backward time-delay into (1) as

$$OSNR_i(t)=\frac{u_i(t-\tau_{i,i}-\tau_{i,B})}{n_{0,i}+\sum\limits_{j\in M}\Gamma_{i,j}u_j(t-\tau_{i,j}-\tau_{i,B})}. \tag{7}$$

Substituting (7) into (5) gives

$$\dot u_i(t)=\mu_i\left\{\frac{\beta_i}{\alpha_i}-\frac{1}{a_i}\left(\frac{n_{0,i}+\sum\limits_{j\in M}\Gamma_{i,j}u_j(t-\tau_{i,j}-\tau_{i,B})}{u_i(t-\tau_{i,i}-\tau_{i,B})}-\Gamma_{i,i}+a_i\right)u_i(t)\right\}.$$

We consider that $u_i(t-\tau_{i,i}-\tau_{i,B})$, the delayed input power, could be stored in memory at each source. We eliminate $u_i(t-\tau_{i,i}-\tau_{i,B})$ by setting $u_i(t)=u_i(t-\tau_{i,i}-\tau_{i,B})$ appropriately in the algorithm and shifting about u_i^* (3),

$$\dot u_i=-\frac{\mu_i}{a_i}\sum_{j\in M}\Gamma_{i,j}u_j(t-\tau_{i,j}-\tau_{i,B})-\frac{\mu_i}{a_i}\left(a_i-\Gamma_{i,i}\right)u_i(t). \tag{8}$$

Notice that if the time-delays $\tau_{i,j}$ and $\tau_{i,B}$ in (8) were all equal to zero, we would get the no-delay system (6).

4. Main result

The main result gives sufficient conditions for the stability of the closed loop system (8). Unlike the work in Paganini et al. (2005), we have no symmetry in our system matrix Γ that we can exploit.

We first state some preliminary results (proofs in the Appendix). Denote by $\mathrm{diag}(v_i)$ the diagonal matrix with elements v_i, and by $\sigma(A)$ the set of eigenvalues of a matrix A. Define the general form of a retarded functional differential equation (Bellman & Cooke, 1963) as

$$\dot x(t)=f(t,x_t) \tag{9}$$

where $x(t)\in\Re$ and $f:\Re\times\mathcal{C}\to\Re^n$ where $\mathcal{C}$ is the set of continuous functions mapping $[-r,0]$ to $\Re^n$. Here, $x_t\in\mathcal{C}$, represents the set of values, $x(\xi)$, for all $\xi\in[t-r,t]$. Let f be linear and time invariant.

Proposition 1. *Consider a unity feedback system with loop transfer function* $L(s)=\mathrm{diag}(\frac{1}{s+\varepsilon_i})F(s)$ *of a linear retarded functional differential equation* (9), *where* $F(s)$ *is an* $n\times n$ *matrix and* $\varepsilon_i\in\Re$. *Suppose:*

(i) $F(s)$ *is analytic in* $\mathrm{Re}[s]>0$ *and bounded in* $\mathrm{Re}[s]\geq 0$.
(ii) $F(0)+\mathrm{diag}(\varepsilon_i)$ *has all of its eigenvalues in the open RHP. If* $\varepsilon_i\leq 0$ *for some* i *then* $F(-\varepsilon_i)$ *has all of its eigenvalues not equal to* 0.
(iii) $-1\notin\sigma(\bar\mu L(\mathrm{j}\omega))$ *for all* $\omega\neq 0$ *for all* $\bar\mu\in(0,1]$.

Then, the closed loop is stable.

Proposition 1 is a slight generalization of results in Paganini et al. (2005).

Lemma 2. *Let* $L_{i,i}(\mathrm{j}\omega)$ *be defined as*

$$L_{i,i}(\mathrm{j}\omega)=\mu_i\frac{\Gamma_{i,i}}{a_i}\frac{\mathrm{e}^{-(\tau_{i,i}+\tau_{i,B})\mathrm{j}\omega}}{\mathrm{j}\omega+\mu_i(1-\frac{\Gamma_{i,i}}{a_i})}. \tag{10}$$

