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Games with coupled propagated constraints in optical networks with multi-link topologies[☆]

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ABSTRACT

We consider games in optical networks in the class of m-player games with coupled utilities and constraints. Nash equilibria of such games can be computed based on recent extension of duality to a game theoretical framework. This work extends previous results on games with coupled constraints in optical links to multi-link topologies. Coupled constraints in optical networks are propagated along links, which introduces additional complexities for analysis. Specifically, convexity of the propagated constraints is no longer automatically ensured. We show that convexity is satisfied for single-sink multi-link topologies. The general case of multi-links with arbitrary sources and sinks is dealt with by a partitioned game with stages. We exploit the single-sink structure of each stage and the ladder-nested form of the game and we discuss iterative computation of equilibria based on a three-level hierarchical algorithm and prove its convergence under certain conditions.

1. Introduction

Work on games with coupled action spaces has been going on for the past 40 years. These are also called generalized Nash games (Harker, 1991), games with coupled constraints (Rosen, 1965), social equilibria (Arrow & Debreu, 1954; Debreu, 1952). Game theoretical formulations of problems and computational approaches towards solving them have been areas of much recent interest, (Facchinei, Fischer, & Piccialli, 2007; Morgan & Romaniello, 2003; Scutari, Barbarossa, & Palomar, 2006). The game-theoretic approach has been used for network flow optimization (congestion control) (Azouzi & Altman, 2003) as well as for network power allocation (power control) (Alpcan, Basar, Srikant, & Altman, 2001; Altman & Altman, 2003; Saraydar, Mandayam, & Goodman, 2002) and channel optical signal-to-noise ratio (OSNR) optimization in optical networks (Pavel, 2006).

The study of conditions for existence and uniqueness of Nash equilibrium (Nash, 1950) in pure strategies continues to be a fundamental issue. Only sufficient conditions for existence are available, while for the study of uniqueness by now only partial results are available. Uniqueness results exist for special classes of games such as for S-modular games (Altman & Altman, 2003), potential games (Scutari et al., 2006), routing games in parallel links (Azouzi & Altman, 2003). Uniqueness of a normalized equilibrium point is studied in Rosen (1965). From a computation point of view, the study of generalized Nash equilibrium presents severe analytical difficulties (Facchinei et al., 2007). Insightful theoretical results have been obtained for computation of equilibria in classes of games with structures, such as two-player polynomial games (Parrilo, 2006), separable games (Stein, Ozdaglar, & Parrilo, 2006) or potential games (Monderer & Shapley, 1996).

Duality has received interest in the study of games. Duality and dual games are studied for repeated two player zero-sum games in de Meyer (1996). The work in Facchinei et al. (2007) shows that a generalized Nash equilibrium can be calculated by solving a variational inequality and the results express conditions in terms of variational inequality problem and Karush–Kuhn–Tucker (KKT) conditions for the pseudo-gradient. Another related work is Morgan and Romaniello (2003) where the authors present a scheme that associates to a generalized variational inequality, a dual problem and KKT conditions, thus allowing to solve primal and dual problems in the spirit of classical Lagrangian duality for constrained optimization problems, using set theoretic concepts and set-valued operators. Recently a procedural method is proposed in Pavel (2007a) for computing a Nash equilibrium, based on an extension of duality to a game theoretical framework. The setting of the construction in Pavel (2007a) uses the two-argument system cost function and relaxes also the constraints

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into a two-argument form. Thus the problem is enlarged into a constrained optimization problem in a space of twice the dimension followed by projection back into a one dimension (with a fixed-point solution). Moreover, for convex constraints, duality leads to hierarchical decomposition into a lower-level game with no coupled constraints and an optimization problem for Lagrangian prices. The results in Pavel (2007a) and their extension herein have provided a way of finding one such Nash equilibrium, assuming that at least one exists but not presuming uniqueness of the Nash equilibrium. The related similar results in Facchinei et al. (2007) that have appeared almost at the same time as (Pavel, 2007a) in parallel and independently indicate continued interest in this area after 40 years.

In a game formulation in optical networks where channel utility is related to maximizing channel OSNR, and players' actions are channel powers (Pavel, 2006), coupled constraints have to be considered. Specifically in optical networks, cascaded amplified spans are present, as well as accumulation and self-generation of optical noise, cross-talk, possible coupling and saturation. Considering these differences and specific realistic physical features, the work in Pavel (2006) concentrates on OSNR modeling and basic Nash game formulation. In order for game methods to be practical, they must incorporate realistic constraints of the underlying network systems. One such important constraint is the power capacity constraint. Such a constraint arises because when all wavelength-multiplexed channels share the optical fiber, the total power launched into the fiber needs to be restricted below the nonlinearity threshold (Agrawal, 2002; Mecozzi, 1998), which is regarded as the *link capacity constraint*. For a single optical link, the coupled capacity constraint was indirectly considered in Pan and Pavel (2005) by modifying each channel cost function, or directly considered based on duality (Pavel, 2007a).

