Distributed Circular Formation Stabilization for Dynamic Unicycles

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Abstract

This paper investigates the problem of designing distributed control laws making a group of dynamic unicycles converge to a common circle of prespecified radius, whose centre is stationary but dependent on the initial conditions, and travel around the circle in a desired direction. The vehicles are required to converge to a formation on the circle, expressed by desired separations and ordering of the unicycles. The information exchange between unicycles is modelled by a directed graph which is assumed to have a spanning tree. A hierarchical approach is proposed which simplifies the control design by decoupling the problem of making the unicycles converge to a common circle from the problem of stabilizing the formation.

I. Introduction

The formation control problem entails designing distributed feedback laws making a group of vehicles move in an ordered manner along a desired reference trajectory or along a geometric path. By *distributed feedback* it is meant a feedback that only requires each vehicle to sense relative information with respect to neighboring vehicles, for instance its relative displacement. The notion of formation can be expressed in a variety of ways. One common description is in terms of geometric relations to be satisfied by the vehicles, such as desired inter-vehicle distances or angles. An alternative description, called behavioral, does not prescribe precise geometric relationships between vehicles, but it simply aims at maintaining cohesion of the group while avoiding collisions.

Interest in the subject of formation control can be traced back to the 1987 work of Craig Reynolds [1] in which an algorithm is presented to simulate the behavior of a flock of birds. The key feature of the algorithm in [1] is its distributed nature. The flight strategy of each bird is based on rules that require sensing of relative displacement and heading of other birds. These simple strategies give rise to interesting group dynamics emulating the behaviors observed by Biologists in nature. This avenue of investigation has given rise to a rich literature on behavioral formations with the objective of understanding the flocking and swarming behavior.

On the Engineering side, interest in formation control exploded in the past decade due, in part, to the increased availability of land and aerial mobile robot platforms. Some influential papers in the area of formation control are [2], [3], [4], [5], [6], [7]. In [8], Gazi-Passino use artificial potential functions to stabilize behavioral formations for kinematic point-mass systems. This approach has become very popular and has been taken by many other researchers, see the recent book [9]. In [10], [11], Marshsall-Broucke-Francis study formations of unicycles in cyclic pursuit. They postulate a control law and show that the resulting equilibrium formations are generalized regular polygons. Using Jacobian linearization and eigenvalue analysis, they characterize the stability type of each equilibrium formation. In [12], Krick-Broucke-Francis use the notion of graph rigidity to formulate a control law stabilizing rigid formations for kinematic point-mass systems. In [13], Lin-Francis-Maggiore prove that distributed stabilization of n unicycles to a common location can be performed if and only if the sensor graph has a spanning tree.

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They also develop feedbacks for formation stabilization to a line and other geometric patterns. Geometric conditions for feasibility of formations of nonholonomic vehicles were investigated in [14]. In [15], Sepulchre-Paley-Leonard investigate problems of synchronization for systems of particles modeled as kinematic unicycles. Potential functions are defined for various tasks and used to generate gradient control laws. In particular, the authors use space potentials to stabilize circular formations, and phase potentials to regulate the spacing of the formations. They show how using different types of phase potentials one obtains different symmetric formations. The results are based on an all-to-all communication assumption. In [16], the same authors generalize their results to different communication topologies using dynamic feedback. This generalization relies on so-called consensus estimators developed by Scardovi-Sarlette-Sepulchre in [17].

In this paper we study a formation control problem similar to that analyzed in [15], [16]. We design distributed feedback laws making a group of dynamic unicycles converge to a common circle, and travel around the circle in formation. In our setting, the formation has a precise geometric description in terms of desired ordering and spacing between vehicles (in [15], [16], one has no direct control over the ordering and spacing). We model the flow of information flow through the formation by a directed graph which is assumed to be static and to have a spanning tree. Our control laws are static and time-invariant (in [16], general graph topologies require dynamic control laws). The solution we propose relies on the observation that the circular formation stabilization problem can be formulated as one of stabilization of a closed set Γ. In order to stabilize Γ, we break down the problem into three decoupled design steps. First, we stabilize a set Γ_1 on which the dynamic unicycles are purely kinematic. Then, on Γ_1 we design a distributed feedback making the kinematic unicycles converge to a common circle. This amounts to stabilizing a set $\Gamma_2 \subset \Gamma_1$. Finally, on Γ_2 we stabilize the desired formation, which amounts to stabilizing a set $\Gamma_3 \subset \Gamma_2$. The set Γ_3 coincides with the original goal set Γ. The principle enabling this decoupled three-step design strategy is a reduction theorem for asymptotic stability of sets, reviewed in the sequel. The main result of this paper, Theorem V.5, presents a solution to the circular formation stabilization problem which makes Γ_1 globally asymptotically stable, and it makes Γ_2 and Γ_3 asymptotically stable. In practice, this means that for suitable initial conditions, the unicycles converge to a common circle and, if in addition the initial condition is near Γ_3, the unicycles also converge to the desired formation. In addition, we show that when the sensor digraph is undirected then the stabilization of Γ_2 is *global*, so that the convergence of the unicycles to a common circle occurs for arbitrary initial conditions. There are practical advantages in a solution which *simultaneously* stabilizes Γ_1, Γ_2, and Γ_3. One of them is the fact that if the unicycles are traveling in formation around a common circle, one may change the formation specification on the fly by changing a parameter in the controller. The unicycles will reconfigure themselves without leaving the circle. A technical challenge of the problem we investigate is the fact that the goal set Γ is not compact. When using Lyapunov functions (or artificial potentials) whose zero level set is Γ, one cannot perform a LaSalle analysis to prove stability, because there is no guarantee that solutions are bounded. Our analysis in this paper is not based on Lyapunov theory or artificial potentials, and it rigorously addresses the issue of boundedness of trajectories.

This paper is organized as follows. In Section II we pose the formation stabilization problem, and in Section III we formulate it in the set stabilization framework. The three-step design methodology to solve the problem is presented in Section IV. This section also presents stability definitions and reviews a so-called reduction theorem for asymptotic stability of sets. In Section V we perform the three design steps and derive distributed feedback laws solving the formation stabilization problem. Simulation results are presented in Section VI. Finally, in Section VII we draw conclusions and discuss open problems.

Notation

If x is a real number, $x \bmod 2\pi$ denotes its value modulo 2π. We will denote by S^1 the set of real numbers modulo 2π, diffeomorphic to the unit circle, and by T^n the n-torus, i.e., the Cartesian product $S^1 \times \cdots \times S^1$, n times. If $\theta, x \in \mathbb{R}$, then the expression $x = \theta \bmod 2\pi$ will be used to state that $x \in$

$\{\theta + 2\pi k, k \in \mathbb{Z}\}$. Similarly, if $x = (x_1, \ldots, x_n), \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, the expression $x = \theta \bmod 2\pi$ will be used to state that $x_i = \theta_i \bmod 2\pi$, $i = 1, \ldots, n$. We will denote by $\mathbf{n}$ the index set $\{1, \ldots, n\}$.

If A and B are two matrices, $\mathrm{blockdiag}(A, B)$ denotes the block-diagonal matrix with blocks A and B. If $a_1, \ldots, a_n$ are scalars, $\mathrm{diag}(a_1, \ldots, a_n)$ is the diagonal matrix with diagonal entries a_i. We denote by $A \otimes B$ the Kronecker product of two matrices A and B, and by $\mathbf{1}$ the n-vector of ones.

We let $\mathsf{Sat}(\mathbb{R})$ denote the class of C^1 functions $\phi : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \mathbb{R}$, $\phi(y) > 0$ and $|\phi(y)y| < 1$. We denote by $\mathsf{Sat}(\mathbb{R}^n)$ the class of functions $\phi : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ defined as $\phi([y_1 \ \cdots \ y_n]^\top) := \mathrm{diag}(\phi_1(y_1), \ldots, \phi_n(y_n))$, with $\phi_i \in \mathsf{Sat}(\mathbb{R})$. We let $\phi_{(2)} := \phi \otimes I_2$. The function $\phi(y)y$ is sometimes referred to as a *decentralized saturation*.

By $\phi(t, x_0)$ we denote the solution of $\dot{x} = f(x)$ with initial condition x_0. Given an interval I of the real line and a set $S \in \mathcal{X}$, denote by $\phi(I, S)$ the set $\phi(I, S) := \{\phi(t, x_0) : t \in I, x_0 \in S\}$. Throughout this paper, unless otherwise stated, we use $\|v\|$ for the two-norm of a vector v. Given a closed nonempty set $S \subset \mathbb{R}^n$, the point-to-set distance $\|x\|_S$ is defined as $\|x\|_S := \inf\{\|x - y\| : y \in S\}$. We use $B_\alpha(x)$ to denote an open ball of radius α centered at x, and $B_\alpha(S)$ the set of points with distance less than α to S. A generic neighbourhood of a set S will be denoted $\mathcal{N}(S)$.