Furthermore, define ω^* *as the frequency value that minimizes* $|\bar\mu L_{i,i}(\mathrm{j}\omega)-(-1)|$ *for any given* $\bar\mu\in(0,1]$. *Then,*

$$1-\bar\mu|L_{i,i}(\mathrm{j}\omega)|\leq\left|\bar\mu L_{i,i}(\mathrm{j}\omega^*)-(-1)\right|\quad\forall\omega\in[0,\omega^*] \tag{11}$$

or equivalently, $\forall\omega\in[0,\omega^*]$,

$$1-\frac{\Gamma_{i,i}}{a_i}\frac{\bar\mu\mu_i}{\sqrt{\omega^2+\mu_i^2(1-\Gamma_{i,i}/a_i)^2}}\leq\left|\bar\mu L_{i,i}(\mathrm{j}\omega^*)-(-1)\right|. \tag{12}$$

Proof of Lemma 2 is immediate based on the triangle inequality and the fact that the magnitude of $L_{i,i}(\mathrm{j}\omega)$ is monotone with respect to ω. From Lemma 2 we infer that (11) is satisfied for any phase in between 0 and $\angle L_{i,i}(\mathrm{j}\omega^*)$.

Lemma 3. *Let* $L_{i,i}(\mathrm{j}\omega)$ *be defined as in* (10) *and* $\tilde\omega=\frac{\pi}{2\tau_i}$, *where* $\tau_i=\tau_{i,i}+\tau_{i,B}$. *Then* $\tilde\omega$ *is guaranteed to satisfy* (11), *i.e.* $\tilde\omega\leq\omega^*$, *if*

$$\frac{0.68244}{(1-\frac{\Gamma_{i,i}}{a_i})\tau_i}\leq\mu_i.$$

We now state and prove the main result; part (i) gives a time-delay dependent stability condition based on the tuning parameters μ_i; part (ii) provides a conservative stability condition independent of time-delay.

Theorem 1. (i) *Provided that the system design parameters* a_i *are selected such that*

$$\sum_{j\neq i}\Gamma_{i,j}<a_i<\Gamma_{i,i}+\sum_j\Gamma_{i,j}\quad\forall i \tag{13}$$

$$\Gamma_{i,i}<a_i\quad\forall i \tag{14}$$

then the closed loop delayed system with the control algorithm (5) *and the delayed model* (7) *is stable if the tuning parameters* μ_i *are selected such that*

$$\frac{0.68244}{(1-\frac{\Gamma_{i,i}}{a_i})\tau_i}\leq\mu_i\quad\forall i \tag{15}$$

$$0<\mu_i<\frac{\pi}{2\tau_i(1-\frac{\Gamma_{i,i}}{a_i})\sqrt{\Gamma_{i,i}^2/(a_i-\sum\limits_j\Gamma_{i,j})^2-1}}\quad\forall i. \tag{16}$$

(ii) *The closed loop delayed system is stable for any time-delays if the parameters* a_i *are selected to satisfy the conservative stability condition:*

$$\Gamma_{i,i}+\sum_j\Gamma_{i,j}<a_i\quad\forall i. \tag{17}$$

Proof. In the frequency domain, the closed loop delayed system (8) becomes

$$U(s)=(I+L(s))^{-1}\mathrm{diag}\left(\frac{1}{s+\mu_i(1-\frac{\Gamma_{i,i}}{a_i})}\right)u(0) \tag{18}$$

where $U(s)$ is a vector of $U_i(s)$ elements with $U_i(s)$ the Laplace transform of $u_i(t)$. The loop transfer function, $L(s)=[L_{i,j}(s)]$, is an $(n\times n)$ matrix

$$L_{i,j}(s)=\mu_i\frac{\Gamma_{i,j}}{a_i}\frac{\mathrm{e}^{-(\tau_{i,j}+\tau_{i,B})s}}{s+\mu_i(1-\frac{\Gamma_{i,i}}{a_i})}. \tag{19}$$

We prove that (18) is stable by applying Proposition 1 to L(s) where we take F(s) with components $F_{i,j}(s)=\mu_i\Gamma_{i,j}\mathrm{e}^{-(\tau_{i,j}+\tau_{i,B})s}/a_i$. Let $\varepsilon_i=\mu_i(1-\frac{\Gamma_{i,i}}{a_i})$, with $\varepsilon_i>0$ $\forall i$ by (14). Condition (i) of Proposition 1 is immediately satisfied. To show condition (ii), we apply Gershgorin's Theorem (Varga, 2004) to $F(0)+\mathrm{diag}(\varepsilon_i)$. Note, based on condition (2) or the left side of (13) we have

$$\mu_i\frac{\Gamma_{i,i}}{a_i}+\mu_i-\mu_i\frac{\Gamma_{i,i}}{a_i}>\sum_{j\neq i}\mu_i\frac{\Gamma_{i,j}}{a_i}\quad\forall i$$

or equivalently, $\sum_{j\neq i}F_{i,j}(0)<|F_{i,i}(0)+\varepsilon_i|$, and (ii) follows. To show that condition (iii) is satisfied for $L(\mathrm{j}\omega)$, (19) for $s=\mathrm{j}\omega$, we apply