This paper extends previous results on games with coupled constraints in optical links (Pavel, 2007a) to general multi-link topologies. The multi-link topology studied in this paper is one type of network topologies in optical networks. A short version of this work appeared in Pan and Pavel (2007). Unlike capacity constraints in flow control or routing (Azouzi & Altman, 2003; Kelly, Maulloo, & Tan, 1998; Low & Lapsley, 1999), coupled constraints in optical networks are propagated along links. This introduces additional complexities. The main contributions of the paper are: (1) analysis of the convexity property of propagated constraints in optical networks with general multi-link topologies; and (2) the multi-link case with arbitrary sources and sinks is dealt with by a partitioned game with stages. For general multi-link topologies with arbitrary sources and sinks, we propose an alternative approach. Specifically, we define a multi-link partitioned game with stages depending on the bifurcation point. The presence of competition in games makes the NE solution inefficient and pricing the system resources appears to be a powerful tool for achieving a more socially desirable result (Saraydar et al., 2002). We exploit the single sink structure of each stage and the ladder-nested form of the game to solve it. We discuss iterative computation of equilibria based on a three-level hierarchical algorithm, for channels, prices and stages.

This paper is organized as follows. Section 2 reviews duality results for games, and a coupled Nash game in a single optical link (Pavel, 2007a). The term "solving a Nash game" indicates finding Nash equilibria of games. Section 3 studies convexity of constraints. Section 4 studies general multi-link topologies. Section 5 gives a numerical example. Section 6 gives conclusions and future work.

2. Background

2.1. Duality for Nash games with coupled constraints

We review first results for solving general Nash games in Pavel (2007a). Consider an m-player Nash game, each player minimizing its individual cost function $J_i : \Omega \to \mathbb{R}$, $i \in \mathcal{M}$, $\mathcal{M} = \{1, \dots, m\}$. Let $\Omega = \prod_{i\in\mathcal{M}} \Omega_i$, $\Omega_i = [u_{\min}, u_{\max}]$. Thus Ω is separable in the sense that each player can take its action independently. Let $u = [u_1, \dots, u_m]^T \in \Omega$ with $u_i \in \Omega_i$ be the vector of player actions. This game is subject to L additional coupled constraints $g_l(u) \le 0$, $l = 1, \dots, L$, or $g(u) \le 0$ with $g(u) = [g_1(u), \dots, g_L(u)]^T$, where $g_l : \Omega \to \mathbb{R}$ is the *constraint function* with the corresponding constraint set $\bar{\Omega}_l = \{u \in \Omega \mid g_l(u) \le 0\}$. The overall action set $\bar{\Omega} \subset \mathbb{R}^m$ is coupled, given as $\bar{\Omega} = \bigcap_{l=1}^{L} \bar{\Omega}_l = \{u \in \Omega \mid g(u) \le 0\}$. A vector $u = (u_{-i}, u_i)$ is called feasible if $u \in \bar{\Omega}$. A Nash equilibrium (NE) solution of this coupled game is defined as follows.

Definition 1. Consider an m-player game with individual cost functions, $J_i : \Omega \to \mathbb{R}$, subject to coupled constraints, $g_l(u) \le 0$, $l = 1, \dots, L$. A feasible vector $u^* = (u^*_{-i}, u^*_i)$ is called an NE solution of this game if

$$J_i(u^*_{-i}, u^*_i) \le J_i(u^*_{-i}, x_i), \quad \forall\, x_i \in \bar{\Omega}_i(u^*_{-i}), \forall i \in \mathcal{M}$$

for every given u^*_{-i}, where $\bar{\Omega}_i(u^*_{-i})$ is the projection set

$$\bar{\Omega}_i(u^*_{-i}) := \{x_i \in \Omega_i | (u^*_{-i}, x_i) \in \bar{\Omega}\}.$$

The following existence theorem (Theorem 4.4, Basar and Olsder (1999)) gives sufficient conditions for existence of an NE solution.