II. The circular formation stabilization problem (CFSP)

Consider a collection of $n \geq 2$ identical dynamic unicycles modelled as vertical rolling disks as in [18],

$$\begin{aligned}
\dot{x}_1^i &= x_5^i \cos x_3^i \\
\dot{x}_2^i &= x_5^i \sin x_3^i \\
\dot{x}_3^i &= x_4^i \\
\dot{x}_4^i &= \frac{1}{J} w_2^i \\
\dot{x}_5^i &= \frac{R}{(I + mR^2)} w_1^i, \qquad i \in \mathbf{n}.
\end{aligned} \tag{1}$$

Referring to the i-th unicycle depicted in Figure 1, R is the radius of the disk and m is its mass; I and

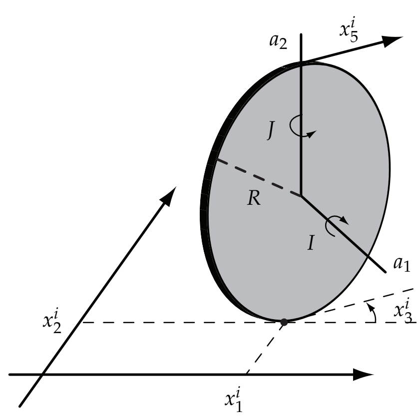

Fig. 1. The i-th dynamic unicycle

J are, respectively the moments of inertia about axes a_1 and a_2 through the centre of mass of the disk. The states (x_1^i, x_2^i) are the coordinates of the point of contact of the disk with the plane; x_3^i is the heading angle; x_4^i is the angular speed around axis a_2; finally, x_5^i is the linear speed of the point of contact of the disk with the plane. The control inputs are the torques w_1^i and w_2^i about axes a_1 and a_2, respectively.

We will denote the state of the i-th unicycle by $x^i = (x_1^i, \ldots, x_5^i) \in \mathcal{X}_i := \mathbb{R}^2 \times S^1 \times \mathbb{R}^2$, and we will let $\chi = (x^1, \ldots, x^n)$ denote the collective state of the n unicycles. The collective state space of the n unicycles is $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. Finally, we will denote $x_3 = (x_3^1, \ldots, x_3^n)$.

We will assume that each unicycle has access to its own absolute heading angle x_3^i, its own speeds x_4^i, x_5^i, and that it can sense relative information of other unicycles. More precisely, the information exchange will be modeled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ called the *sensor digraph*. Each node of $v_i \in \mathcal{V}$ of $\mathcal{G}$ represents a unicycle. An edge in $\mathcal{E}$ from node i to node j of $\mathcal{G}$ signifies that unicycle i has access to its relative displacement, relative heading, and relative linear and angular speeds with respect to unicycle j. We will let L denote the Laplacian of the digraph of $\mathcal{G}$. We will use the notation L^i for the i-th row of L, and we will denote $L_{(2)} := L \otimes I_2$ (Kronecker product), $L^i_{(2)} := L^i \otimes I_2$, where I_2 is the 2×2 identity matrix. A node v_i of $\mathcal{G}$ is said to be *globally reachable* [13] if for all $j \in \mathbf{n}$, $j \neq i$, there exists a walk in $\mathcal{G}$ from node v_j to node v_i. A graph $\mathcal{G}$ has a globally reachable node if and only if $\mathcal{G}$ has a spanning tree, i.e., if using some of the edges of $\mathcal{G}$ one can form a tree containing all the nodes of $\mathcal{G}$. An algebraic characterization of this property, used in the sequel, was given in [13] and is repeated here for convenience.

Lemma II.1 (Lemma 2, [13])**.** A digraph $\mathcal{G}$ has a globally reachable node (or, equivalently, a spanning tree) if and only if 0 is a simple eigenvalue of L.

In this paper we investigate the following

Circular Formation Stabilization Problem (CFSP). Consider the n unicycles in (1). For a given sensor digraph $\mathcal{G}$ with a globally reachable node (or, equivalently, with a spanning tree), design a control law meeting the following specifications:

(i) *Circular path following.* For a suitable set of initial conditions, the unicycles should converge to a common circle of radius $r > 0$, whose centre is stationary but dependent on the initial conditions, and move along the circle in a desired direction (clockwise or counter-clockwise). In steady-state, all unicycles should have a linear speed $x_5^i = v > 0$, and angular speed $x_4^i = v/r$, for all $i \in \mathbf{n}$.

(ii) *Formation stabilization.* On the circle in part (i) of the problem, the n-unicycles are required to locally converge to a formation expressed by desired distances and ordering of unicycles.

(iii) *Distributed feedback.* The feedback law must be consistent with the sensor digraph, as follows. In computing its own feedback law, unicycle i has only access to the following variables:
- The relative displacement of unicycles that are visible to unicycle i according to $\mathcal{G}$, measured in the local frame of unicycle i (see Figure 2).
- The relative heading of unicycles that are visible to unicycle i according to $\mathcal{G}$ (see Figure 2).
- The angular and linear speeds x_4^i, x_5^i.
- The relative angular speed of unicycles that are visible to unicycle i according to $\mathcal{G}$.

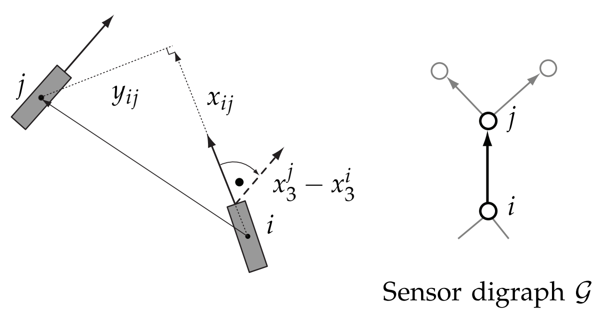

Fig. 2. Illustration of the relative kinematic variables sensed by unicycle i if there is an edge from node i to node j in the sensor digraph.

One could pose CFSP for kinematic unicycles. In this case the state of each unicycle would be (x_1^i, x_2^i, x_3^i), and the control inputs would be $u_1^i = x_5^i$ and $u_2^i = x_4^i$. Moreover, unicycle i would have access to its relative displacement (measured in its local frame) and relative heading angle with respect to neighboring unicycles. A byproduct of the hierarchical approach to solving CFSP presented in Section IV is a solution to the kinematic version of this problem.

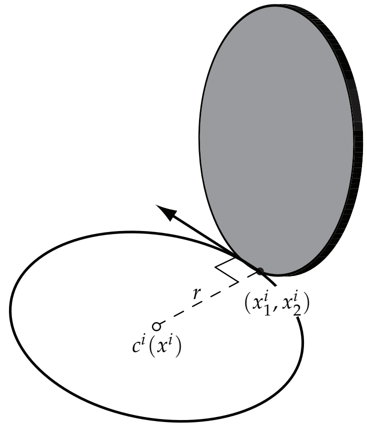

Fig. 3. The centre of rotation $c^i(x^i)$ of unicycle i if $|x_5^i| > 0$ and $x_4^i = x_5^i/r$.

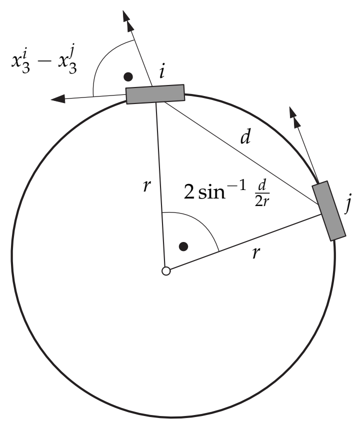

Fig. 4. Formation on the circle

III. CFSP as a set stabilization problem

In this section we show that CFSP can be reformulated as a set stabilization problem. For each $i \in \mathbf{n}$, define the function $c^i(x^i)$ as

$$c^i(x^i) = \begin{bmatrix} x_1^i - r\sin x_3^i & x_2^i + r\cos x_3^i \end{bmatrix}^\top, \tag{2}$$

and denote $c(\chi) = \begin{bmatrix} c^1(x^1)^\top & \cdots & c^n(x^n)^\top \end{bmatrix}^\top$. As shown in Figure 3, the point $c^i(x^i)$ lies at a distance r from the point of contact (x_1^i, x_2^i) of unicycle i with the plane, and the vector $\begin{bmatrix} x_1^i & x_2^i \end{bmatrix}^\top - c^i(x^i)$ is orthogonal to the normalized velocity vector $\begin{bmatrix} \cos x_3^i & \sin x_3^i \end{bmatrix}^\top$ of unicycle i. Therefore, the point $c^i(x^i)$ is the centre of the circle that the contact point of unicycle i would describe if its linear speed x_5^i and angular speed x_4^i were chosen so that $|x_5^i| > 0$ and $x_4^i = x_5^i/r$. In light of the above, meeting specification (i) of CFSP is equivalent to making the set

$$\begin{aligned} \mathcal{C} = \{\chi \in \mathcal{X} : & c^1(x^1) = \cdots = c^n(x^n),\, x_4^i = v/r,\, x_5^i = v, \\ & i = 1, \ldots, n\} \end{aligned}$$

attractive. The notions of stability and attractivity of sets used in this paper are defined precisely in Section IV-B. We now turn our attention to specification (ii). Consider a formation where unicycles i and j both travel along a common circle of radius r in the counter-clockwise direction, and unicycle j travels at distance $d \in [0, 2r]$ from unicycle i, as shown in Figure 4. This formation constraint can be equivalently expressed as $x_3^i - x_3^j = 2\sin^{-1}\left(\frac{d}{2r}\right) \mod 2\pi$, so to specify a formation on a circle with