Gershgorin's Theorem (Varga, 2004) to show that no Gershgorin disc of $\bar{\mu}L(j\omega)$ contains the value -1 for any $\bar{\mu} \in (0, 1]$. This is ensured if for all $\bar{\mu} \in (0, 1]$

$$\frac{\bar{\mu}\mu_i}{\sqrt{\omega^2+\mu_i^2(1-\Gamma_{i,i}/a_i)^2}}\sum_{j\neq i}\frac{\Gamma_{i,j}}{a_i} < \left|\bar{\mu}L_{i,i}(j\omega)+1\right| \qquad (20)$$

$\forall i, \forall \omega \geq 0$ where $L_{i,i}(j\omega)$ is defined in (10). The RHS of (20) is the distance from the critical point, -1, to the Nyquist curve of $\bar{\mu}L_{i,i}(j\omega)$. From the LHS of (20),

$$\frac{\bar{\mu}\mu_i}{\sqrt{\omega^2+\mu_i^2(1-\Gamma_{i,i}/a_i)^2}}\sum_{j\neq i}\frac{\Gamma_{i,j}}{a_i} \leq \frac{1}{(1-\frac{\Gamma_{i,i}}{a_i})}\sum_{j\neq i}\frac{\Gamma_{i,j}}{a_i}$$

then if the following, (21), is satisfied, (20) will follow.

$$\frac{1}{(1-\frac{\Gamma_{i,i}}{a_i})}\sum_{j\neq i}\frac{\Gamma_{i,j}}{a_i} < \left|\bar{\mu}L_{i,i}(j\omega)+1\right| \quad \forall i, \forall\omega \geq 0. \qquad (21)$$

(i) Let ω^* be the value of ω that minimizes the RHS of (21). Then if the following holds,

$$\frac{1}{(1-\frac{\Gamma_{i,i}}{a_i})}\sum_{j\neq i}\frac{\Gamma_{i,j}}{a_i} < \left|\bar{\mu}L_{i,i}(j\omega^*)+1\right| \quad \forall i \qquad (22)$$

it follows that (21), hence (20) holds, so $-1 \notin \sigma(\bar{\mu}L(j\omega))$. We continue this approach and rewrite the RHS of (22) into a more useful form based on Lemmas 2 and 3. From Lemmas 2 and 3, we have that $\tilde{\omega} = \frac{\pi}{2\tau_i}$ satisfies (12), as long as (15) is satisfied. Next, (13), (15), and (16) guarantee that

$$\frac{1}{(1-\frac{\Gamma_{i,i}}{a_i})}\sum_{j\neq i}\frac{\Gamma_{i,j}}{a_i} < 1-\frac{\bar{\mu}\mu_i\Gamma_{i,i}}{a_i\sqrt{\pi^2/(4\tau_i^2)+\mu_i^2(1-\Gamma_{i,i}/a_i)^2}}. \qquad (23)$$

Thus by (23) and (12) it will follow that (22) and (21) hold, and hence $-1 \notin \sigma(\bar{\mu}L(j\omega))$. To show (23), note that setting $\bar{\mu} = 1$ gives the smallest value on the RHS of (23). After some manipulation, this can be rewritten as

$$\mu_i^2\left(1-\frac{\Gamma_{i,i}}{a_i}\right)^2\left[\frac{\Gamma_{i,i}^2}{\left(a_i-\sum_j\Gamma_{i,j}\right)^2}-1\right] < \frac{\pi^2}{4\tau_i^2}. \qquad (24)$$

By (13) the third term on the LHS of (24) is positive and (24) can be rearranged as in (16). Then (24) holds along with (23) and therefore $-1 \notin \sigma(\bar{\mu}L_{ii}(j\omega))$. By Proposition 1, the τ-dependent stability is proved.