Theorem 1. *Let $\bar{\Omega} \subset \mathbb{R}^m$ be compact and convex and for each $i \in \mathcal{M}$, let the cost $J_i : \bar{\Omega} \to \mathbb{R}$ be continuous on $\bar{\Omega}$ and convex in u_i, for every $u_j \in \bar{\Omega}_j$, $j \in \mathcal{M}$, $j \ne i$. Then the associated Nash game admits an NE solution.*

An NE solution can also be defined by using the concept of *system-like* cost function first introduced in Basar and Olsder (1999). This is a two-argument function $\widetilde{J} : \Omega \times \Omega \to \mathbb{R}$ called the *Nash game (NG) cost function*. It is defined as

$$\widetilde{J}(u; x) := \sum_{i=1}^{m} J_i(u_{-i}, x_i), \quad \forall x \in \Omega. \tag{1}$$

This augmented function defined on a space of twice the dimension of the original game is instrumental in what follows. Our aim of using $\widetilde{J}$ based on Basar and Olsder (1999) is that of finding a solution of the original Nash game with coupled constraints by solving a constrained optimization problem for $\widetilde{J}$ and searching for a fixed-point solution. Firstly, the NG cost function is separable in the second argument x for every given u, i.e., each component cost function in $\widetilde{J}(u; x)$ is decoupled in x. Similarly, the constraints g can be augmented into a separable two-argument form, $\widetilde{g}$,

$$\widetilde{g}(u; x) = \sum_{i=1}^{m} g(u_{-i}, x_i), \tag{2}$$

thus enlarging the search set. *NG-feasibility* is equivalent to $\widetilde{g}(u; u) \le 0$. A Nash game with coupled constraints is related to a *constrained* minimization of $\widetilde{J}$, (1), (2), with respect to the second argument x, that admits a fixed-point solution (Lemma 1 in Pavel (2007a)). A solution u^* to this constrained minimization satisfies

$$\widetilde{J}(u^*; u^*) \le \widetilde{J}(u^*; x), \quad \forall x \in \Omega,\ \widetilde{g}(u^*; x) \le 0, \tag{3}$$

with $\widetilde{g}(u^*; u^*) \le 0$. Proposition 1 in Pavel (2007a) shows that individual components of the solution $u^* = [u_i^*]$ constitute an NE solution in the sense of Definition 1. As in standard optimization (Bertsekas, 1999), a two-argument Lagrangian can be defined for $\widetilde{J}$ and $\widetilde{g}$,

$$\widetilde{L}(u; x; \mu) = \widetilde{J}(u; x) + \mu^T \widetilde{g}(u; x). \tag{4}$$

For u^*, a fixed-point solution to the minimization of $\widetilde{L}$ over $x \in \Omega$, i.e., such that $\widetilde{L}(u^*; u^*; \mu) \le \widetilde{L}(u^*; x; \mu), \forall x \in \Omega$. In the Lagrangian optimality condition, u^* is obtained by first minimizing the augmented Lagrangian function $\widetilde{L}(u; x; \mu)$ with respect to the second argument x, which gives $x = \phi(u)$ for every given u. The next step involves finding a fixed-point solution u^* of ϕ by setting $x = u$, i.e., solving $u = \phi(u)$. We write in a compact notation

$$\widetilde{L}(u^*; u^*; \mu) = \left[\min_{x \in \Omega} \widetilde{L}(u; x; \mu)\right]\Big|_{x=u}. \tag{5}$$

Such an u^* is a solution to (3) and hence an NE solution if the complementary slackness condition in Theorem 2 below holds, (Proposition 2 in Pavel (2007a)). Note that u^* thus obtained depends on μ, $u^*(\mu)$.

Then a dual cost function $D(\mu)$ is defined as

$$D(\mu) := \widetilde{L}(u^*; u^*; \mu), \tag{6}$$

and the dual optimal value is defined as

$$D^* = \max_{\mu \ge 0} D(\mu). \tag{7}$$

The primal and dual optimal solution pairs are characterized by Theorem 2 in Pavel (2007a), which we restate here for completeness as Theorem 2.

Theorem 2. *$(u^*; \mu^*)$ is an optimal NE solution-Lagrange multiplier pair in the sense of (3), (7), if and only if:*

(1) $u^* \in \Omega$ *and* $\widetilde{g}_l(u^*; u^*) \le 0$ *(NG-feasibility);*
(2) $\mu^* \ge 0$ *(Dual feasibility);*
(3) $u^* = \arg\{\min_{x \in \Omega} \widetilde{L}(u; x; \mu^*)|_{x=u}\}$ *(Lagrangian optimality);*
(4) $\mu^{*T} \widetilde{g}(u^*; u^*) = 0$ *(Complementary slackness).*

To summarize, we show that a Nash equilibrium for games with coupled constraints can be found by solving a constrained optimization problem for the two-argument system cost function. We use this as a convenient vehicle for computation of a Nash equilibrium. We relax the problem into an optimization problem in a space of twice the dimension and use a fixed-point solution to project back into the original space and game problem. The formalism of standard optimization enables us to set up a Lagrangian extension and duality results. We do not presume uniqueness of the Nash equilibrium, but we address the problem of finding one such solution. The conditions for existence of an NE solution are given in Theorem 1, which are same as the ones for the existence of a normalized equilibrium in Rosen (1965). In fact, $u^*(\mu^*)$ is a normalized equilibrium point with all r_i in Rosen (1965) equal to 1.