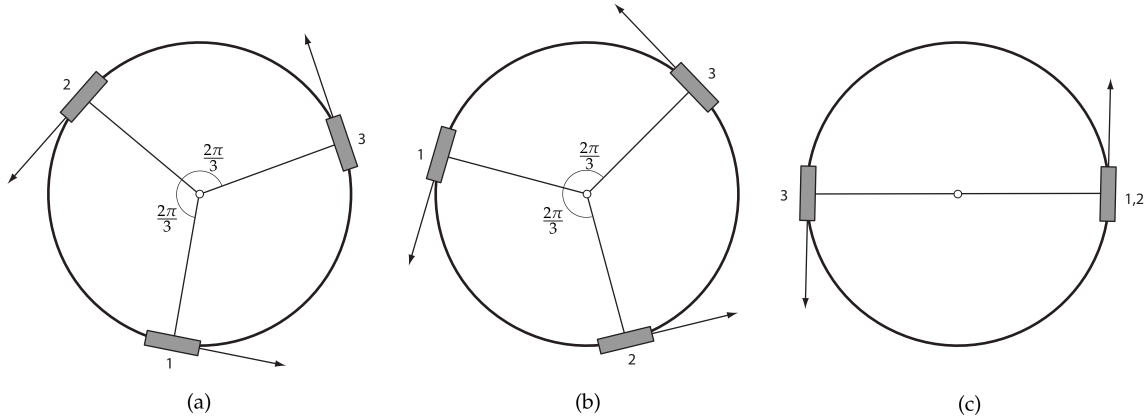

Fig. 5. Sample formations of three unicycles on a common circle.

given ordering and distances between unicycles one should specify angles $\theta_1, \ldots, \theta_{n-1}$ and enforce the relations $x_3^i - x_3^{i+1} = \theta_i \bmod 2\pi$, $i = 1, \ldots n-1$. Meeting specification (ii) in CFSP is equivalent to making the set $\Gamma = \{\chi \in \mathcal{C} : x_3^i - x_3^{i+1} = \theta_i \bmod 2\pi,\ i = 1, \ldots, n-1\}$ attractive. Letting $\alpha_n = 0$, $\alpha_i = \sum_{j=i}^{n-1} \theta_j$, $i = 1, \ldots, n-1$, and $\alpha = \begin{bmatrix}\alpha_1 & \cdots & \alpha_n\end{bmatrix}^\top$, the set Γ can be equivalently expressed as

$$\Gamma = \{\chi \in \mathcal{C} : L(x_3 - \alpha) = 0 \bmod 2\pi\}.$$

Notice, indeed, that since we are assuming that the sensor digraph $\mathcal{G}$ has a globally reachable node, by Lemma II.1 the Laplacian matrix L has a simple eigenvalue at 0. The corresponding eigenvector is $\mathbf{1}$, and so $\ker L = \operatorname{span} \mathbf{1}$. Therefore, using our definition of α above we have

$$\begin{aligned}\{\chi :& L(x_3 - \alpha) = 0 \bmod 2\pi\} \\ &= \{\chi : x_3^1 - \alpha_1 = \cdots = x_3^n - \alpha_n \bmod 2\pi\} \\ &= \{\chi : x_3^i - x_3^{i+1} = \alpha_i - \alpha_{i+1} \bmod 2\pi\} \\ &= \{\chi : x_3^i - x_3^{i+1} = \theta_i \bmod 2\pi\},\end{aligned}$$

as required. The vector of n angles α will be called a *formation vector.* Note that any two formation vectors α_1 and α_2 differing by a constant angle θ define the same formation because $L(x_3 - \alpha_1) = L(x_3 - \alpha_2 - \theta\mathbf{1}) = L(x_3 - \alpha_2)$. In other words, the vector α defines a unique formation up to rotation on the circle.

From a practical viewpoint, it is desirable to require Γ to be asymptotically stable in addition to being attractive. For instance, if a perturbation of finite duration steers the unicycles away from a circular formation, it is desirable that during the ensuing transient the unicycles remain close to such formation. Adding the stability requirement, specifications (i) and (ii) of CFSP become the problem of asymptotically stabilizing the goal set

$$\begin{aligned}\Gamma = \{\chi \in \mathcal{X} : c^1(x^1) = \cdots = c^n(x^n),\ x_4^i = v/r,\ x_5^i = v,\ i \in \mathbf{n}, \\ L(x_3 - \alpha) = 0 \bmod 2\pi\}.\end{aligned} \tag{3}$$

Example 1. Consider $n = 3$ unicycles and an arbitrary sensor digraph with a globally reachable node. Suppose we want to stabilize a circular formation where the unicycles are equidistant on the circle and such that 2 is behind 1, and 1 is behind 3, as shown in part (a) of Figure 5. Stabilizing this formation on the circle corresponds to stabilizing the set Γ in (3) with formation vector $\alpha = \begin{bmatrix}4\pi/3 & 2\pi/3 & 0\end{bmatrix}^\top$.

Consider now the formation shown in part (b) of Figure 5, where the inter-agent distances are the same, but now 2 is behind 3, and 3 is behind 1. This time we have $\alpha = \begin{bmatrix}2\pi/3 & 4\pi/3 & 0\end{bmatrix}^\top$. Note that the formation vectors $\alpha_1 = \begin{bmatrix}0 & 2\pi/3 & -2\pi/3\end{bmatrix}^\top$ and $\alpha_2 = \begin{bmatrix}0 & -4\pi/3 & 4\pi/3\end{bmatrix}^\top$ define the same formation since α and α_1 differ by the constant angle $2\pi/3$ and $\alpha_1 = \alpha_2 \bmod 2\pi$.

Finally, suppose that we want unicycles 1 and 2 to coincide on the circle, and we want unicycle 3 to be behind 1 and 2 on the antipodal point of the circle, as shown in part (c) of Figure 5. In this case, we have $\alpha = \begin{bmatrix}\pi & \pi & 0\end{bmatrix}^\top$. To illustrate the significance of the Laplacian matrix in the definition of the goal set Γ, suppose unicycle 1 can see unicycle 2, unicycle 2 can see unicycle 3, and unicycle 3 can see unicycle 1. In this case the Laplacian is circulant,

$$L = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Consider the formation in part (b) of Figure 5, where $\alpha = \begin{bmatrix}2\pi/3 & 4\pi/3 & 0\end{bmatrix}^\top$. Writing $L(x_3 - \alpha) = 0 \bmod 2\pi$ we get

$$\begin{aligned} x_3^1 - x_3^3 &= 2\pi/3 \bmod 2\pi \\ x_3^2 - x_3^3 &= 4\pi/3 \bmod 2\pi \\ x_3^2 - x_3^1 &= 2\pi/3 \bmod 2\pi. \end{aligned}$$

These relations identify precisely the formation in part (b) of Figure 5. One can readily check that the expressions $L(x_3 - \alpha_1) = 0 \bmod 2\pi$ and $L(x_3 - \alpha_2) = 0 \bmod 2\pi$, with α_1, α_2 defined as above, give the same result. $\triangle$

IV. Methodology to solve CFSP

In the previous section we have seen that solving CFSP amounts to designing distributed feedback laws for the n unicycles that stabilize the set Γ in (3). In this section we present the general methodology we will follow in solving this problem. Our approach is hierarchical and rests upon a so-called reduction theorem for asymptotic stability of sets.

A. Hierarchical control design approach

To manage the complexity of CFSP, we break down its solution into three simpler steps. Consider the following hierarchy of control specifications encoded in three nested goal sets $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3$:

spec 1: Design feedbacks $w_1^i(\chi), w_2^i(\chi)$, $i \in \mathbf{n}$, stabilizing a desired "kinematic behavior." In other words, stabilize the set

$$\Gamma_1 = \{\chi \in \mathcal{X} : x_4^i = u_2^i(\chi),\ x_5^i = u_1^i(\chi),\ i \in \mathbf{n}\}, \tag{4}$$

where $u_1^i(\chi)$, $u_2^i(\chi)$ are C^1 functions defined later. On Γ_1, the dynamic unicycles become purely kinematic, with new inputs u_1^i, u_2^i.

spec 2: Considering the kinematic motion on Γ_1, make the unicycles follow a common circle. This corresponds to stabilizing the set $\Gamma_2 \subset \Gamma_1$ defined as

$$\Gamma_2 = \{\chi \in \Gamma_1 : c^1(x^1) = \cdots = c^n(x^n),\ i \in \mathbf{n}\}, \tag{5}$$

and making sure that, on it, $u_2^i(\chi) = u_1^i(\chi)/r$, for all $i \in \mathbf{n}$.

spec 3: On Γ_2, make the unicycles locally converge to the desired formation with the required speed specifications. This corresponds to stabilizing the set $\Gamma_3 \subset \Gamma_2$ defined as

$$\begin{gathered}\Gamma_3 = \{\chi \in \Gamma_2 : L(x_3 - \alpha) = 0 \bmod 2\pi,\ x_4^i = v/r, \\ x_5^i = v, i \in \mathbf{n}\}.\end{gathered} \tag{6}$$

Note that Γ_3 is precisely the goal set Γ in (3).