(ii) We return to (21). On the RHS, we have $\left|\bar{\mu}L_{i,i}(j\omega)+1\right| \geq 1-\left|L_{i,i}(0)\right|$ based on $0 < \bar{\mu} \leq 1$ and the monotonicity of $|L_{i,i}(j\omega)|$ with respect to ω. Thus, $\left|\bar{\mu}L_{i,i}(j\omega)+1\right| \geq 1-\Gamma_{i,i}/(a_i-\Gamma_{i,i})$ and from condition (17) it follows that (21) holds, which implies (20) holds, and $-1 \notin \sigma(\bar{\mu}L(j\omega))$. Thus, part (ii) is proved.

Remark. Conditions (15) and (17) for a_i cover the entire valid range in (2). For large a_i, there is less preference for higher OSNR values and the Nash equilibrium becomes $u_i^* = \frac{\beta_i}{\alpha_i}$ for all i, which is decoupled. Thus, we want to keep a_i as small as possible.

Remark. It can easily be shown that both (15) and (16) can only be satisfied simultaneously if $\Gamma_{i,i}/\delta + \sum_j \Gamma_{i,j} \leq a_i$ where $\delta = \sqrt{\left(\frac{\pi}{2(0.68244)}\right)^2+1}$.

Table 1
Simulation parameters.

Parameter	Value	Units
OSNR target range	21–25	dB
OAs per link	10	
Wavelength spacing	1	nm
P_0	8.3	dBm
Span loss	10	dB
$n_{0,i}$	0.1	% signal

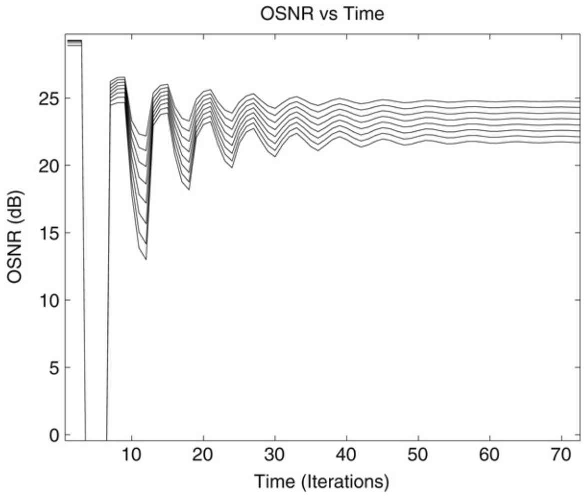

Fig. 2. Conservative case, time-delay 10 ms, $\mu_i = 1$.

5. Simulations and discussion

Table 1 lists the parameters common to all simulations. For the non-conservative case, we select $a_i = \frac{\Gamma_{i,i}}{\delta} + \sum_j \Gamma_{i,j}$. In the conservative case, we pick $a_i = 1.2 \times (\Gamma_{i,i} + \sum_j \Gamma_{i,j})$. We use a parabolic shape for the optical amplifier gain spectrum given by the expression $G = -4 \times 10^{16} \times (\lambda - 1555 \times 10^{-9})^2 + 15$ dB, where λ denotes the channel wavelength. We arbitrarily choose $\beta_i = 1$ for all i. We select α_i according to the proportional pricing scheme in Pavel (2006). The speed of light in an optical cable is approximately 200,000 km/s. This results in a 1 ms round-trip delay for a 100 km optical span. The channel powers are updated every 5 ms.

We first verify Theorem 1 for the 8 channel, single link conservative stability case. We select $\mu_i = 1$. Fig. 2 depicts the OSNR for round trip delays of 10 ms. The system is stable for any delays. Figs. 3 and 4 depict the time-delay dependent case for the single link network. These plots use the parameters $\mu_i = 0.1$ and $\mu_i = 0.95$ for all i. The calculated values for μ_i to fulfill Theorem 1 are all between 0.366 and 0.4 for the time delay of 10 ms. Instability does not occur until μ_i approaches 1. This indicates that Theorem 1 is conservative.

We simulate a 2 channel network with four links where the inputs u_i enter one node on the network from separate links, but then share one of the remaining two links. The output $OSNR_i$ values are measured from two separate points in the last two links. Fig. 5 depicts the case $\mu_i = 0.4$. According to Theorem 1, the calculated values of μ_i are 0.36 and 0.3 for channels 1 and 2. Instability occurs at $\mu_i = 0.65$ showing that the predicted values are conservative.

6. Conclusions and future work

We analytically studied the effects of time-delay on the stability of an optical communication network. We incorporated