The separability in the second argument of both NG cost function and constraints ensures that $D(\mu)$ in (6) can be decomposed.

Corollary 1 (*Pavel, 2007a*). *For a Nash game with cost J_i and constraints g_l, $l = 1, \ldots, L$, where J_i and g_l are continuously differentiable and convex, the dual cost $D(\mu)$ (6), (4), can be decomposed as*

$$D(\mu) = \sum_{i=1}^{m} [\min_{x_i \in \Omega_i} L_i(u_{-i}, x_i, \mu)]|_{x_i=u_i} = \sum_{i=1}^{m} L_i(u_{-i}^*, u_i^*, \mu) \tag{8}$$

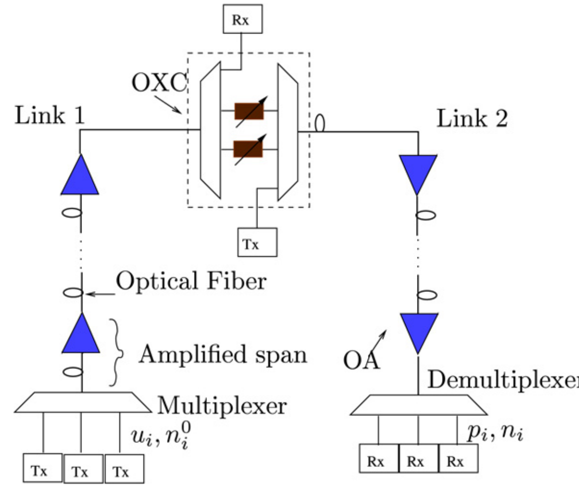

Fig. 1. Optical networks: 2-link configuration.

where

$$L_i(u_{-i}, x_i, \mu) = J_i(u_{-i}, x_i) + \mu^T g(u_{-i}, x_i) \tag{9}$$

and $u^*(\mu) = [u_i^*(\mu)] \in \Omega$ is a fixed-point solution to the set of minimizations on the right-hand side (RHS) of (8).

The duality approach offers a natural way to hierarchically decompose a Nash game with coupled convex constraints into a lower-level Nash game without coupled constraints with individual cost functions $L_i(u_{-i}, x_i, \mu)$, (9), and a higher-level system optimization problem. In effect the interpretation is that a procedural method for finding a solution to a Nash game with coupled constraints can be based on solving a modified game with no coupled constraints and an optimization problem.

2.2. OSNR model and Nash game formulations

We next review the network OSNR model and coupled OSNR Nash game. Consider a multi-link optical network, with a set of optical links $\mathcal{L} = \{1, \ldots, L\}$ interconnected via optical switching nodes (e.g. optical cross-connects (OXCs)), and a set of channels, $\mathcal{M} = \{1, \ldots, m\}$, transmitted from transmitter sites (Txs) to corresponding receiver sites (Rxs) by intensity modulation and wavelength-multiplexing (Agrawal, 2002). Fig. 1 shows an optical network configuration with two optical links. A link $l \in \mathcal{L}$ is composed of N_l cascaded optically amplified spans via optical amplifiers (OAs). An OA amplifies the optical power of all channels simultaneously, at the expense of introducing amplified spontaneous emission (ASE) noise. For each channel $i \in \mathcal{M}$, let $G_{l,v,i}$ and $ASE_{l,v,i}$ denote its gain profile and generated ASE noise power on the vth OA in link l, respectively. As in Forghieri, Tkach, and Favin (1998) and Pavel (2006), the following non-restrictive assumptions are used: in link l, all spans have equal length and all OAs have the same spectral shape, i.e., for channel i, $G_{l,v,i} = G_{l,i}$. We denote by $\mathcal{M}_l$ the set of channels transmitted over link l. For each channel $i \in \mathcal{M}$, let $u_{l,i}$ ($n_{l,i}^{in}$) denote input signal (noise) optical power on link l. Let $p_{l,i}$ ($n_{l,i}^{out}$) be output signal (noise) optical power on link l, while u_i (n_i^0) and p_i (n_i) are input and output signal (noise) power at Tx and Rx, respectively. For channel i, $\mathcal{R}_i$ denotes its optical path (a subset of optical links) from Tx to Rx. Different channels can be transmitted via different optical paths and different links may have different number of channels.

OAs are operated in automatic power control mode such that a target total power is maintained, which compensates for variations in span loss over the link. Total power target is selected below the threshold for nonlinear effects (Forghieri et al., 1998). For spans of