The specifications above are hierarchical in the sense that, for $i = 2, 3$, specification i is met (i.e., $\chi \in \Gamma_i$) only if specification $i-1$ is met. Our control design will reflect the hierarchical nature of the specifications, and it will unfold in three steps:

step 1: Design feedbacks $w_1^i(\chi), w_2^i(\chi)$, $i \in \mathbf{n}$, that asymptotically stabilize Γ_1 in (4).

step 2: Set $u_2^i(\chi) = \frac{u_1^i(\chi)}{r} + \tilde{u}^i(\chi)$, $i \in \mathbf{n}$, and design $\tilde{u}^i(\chi)$ to stabilize Γ_2 in (5) relative[1] to Γ_1.
step 3: Finally, design $u_1^i(\chi)$, $i \in \mathbf{n}$, to asymptotically stabilize Γ_3 in (6) relative to Γ_2.

The question arising in this context is whether the three properties of Γ_1 being asymptotically stable, Γ_2 being asymptotically stable relative to Γ_1, and Γ_3 being asymptotically stable relative to Γ_2 imply that Γ_1, Γ_2, and Γ_3 are asymptotically stable for the closed-loop system. The answer is yes, under certain conditions. To make this precise, we will introduce basic notions of stability and boundedness, and we will review a so-called reduction theorem addressing the question posed above. Then, in Section V, we will follow the hierarchical design approach just outlined to solve CFSP.

B. Stability definitions and reduction theorems

Consider a dynamical system described by

$$\Sigma: \ \dot{x} = f(x) \tag{7}$$

with state space a domain $\mathcal{X} \subset \mathbb{R}^n$. Assume that f is locally Lipschitz on $\mathcal{X}$ and let $\mathbb{R}^+ = [0, +\infty)$. Let $\Gamma \subset \mathcal{X}$ be a closed positively invariant set for Σ in (7).

Definition IV.1 (Set stability and attractivity)**.** (i) Γ is *stable* for Σ if for all $\varepsilon > 0$ there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.

(ii) Γ is an *attractor* for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\lim_{t\to\infty} \|\phi(t, x_0)\|_\Gamma = 0$ for all $x_0 \in \mathcal{N}(\Gamma)$.

(iii) Γ is a *global attractor* for Σ if it is an attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$.

(iv) Γ is [*globally*] *asymptotically stable* for Σ if it is stable and attractive [globally attractive] for Σ.

Definition IV.2 (Local stability and attractivity near a set)**.** Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be closed positively invariant sets. The set Γ_2 is *locally stable near* Γ_1 if for all $x \in \Gamma_1$, for all $c > 0$, and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma_1)$ and all $t > 0$, whenever $\phi([0, t], x_0) \subset B_c(x)$ one has that $\phi([0, t], x_0) \subset B_\varepsilon(\Gamma_2)$. The set Γ_2 is *locally attractive near* Γ_1 if there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that, for all $x_0 \in \mathcal{N}(\Gamma_1)$, $\phi(t, x_0) \to \Gamma_2$ at $t \to +\infty$.

The definition of local stability can be rephrased as follows. Given an arbitrary ball $B_c(x)$ centred at a point x in Γ_1, trajectories originating in $B_c(x)$ sufficiently close to Γ_1 cannot travel far away from Γ_2 before first exiting $B_c(x)$. It is immediate to see that if Γ_1 is stable, then Γ_2 is locally stable near Γ_1, and therefore local stability of Γ_2 near Γ_1 is a necessary condition for the stability of Γ_1.

Definition IV.3 (Relative set stability and attractivity)**.** Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be closed positively invariant sets. We say that Γ_1 is *stable relative to* Γ_2 for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$. Similarly, one modifies all other notions in Definitions IV.1 and IV.2 by restricting initial conditions to lie in Γ_2.

Definition IV.4 (Local uniform boundedness (LUB))**.** System Σ is *locally uniformly bounded near* Γ *(LUB)* if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$.

We are now ready to state a result from [19], [20], [21] addressing the following question. Consider the dynamical system Σ in (7), and suppose that two closed sets $\Gamma_2 \subset \Gamma_1 \subset \mathcal{X}$ are positively invariant, and that Γ_2 is asymptotically stable relative to Γ_1. Under what conditions is Γ_2 asymptotically stable relative to the state space $\mathcal{X}$?

Theorem IV.5 (Reduction Principle for Asymptotic stability)**.** Let Γ_1 and Γ_2, $\Gamma_2 \subset \Gamma_1 \subset \mathcal{X}$, be two closed positively invariant sets for Σ in (7). Then, Γ_2 is [globally] asymptotically stable if the following conditions hold:

[1]By asymptotic stability of Γ_2 relative to Γ_1 it is meant that Γ_2 is asymptotically stable when the initial conditions are restricted to lie on Γ_1. This notion is defined precisely in Section IV-B.

(i) Γ_2 is [globally] asymptotically stable relative to Γ_1,
(ii) Γ_1 is locally stable near Γ_2,
(iii) Γ_1 is locally attractive near Γ_2 [Γ_1 is globally attractive],
(iv) if Γ_2 is unbounded, then Σ is LUB near Γ_2,
(v) [All trajectories of Σ are bounded]

Remark 1. Conditions (i), (ii), and (iii) in the theorem above are necessary. Additionally, one can show that if in the theorem above one has that $\Gamma_2 \subset \Gamma_1 \subset \Xi$, where Ξ is a closed positively invariant subset of $\mathcal{X}$, and conditions (ii)-(v) are relaxed by assuming that they hold relative to Ξ, then the conclusions of Theorem IV.5 hold relative to Ξ.

The reduction theorem above yields the following useful corollary, answering the question that was posed at the end of Section IV-A. Its simple proof is omitted.

Corollary IV.6. Let $\Gamma_3 \subset \Gamma_2 \subset \Gamma_1$ be closed subsets of $\mathcal{X}$ that are positively invariant for Σ in (7), and consider the following conditions:
(i) Γ_1 is asymptotically stable and, for $i = 1, 2$, Γ_{i+1} is asymptotically stable relative to Γ_i.
(ii) Γ_1 is globally asymptotically stable, Γ_2 is globally asymptotically stable relative to Γ_1, and Γ_3 is asymptotically stable relative to Γ_2.
(iii) Σ is LUB near Γ_3.
(iv) All trajectories of Σ are bounded.

Then, the following implications hold:
(a) (i) $\wedge$ (iii) $\implies$ $\Gamma_1, \Gamma_2, \Gamma_3$ are asymptotically stable for Σ.
(b) (ii) $\wedge$ (iii) $\wedge$ (iv) $\implies$ Γ_1, Γ_2 are globally asymptotically stable, and Γ_3 is asymptotically stable for Σ.

V. Control design

In this section we solve CFSP by following the three-step hierarchical approach presented in Section IV-A, and appealing to Corollary IV.6 to prove asymptotic stability of $\Gamma_1, \Gamma_2, \Gamma_3$ for the closed-loop system.

A. Specification 1: asymptotic stabilization of Γ_1

Recall the set Γ_1 in (4),

$$\Gamma_1 = \{\chi \in \mathcal{X} : x_4^i = u_2^i(\chi),\ x_5^i = u_1^i(\chi),\ i = 1 \in \mathbf{n}\},$$

where $u_1^i(\chi), u_2^i(\chi)$, $i \in \mathbf{n}$, are smooth functions to be assigned later. Let $e^i(\chi) = \begin{bmatrix} e_1^i(\chi) & e_2^i(\chi)) \end{bmatrix}^\top = \begin{bmatrix} x_4^i - u_2^i(\chi) & x_5^i - u_1^i(\chi) \end{bmatrix}^\top$, $i \in \mathbf{n}$, and $e(\chi) = \begin{bmatrix} e^1(\chi)^\top & \cdots & e^n(\chi)^\top \end{bmatrix}^\top$. Consider the following candidate Lyapunov function

$$V_1 = \frac{1}{2} \sum_{i=1}^{n} \|e^i\|^2.$$

Taking the time derivative of V along the dynamics (1) we get

$$\begin{aligned} \dot{V}_1 &= \sum_{i=1}^{n} (e^i)^\top \left[\dot{x}_4^i - \dot{u}_2^i(\chi) \quad \dot{x}_5^i - \dot{u}_1^i(\chi)\right]^\top \\ &= \sum_{i=1}^{n} (e^i)^\top \left[\frac{1}{J} w_2^i - \dot{u}_2^i(\chi) \quad \frac{R}{I + mR^2} w_1^i - \dot{u}_1^i(\chi)\right]^\top, \end{aligned}$$

where $\dot{u}_j^i(\chi) := \frac{\partial u_j^i(\chi)}{\partial \chi} \dot{\chi}$, $i \in \mathbf{n}$, $j = 1, 2$, and $\dot{\chi}$ is given by the dynamics in (1). By setting

$$\begin{aligned} w_1^i &= \frac{I + mR^2}{R} \left[\dot{u}_1^i(\chi) - K_i(x_5^i - u_1^i(\chi))\right] \\ w_2^i &= J \left[\dot{u}_2^i(\chi) - K_i(x_4^i - u_2^i(\chi))\right],\ i \in \mathbf{n}, \end{aligned} \tag{8}$$

where $K_i > 0$ are design constants, we get $\dot{V}_1 = -\sum_{i=1}^n K_i \|e^i\|^2$, and Γ_1 is globally asymptotically (in fact, exponentially) stable for the closed-loop system (1), (8) *provided* that the latter has no finite escape times. To this end, suppose that $u_1(\chi), u_2(\chi)$ are chosen to be uniformly bounded on $\mathcal{X}$. Our choice of feedback $w_1^i(\chi), w_2^i(\chi)$ guarantees that for any initial condition $\chi(0)$, $e^i(t)$, $i \in \mathbf{n}$ are bounded, and thus that $x_4^i(t), x_5^i(t)$ are bounded as well. In turn, referring to (1), this implies that $\dot{x}_1^i(t), \dot{x}_2^i(t), \dot{x}_3^i(t)$ are bounded, and therefore that $x_1^i(t), x_2^i(t), x_3^i(t)$ are well-defined for all $t \geq 0$ and all $i \in \mathbf{n}$, excluding the possibility of finite escape times. These observations are summarized in the next proposition.

Proposition V.1. Consider system (1) and let $u_1(\chi), u_2(\chi)$ be C^1 functions that are uniformly bounded on $\mathcal{X}$. Then, the feedbacks $w_1^i(\chi), w_2^i(\chi)$ in (8) guarantee that all solutions are globally defined, and the set Γ_1 is globally asymptotically stable for the closed-loop system (1), (8).

B. Specification 2: asymptotic stabilization of Γ_2 relative to Γ_1

The objective now is to design $u_1^i(\chi), u_2^i(\chi)$, $i \in \mathbf{n}$, so as to stabilize the set

$$\Gamma_2 = \{\chi \in \Gamma_1 : c^1(x^1) = \cdots = c^n(x^n),\ i \in \mathbf{n}\},$$

relative to Γ_1. To this end, note that the dynamics of (1) restricted to the set Γ_1 are those of n kinematic unicycles

$$\begin{aligned} \dot{x}_1^i &= u_1^i \cos x_3^i \\ \dot{x}_2^i &= u_1^i \sin x_3^i \\ \dot{x}_3^i &= u_2^i, \qquad i \in \mathbf{n} \end{aligned} \tag{9}$$

where now u_1^i, u_2^i are viewed as control inputs. For $i \in \mathbf{n}$, let

$$u_2^i(\chi) = \frac{u_1^i(\chi)}{r} + \tilde{u}^i(\chi),$$

where $\tilde{u}^i(\chi)$ will be now designed to stabilize Γ_2 for system (9), while $u_1^i(\chi)$, $i \in \mathbf{n}$, will be designed in the next section to stabilize Γ_3 relative to Γ_2. We will begin by solving the problem in the special case when the sensor digraph is undirected. Then we will generalize to the case of directed graphs.

When $\tilde{u}^i(\chi) = 0$ for all $i \in \mathbf{n}$, we have $u_2^i(\chi) = u_1^i(\chi)/r$, and thus the n unicycles rotate around circles of radius r, so that their centres of rotation $c^i(x^i)$ in (2) remain constant. This shows, in particular, that the set Γ_2 is invariant for any choice of $u_1^i(\chi)$, $i \in \mathbf{n}$, as long as $\tilde{u}^i(\chi) = 0$ on Γ_2 for all $i \in \mathbf{n}$. In order to stabilize Γ_2, we write the dynamics of the centres of rotation $c^i(x^i)$,

$$\dot{c} = -rS(x_3)^\top \tilde{u}(\chi),$$

where

$$S(x_3) = \text{blockdiag}\left\{[\cos x_3^1\ \ \sin x_3^1], \cdots, [\cos x_3^n\ \ \sin x_3^n]\right\}.$$

In order to design $\tilde{u}(\chi)$, we will begin by assuming that the sensor graph $\mathcal{G}$ is undirected, i.e., its Laplacian matrix L is symmetric. In this case, letting $V_2(\chi) = c(\chi)^\top L_{(2)} c(\chi)$, we have

$$\dot{V}_2(\chi) = -rc(\chi)^\top L_{(2)} S(x_3)^\top \tilde{u}(\chi).$$

If we define $\tilde{y} = -rS(x_3)L_{(2)}c(\chi)$, we see that $\dot{V}_2 = \tilde{y}^\top \tilde{u}$, and so system (9) with input $\tilde{u}$ and output $\tilde{y}$ is passive with storage function V. Moreover, the set $\{\chi \in \Gamma_1 : V_2(\chi) = 0\}$ is precisely Γ_2, the set we wish to stabilize. This observation suggests the definition of a passivity-based feedback $\tilde{u}(\chi) = K\phi(y)y$, where $y = S(x_3)L_{(2)}c(\chi)$, $K > 0$, and $\phi \in \mathsf{Sat}(\mathbb{R}^n)$. With this definition, we get

$$\dot{V}_2 = -rKy^\top \phi(y) y \leq 0.$$

Since

$$\dot{x}_3^i = u_2^i(\chi) = \frac{u_1^i(\chi)}{r} + K\phi_i(y_i)y_i,$$

we have

$$\dot{x}_3^i \geq \frac{\inf_{\chi\in\Gamma_1} u_1^i(\chi)}{r} - K.$$

If $u_1(\chi)$ is chosen so that $\inf_{\chi\in\Gamma_1} u_1^i(\chi) > v/2$, $\sup_{\chi\in\Gamma_1} u_1^i(\chi) < \infty$ for all $i \in \mathbf{n}$, and if $K < v/(2r)$, then there exist $\mu_1, \mu_2 > 0$ such that $0 < \mu_1 < \dot{x}_3^i < \mu_2$ for all $i \in \mathbf{n}$. Moreover, the boundedness of u_1 and the fact that x_3^i belongs to a compact set imply that all solutions of (9) are defined for all $t \geq 0$. This fact and the fact that $\dot{V}_2 \leq 0$ imply that the set $\{\chi \in \Gamma_1 : V_2(\chi) = 0\}$ is stable for (9). Let $\chi(0) \in \Gamma_1$ be arbitrary, and let $\chi(t)$ be the associated solution. One can easily see that the boundedness of $V_2(\chi(t))$ and the fact that $\dot{x}_3^i < \mu_2$ imply that $\ddot{V}_2(\chi(t))$ is bounded. By Barbalat's lemma, $y(t) = S(x_3(t))L_{(2)}c(\chi(t)) \to 0$ as $t \to \infty$. In components, $y_i(t) = \begin{bmatrix} \cos x_3^i(t) & \sin x_3^i(t) \end{bmatrix} L_{(2)}^i c(\chi(t)) \to 0$. Thus, $L_{(2)}^i c(\chi(t)) \to a(t) \begin{bmatrix} -\sin x_3^i(t) & \cos x_3^i(t) \end{bmatrix}^\top$, where $a(t)$ is a continuous scalar function. Since the function $t \mapsto V_2(\chi(t))$ is continuous, bounded from below, and nonincreasing, it has a finite limit, implying that $L_{(2)}c^i(\chi(t))$ has a finite limit. Since $\dot{x}_3^i > \mu_1 > 0$, the only way that $a(t)\begin{bmatrix} -\sin x_3^i(t) & \cos x_3^i(t) \end{bmatrix}^\top$ may have a finite limit is that $a(t) \to 0$ as $t \to \infty$. This proves that for any $\chi(0) \in \Gamma_1$, $L_{(2)}c(\chi(t)) \to 0$ as $t \to \infty$ or, what is the same, the set $\{\chi \in \Gamma_1 : V_2(\chi) = 0\}$ is globally attractive for (9), and hence also globally asymptotically stable. In conclusion, Γ_2 is globally asymptotically stable relative to Γ_1 for (1). These considerations are summarized in the next proposition.

Proposition V.2. Consider the kinematic unicycles in (9) representing the motion of the dynamic unicycles on the set Γ_1, and assume that the sensor digraph $\mathcal{G}$ is undirected. For $i \in \mathbf{n}$, let $u_1^i(\chi)$ be a C^1 function such that $\inf_{\chi\in\Gamma_1} u_1^i(\chi) > v/2$ and $\sup_{\chi\in\Gamma_1} u_1^i(\chi) < \infty$, and let $\phi \in \mathsf{Sat}(\mathbb{R}^n)$. Then, for all $K \in \big(0, v/(2r)\big)$, the feedback law

$$\begin{aligned} u_1 &= u_1(\chi) \\ u_2 &= \frac{u_1(\chi)}{r} + K\phi(y)y, \quad y = S(x_3)L_{(2)}c(\chi), \end{aligned} \tag{10}$$

globally asymptotically stabilizes the set $\{\chi : c^1(\chi) = \cdots = c^n(\chi)\}$ for (9). Therefore, the feedback laws in (8), with $u_1(\chi), u_2(\chi)$ as in (10), render Γ_2 globally asymptotically stable relative to Γ_1 for (1).

Remark 2. In (10), the feedback $u_2^i(\chi)$ for unicycle i is given by

$$u_2^i(\chi) = \frac{u_1^i(\chi)}{r} + K\phi_i(y_i)y_i, \; y_i = \begin{bmatrix} \cos x_3^i & \sin x_3^i \end{bmatrix} L_{(2)}^i c(\chi).$$

If, for each $i \in \mathbf{n}$, we denote by $N(i)$ the set of nodes in $\mathcal{G}$ connected to node i through an edge with tail at i, then $y_i = \begin{bmatrix} \cos x_3^i & \sin x_3^i \end{bmatrix} \sum_{j\in N(i)} c^i(\chi) - c^j(\chi)$. Using the definition of c^i in (2), one has that

$$y_i = \sum_{j\in N(i)} -x_{ij} + r\sin(x_3^j - x_3^i),$$

where x_{ij} and $x_3^j - x_3^i$ are depicted in Figure 2. We see that the computation of the term $\phi_i(y_i)y_i$ in $u_2^i(\chi)$ requires only the measurement of the relative heading angle of unicycles that are visible to unicycle i, and of the projection of the relative displacement of said unicycles onto the heading axis of unicycle i.

When the sensor graph is directed, $\tilde{u}(\chi)$ defined earlier is no longer a passivity-based feedback and one cannot use the foregoing analysis to generalize Proposition V.2. Indeed, the storage function becomes

$$V_2(\chi) = \frac{1}{2}c(\chi)^\top (L_{(2)} + L_{(2)}{}^\top)c(\chi),$$

and it is no longer true, in general, that $\{\chi \in \Gamma_1 : V_2(\chi) = 0\} = \Gamma_2$ because $L^\top$ may not be the Laplacian of a graph. It turns out [22] that under our connectivity assumption, $\{\chi \in \Gamma_1 : V_2(\chi) = 0\} = \Gamma_2$ if and only if the sensor digraph is balanced, i.e, if the in-degree of each node is equal to its out-degree. The second, more fundamental, issue is that the passive output corresponding to the storage above is $y = -rS(x_3)(L_{(2)} + L_{(2)}{}^\top)c(\chi)$. The computation of this function would require information which is not compatible with the sensor digraph, and so a passivity-based feedback would not be distributed! Nonetheless, we will now show that the feedback in (10) can still be used to asymptotically stabilize Γ_2 relative to Γ_1. Let again

$$\tilde{u}(\chi) = K\phi(y)y, \quad y = S(x_3)L_{(2)}c(\chi), \tag{11}$$

where $K \in \big(0, v/(2r)\big)$ and $\phi \in \mathsf{Sat}(\mathbb{R}^n)$. The dynamics of the centres of rotation read as

$$\dot{c} = -rK\phi_{(2)}(S(x_3)L_{(2)}c)R(x_3)L_{(2)}c, \tag{12}$$

where $R(x_3) = \text{blockdiag}(R_1(x_3^1), \ldots, R_n(x_3^n))$ and

$$R_i(x_3^i) = \begin{bmatrix} \cos^2 x_3^i & \sin x_3^i \cos x_3^i \\ \sin x_3^i \cos x_3^i & \sin^2 x_3^i \end{bmatrix}.$$

We will view (12) as a time-varying system, where the time dependence of the vector field is brought about by the signal $x_3(t)$. We will use averaging theory [23], [24] to show that the subspace $\{c : c^1 = \cdots = c^n\}$ is asymptotically stable for (12). To this end, for an arbitrary $\chi(0) \in \Gamma_1$ consider the averaged system

$$\dot{c}_{\text{avg}} = -rK\bar{R}(L_{(2)}c_{\text{avg}})L_{(2)}c_{\text{avg}}, \tag{13}$$

where

$$\bar{R}(L_{(2)}c_{\text{avg}}) := \text{blockdiag}\{\bar{R}_1(L_{(2)}^1 c_{\text{avg}}), \cdots, \bar{R}_n(L_{(2)}^n c_{\text{avg}})\},$$

and

$$\begin{aligned}\bar{R}_i(L_{(2)}^i c_{\text{avg}}) := \lim_{T\to\infty} \frac{1}{T}\int_0^T & \phi_i([\cos x_3^i(\tau) \;\; \sin x_3^i(\tau)]L_{(2)}^i c_{\text{avg}})\cdot \\ & \cdot R_i(x_3^i(\tau))d\tau.\end{aligned}$$

We will now show that for each fixed c_{avg} and for all $i \in \mathbf{n}$, $\bar{R}_i(L_{(2)}^i c_{\text{avg}})$ is a positive definite matrix. Fix c_{avg} and consider the integral $(1/T)\int_0^T \phi_i(\cdot)\cos^2(x_3^i(\tau))d\tau$ in the expression of $\bar{R}_i$. Let $\underline{\phi}_i := \min_{x_3^i \in S^1} \phi_i([\cos x_3^i(\tau) \;\; \sin x_3^i(\tau)]L_{(2)}^i c_{\text{avg}})$. Since $\phi_i > 0$ and S^1 is compact, it holds that $\underline{\phi}_i > 0$. Since $x_3^i(t)$ appears inside a cosine in the argument of the integral, we can take $x_3^i(t)$ to be a function $\mathbb{R} \to \mathbb{R}$, rather than a function $\mathbb{R} \to S^1$. Then, the fact that $\dot{x}_3^i \geq \mu_1 > 0$ implies that the map $t \mapsto x_3^i(t)$ is a diffeomorphism, and so we can perform a change of variables $s = x_3^i(t)$, obtaining

$$\begin{aligned}&\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi_i(\cdot)\cos^2(x_3^i(\tau))d\tau \\ &\qquad \geq \underline{\phi}_i \lim_{T\to\infty} \frac{1}{T}\int_{x_3^i(0)}^{x_3^i(T)} \frac{\cos^2 s}{\dot{x}_3^i((x_3^i)^{-1}(s))}ds.\end{aligned}$$

Using the inequality $\dot{x}_3^i \leq \mu_2$ and the fact that $x_3^i(T) \to \infty$ as $T \to \infty$, we conclude that, for all $i \in \mathbf{n}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi_i(\cdot)\cos^2(x_3^i(\tau))d\tau \geq \frac{1}{2\mu_2}.$$

Now we show that $\det \bar{R}_i(\cdot) > 0$. Since $\phi_i(\cdot) > 0$, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}&\left(\int_0^T \phi_i(\cdot)\sin x_3^i(\tau)\cos x_3^i(\tau)d\tau\right)^2 \\ &\leq \left(\int_0^T \phi_i(\cdot)\sin^2 x_3^i(\tau)d\tau\right)\left(\int_0^T \phi_i(\cdot)\cos^2 x_3^i(\tau)d\tau\right),\end{aligned} \tag{14}$$

with equality holding if and only if $\sin x_3^i(t) = \cos x_3^i(t)$ for all t. However, $\dot{x}_3^i > \mu_1 > 0$ so the latter identity cannot hold, proving that relation (14) holds with strict inequality. This implies that for each fixed $c_{\text{avg}} \in \mathbb{R}^{2n}$, the 2×2 symmetric matrix $\bar{R}_i(L_{(2)}^i c_{\text{avg}})$ is positive definite. Moreover, since $\dot{x}_3^i$ is uniformly bounded away from zero for all $\chi(0) \in \Gamma_1$, so are the eigenvalues of $\bar{R}(L_{(2)}c_{\text{avg}})$ for each fixed c_{avg}.

By Lemma II.1, the Laplacian matrix L has a simple eigenvalue at 0, and all its other eigenvalues have positive real part, so $-L_{(2)}$ has 2 eigenvalues at 0 with geometric multiplicity 2, and its remaining eigenvalues have negative real part. Since, as we have seen, for each fixed $c_{\rm avg} \in \mathbb{R}^{2n}$ the matrix $\bar{R}(L_{(2)}c_{\rm avg})$ is positive definite, and since the digraph $\mathcal{G}$ has a globally reachable node, Lemma 4 in [13] implies that for each fixed $c_{\rm avg}$ the matrix $-\bar{R}(L_{(2)}c_{\rm avg})L_{(2)}$ has the same spectral properties of $-L_{(2)}$, namely it has two eigenvalues at 0 with geometric multiplicity 2 and all other eigenvalues with negative real part. We now show that this fact implies the exponential stability of Γ_2. Consider the coordinate transformations

$$z = P^{-1}c, \; z_{\rm avg} = P^{-1}c_{\rm avg},$$

where $P \in \mathbb{R}^{2n\times 2n}$ is defined as $P = [\mathbf{1}\; e_2 \; \cdots \; e_n] \otimes I_2$, where $e_2, \ldots, e_n$ are the last $n-1$ vectors in the natural basis of $\mathbb{R}^n$. System (12) after coordinate transformation becomes

$$\dot{z} = -rKP^{-1}\Big(\phi_{(2)}(S(x_3)L_{(2)}Pz)R(x_3)L_{(2)}\Big)Pz, \tag{15}$$

while the averaged system in (13) becomes

$$\dot{z}_{\rm avg} = -rKP^{-1}\Big(\bar{R}(L_{(2)}c_{\rm avg})L_{(2)}\Big)Pz_{\rm avg}.$$

The matrices $P^{-1}\phi_{(2)}(\cdot)R(x_3)L_{(2)}P$ and $P^{-1}\bar{R}(\cdot)L_{(2)}P$ have the following structure

$$\begin{aligned} P^{-1}\phi_{(2)}(\cdot)R(x_3)L_{(2)}P &= \begin{bmatrix} 0_{2\times 2} & A_{12}(x_3(t), L_{(2)}Pz) \\ 0_{2n-2\times 2} & A_{22}(x_3(t), L_{(2)}Pz) \end{bmatrix}, \\ P^{-1}\bar{R}(\cdot)L_{(2)}P &= \begin{bmatrix} 0_{2\times 2} & \bar{A}_{12}(L_{(2)}Pz_{\rm avg}) \\ 0_{2n-2\times 2} & \bar{A}_{22}(L_{(2)}Pz_{\rm avg}) \end{bmatrix}. \end{aligned}$$

Partitioning z and $z_{\rm avg}$ as $z = \begin{bmatrix}\bar{z}^\top & \tilde{z}^\top\end{bmatrix}^\top$, $z_{\rm avg} = \begin{bmatrix}\bar{z}_{\rm avg}{}^\top & \tilde{z}_{\rm avg}{}^\top\end{bmatrix}^\top$, with $\bar{z}, \bar{z}_{\rm avg} \in \mathbb{R}^2$ and $\tilde{z}, \tilde{z}_{\rm avg} \in \mathbb{R}^{2n-2}$, and using the fact that the terms $L_{(2)}Pz$ and $L_{(2)}Pz_{\rm avg}$ are independent of $\bar{z}$ and $\bar{z}_{\rm avg}$, we have

$$\begin{aligned} \dot{\bar{z}} &= -rKA_{12}(x_3(t), \tilde{z})\tilde{z} \qquad & \dot{\bar{z}}_{\rm avg} &= -rK\bar{A}_{12}(\tilde{z}_{\rm avg})\tilde{z}_{\rm avg} \\ \dot{\tilde{z}} &= -rKA_{22}(x_3(t), \tilde{z})\tilde{z} \qquad & \dot{\tilde{z}}_{\rm avg} &= -rK\bar{A}_{22}(\tilde{z}_{\rm avg})\tilde{z}_{\rm avg}. \end{aligned}$$

In light of the discussion above, for each fixed $\tilde{z}_{\rm avg}$ the matrix $-\bar{A}_{22}(\tilde{z}_{\rm avg})$ is Hurwitz. For all $K > 0$, the equilibrium $\tilde{z}_{\rm avg} = 0$ is exponentially stable for the $\tilde{z}_{\rm avg}$ subsystem since its linearization is

$$\dot{\tilde{z}}_{\rm avg} = -rK\bar{A}_{22}(0)\tilde{z}_{\rm avg},$$

and the matrix $-\bar{A}_{22}(0)$ is Hurwitz. By the averaging theorem [23], [24], there exists $K^\star \in (0, v/(2r))$ such that for all $K \in (0, K^\star)$, the origin of the $\tilde{z}$ subsystem is exponentially stable, implying that the subspace $\{(\bar{z}, \tilde{z}) : \tilde{z} = 0\}$ is exponentially stable. Note that the matrix-valued function $(x_3, \tilde{z}) \mapsto A_{12}(x_3, \tilde{z})$ is bounded because $\phi_{(2)}(\cdot)$ and $R(x_3)$ in (15) are bounded functions. Therefore, the exponential convergence of $\tilde{z}(t)$ to zero implies that $\bar{z}(t)$ has a finite limit, so that all solutions in a neighbourhood of the subspace $\{(\bar{z}, \tilde{z}) : \tilde{z} = 0\}$ converge to a point on the subspace. Returning to dynamics in c-coordinates, for all $K \in (0, K^\star)$, the subspace $\{c : c^1 = \cdots = c^n\}$ is exponentially stable for (12) and all solutions in a neighbourhood of said subspace converge to a point on it. The boundedness of $c(\chi(t))$ rules out finite escape times for (9), and so the exponential stability of $\{c : c^1 = \cdots = c^n\}$ for (12) implies the exponential asymptotic stability of $\{\chi : c^1(x^1) = \cdots = c^n(x^n)\}$ for (9). In turn, this implies that Γ_2 is exponentially stable relative to Γ_1 for (1). The considerations above are summarized in the next

Proposition V.3. Consider the kinematic unicycles in (9). For $i \in \mathbf{n}$, let $u_1^i(\chi)$ be a C^1 function such that $\inf_{\chi\in\Gamma_1} u_1^i(\chi) > v/2$ and $\sup_{\chi\in\Gamma_1} u_1^i(\chi) < \infty$, and let $u_1(\chi), u_2(\chi)$ be defined as in (10). Then, there exists $K^\star \in \big(0, v/(2r)\big)$ such that for all $K \in (0, K^\star)$ the set $\{\chi : c^1(x^1) = \cdots = c^n(x^n)\}$ is asymptotically stable for (9). Therefore, the feedback laws in (8), with $u_1(\chi), u_2(\chi)$ as in (10), render Γ_2 asymptotically stable relative to Γ_1 for (1).

Remark 3. While the above proposition only asserts asymptotic stability of Γ_2, we in fact conjecture that Γ_2 is *globally* asymptotically stable. In the special case when the sensor graph $\mathcal{G}$ is undirected, Proposition V.2 shows that our conjecture is correct. For the general case of directed graphs, a proof is not known. We envision two possible lines of attack to prove our conjecture. First, one could show that the equilibrium $\tilde{z}_{\text{avg}} = 0$ is globally asymptotically stable for the system $\dot{\tilde{z}}_{\text{avg}} = -rK\bar{A}_{22}(\tilde{z}_{\text{avg}})\tilde{z}_{\text{avg}}$. Alternatively, if one could prove that the matrices $\bar{R}_i$ are diagonal, i.e., that $\lim_{T\to\infty}(1/T)\int_0^T \phi_i(\cdot)\sin x_3^i(\tau)\cos x_3^i(\tau)d\tau = 0$, then the averaged system (13) would satisfy the strict subtangentiality assumption of [25], and Theorem 3.8 in [25] would imply that the subspace $\{c_{\text{avg}} : c^1_{\text{avg}} = \cdots = c^n_{\text{avg}}\}$ is globally asymptotically stable for (13). From this, our existing analysis would allow one to conclude global asymptotic stability of Γ_2.

C. *Specification 3: asymptotic stabilization of* Γ_3 *relative to* Γ_2

Given a formation vector α, we are now left with the objective of designing $u_1^i(\chi)$, $i \in \mathbf{n}$, so as to stabilize the set

$$\Gamma_3 = \{\chi \in \Gamma_2 : L(x_3 - \alpha) = 0 \bmod 2\pi, x_4^i = v/r, x_5^i = v, i \in \mathbf{n}\},$$

relative to Γ_2. To this end, note that on Γ_2 all unicycles lie on a common circle of radius r with fixed centre. Their motion, therefore, is completely characterized by their displacements along the circle or, equivalently, by their heading angles x_3^i. More precisely, note that if the functions $u_1^i(\chi)$, $u_2^i(\chi)$ do not depend on (x_4^i, x_5^i), $i \in \mathbf{n}$, then the map $\mathcal{X} \to (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2) \times \cdots \times (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2)$ defined as

$$\begin{aligned}
&\chi = (x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^n)\\
&y^i = (c^i(x^i), x_3^i, e^i(\chi)),\ e^i(\chi) = (x_4^i - u_2^i(\chi), x_5^i - u_1^i(\chi)),
\end{aligned}$$

is a diffeomorphism. In new coordinates, the motion of the unicycles on Γ_2 is given by

$$\begin{aligned}
\dot{c}^i &= 0\\
\dot{x}_3^i &= \frac{u_1^i(\chi)}{r}\\
\dot{e}^i &= 0, \quad i \in \mathbf{n}.
\end{aligned}$$

Thus, to stabilize Γ_3 relative to Γ_2 we need to design $u_1(\chi)$ to meet two objectives:

(a) Stabilize the set $\mathcal{S} = \{x_3 : L(x_3 - \alpha) = 0 \bmod 2\pi\}$ for the system $\dot{x}_3 = u_1/r$. As we have seen in Section III, this objective corresponds to making the relative heading angles $x_3^i - x_3^{i+1}$ converge to fixed constants. This is a consensus problem on the n-torus, a problem studied, for instance, in [17], [26].

(b) On the set $\{\chi \in \Gamma_2 : x_3 \in \mathcal{S}\}$ it must hold that $u_1^i(\chi) = v$ for all $i \in \mathbf{n}$. This implies that, on this set, $u_2^i(\chi) = v/r$, and therefore that $x_4^i = v/r$, $x_5^i = v$. In other words, this ensures that the stabilization of the set $\mathcal{S}$ for system $\dot{x}_3 = u_1/r$ is equivalent to the stabilization of Γ_3 relative to Γ_2.

To meet the two objectives above we propose the control law

$$u_1^i(\chi) = v - k_i \sin(L^i(x_3 - \alpha)), \quad i \in \mathbf{n}, \tag{16}$$

where $k_i > 0$ is a design parameter. It is obvious that this control law meets objective (b). Proposition V.4 below shows that it also meets objective (a).

Proposition V.4. For any $k_i > 0$, the control law (16) stabilizes the set $\mathcal{S} = \{x_3 : L(x_3 - \alpha) = 0 \bmod 2\pi\}$ for the system $\dot{x}_3 = u_1/r$. Therefore, the feedback laws in (8), with $u_1(\chi)$ as in (16) and $u_2(\chi)$ as in (10), render Γ_3 asymptotically stable relative to Γ_2 for the dynamic unicycles in (1).

Remark 4. When $\alpha = 0$, the set Γ_3 coincides with the consensus manifold $\{x_3 \in T^n : x_3^1 = \cdots = x_3^n\}$. Therefore, in the consensus framework the result above can be restated as follows: for any $k_i > 0$, the control laws $u^i = -k_i \sin(L^i x_3)$, $i \in \mathbf{n}$, asymptotically stabilize the set $\{x_3 \in T^n : x_3^1 = \cdots = x_3^n\}$, and hence achieve consensus, provided that the sensor digraph with Laplacian matrix L has a globally reachable node. Note that in place of (16), one could let $u_1^i(\chi) = v - k_i \sum_{j\in N(i)} \sin(x_3^i - x_3^j - \alpha_i + \alpha_j)$ to obtain the dynamics of the Kuramoto coupled oscillator model used in [17], [26], with an offset to control the spacing of unicycles. The fact that such a control law asymptotically stabilizes Γ_3 for general digraphs with a globally reachable node was proved, e.g., in [25].

Remark 5. To compute the control law in (16), unicycle i need only measure its relative heading angle with respect to unicycles that are visible to it.

Proof of Proposition V.4: Let $s(x_3) = \begin{bmatrix}\sin(L^1(x_3-\alpha)) & \cdots & \sin(L^n(x_3-\alpha))\end{bmatrix}^\top$. The derivative of this function along solutions of $\dot{x}_4 = u_1(\chi)/r$ is given by

$$\dot{s} = \frac{1}{r}\begin{bmatrix}\cos(L^1(x_3-\alpha)) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \cos(L^n(x_3-\alpha))\end{bmatrix} L u_1(\chi).$$

Now we substitute in the control law in (16), which in vector form reads as $u_1(\chi) = v\mathbf{1} - ks(x_3)$, where $k = \mathrm{diag}(k_1,\dots,k_n)$. Since $\mathbf{1}$ is an eigenvector of L associated to the eigenvalue at 0, we have

$$\dot{s} = -\frac{k}{r}\begin{bmatrix}\cos(L^1(x_3-\alpha)) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \cos(L^n(x_3-\alpha))\end{bmatrix} Ls,$$

which can be rewritten by isolating the linear part of the vector field,

$$\dot{s} = -\frac{k}{r}Ls - \frac{k}{r}\Delta(x_3)Ls,$$

where $\Delta(x_3) = \mathrm{diag}(\cos(L^1(x_3-\alpha))-1,\dots,\cos(L^n(x_3-\alpha))-1)$ vanishes on $\mathcal{S}$. Recall that by our connectedness assumption on the sensor digraph and Lemma II.1, L has one eigenvalue at zero and all its other eigenvalues have positive real part. Since k is diagonal and positive definite, Lemma 4 in [13] implies that kL has the same properties. As we did in Section V-B, we now define a coordinate transformation to quotient out the dynamics associated to the zero eigenvalue of L. Let $P = [\mathbf{1}\ e_2\ \cdots\ e_n]$, and define the transformation $\mathbb{R}^n \to \mathbb{R}\times\mathbb{R}^{n-1}$, $s \mapsto \begin{bmatrix}\bar{s} & \tilde{s}^\top\end{bmatrix}^\top = P^{-1}s$ which gives

$$\begin{aligned}\dot{\bar{s}} &= A_{21}\tilde{s} + \Delta_1(x_3)\tilde{s}\\ \dot{\tilde{s}} &= A_{22}\tilde{s} + \Delta_2(x_3)\tilde{s},\end{aligned}$$

where $\begin{bmatrix}\Delta_1(x_3) & \Delta_2(x_3)^\top\end{bmatrix}^\top = -(k/r)P^{-1}\Delta(x_3)LP$ and the matrix A_{22} is Hurwitz. Since Δ_1, Δ_2 are uniformly bounded functions, all solutions of the system above are defined for all $t \geq 0$. The $\tilde{s}$ subsystem is composed of two terms: an asymptotically stable LTI nominal part, $A_{22}\tilde{s}$, and a perturbation, $\Delta_2(x_3)\tilde{s}$, with the property that $\Delta_2(x_3) = 0$ when $L(x_3-\alpha) = 0$. Letting

$$\mathcal{N} = \{x_3 : \cos(L^i(x_3-\alpha)) > 0,\ i = 1,\dots,n\},$$

we claim that

$$\mathcal{S} = \{x_3 : L(x_3-\alpha) = 0 \bmod 2\pi\} = \{x_3 \in \mathcal{N} : \tilde{s}(x_3) = 0\}.$$

Obviously, $\{x_3 : L(x_3-\alpha) = 0 \bmod 2\pi\} \subset \{x_3 \in \mathcal{N} : \tilde{s}(x_3) = 0\}$. Suppose that $x_3 \in \mathcal{N}$ is such that $\tilde{s}(x_3) = 0$. Then, $s(x_3) = \lambda\mathbf{1}$, for some $\lambda \in \mathbb{R}$ or, since $x_3 \in \mathcal{N}$, $L(x_3-\alpha) = \arcsin\lambda\mathbf{1}$. Let p denote the left eigenvector of L corresponding to the zero eigenvalue, so that $p^\top L = 0$. Then, p has nonnegative entries (for instance, see Lemma 4 in [27]) and therefore $p^\top\mathbf{1} > 0$, implying that $\arcsin\lambda = 0$, so that $L(x_3-\alpha) = 0 \bmod 2\pi$. This proves the claim.

Since $\mathcal{N}$ is a neighborhood of the set $\mathcal{S}$ in the n-torus, the result we just proved implies that the asymptotic stability of $\mathcal{S}$ for $\dot{x}_3 = u_1(\chi)/r$ is equivalent to the asymptotic stability of $\tilde{s} = 0$ for the $\tilde{s}$

subsystem. Let R be the positive definite solution of Lyapunov's equation $A_{22}{}^{\top}R + RA_{22} = -I_{n-1}$, and define $W(x_3) = \tilde{s}(x_3)^{\top}R\tilde{s}(x_3)$. We have

$$\begin{aligned}\dot{W} &= -\|\tilde{s}\|^2 + 2\tilde{s}^{\top}R\Delta_2(x_3)\tilde{s} \\ &\le -\left[1 - M\|\Delta_2(x_3)\|_{\infty}\right]\|\tilde{s}\|^2\end{aligned}$$

for some positive scalar M. Since $\Delta_2(x_3) = 0$ on $\mathcal{S}$, and since we have shown that $\mathcal{S} = \{x_3 \in \mathcal{N} : \tilde{s}(x_3) = 0\}$, there exists a neighbourhood $\mathcal{W}$ of $\mathcal{S}$, with $\mathcal{S} \subset \mathcal{W} \subset \mathcal{N}$, such that $\|\Delta_2(x_3)\|_{\infty} < 1/M$ on $\mathcal{W}$, and therefore the set $\{x_3 \in \mathcal{N} : \tilde{s}(x_3) = 0\}$ is asymptotically stable for the $\tilde{s}$ subsystem or, what is the same, $\mathcal{S}$ is asymptotically stable for $\dot{x}_3 = u_1(\chi)/r$. ∎

D. Solution of CFSP

In the previous sections we have designed feedbacks inducing the following three properties on the dynamic unicycles:

(a) The set Γ_1 in (4) is globally asymptotically stable (Proposition V.1). Hence, for all initial conditions, the dynamic unicycles converge to a desired "kinematic behavior."

(b) When the sensor digraph is undirected, the set Γ_2 in (5) is globally asymptotically stable relative to Γ_1 (Proposition V.2), and in particular for all initial conditions on Γ_1 the unicycles converge to a common circle whose centre depends on the initial condition. When the sensor digraph is directed, Proposition V.3 proves that Γ_2 is asymptotically stable relative to Γ_1.

(c) The set Γ_3 in (6) is asymptotically stable relative to Γ_2 (Proposition V.4). Hence, for all initial conditions in Γ_2 near Γ_3, the unicycles converge to a desired formation expressed by a desired ordering and spacing on the circle. In so doing, the unicycles do not leave the circle.

Now, using Corollary IV.6, we are ready to solve CFSP.

Theorem V.5. Consider the dynamic unicycles in (1), and assume that the sensor digraph $\mathcal{G}$ has a globally reachable node. Let $\phi \in \mathsf{Sat}(\mathbb{R}^n)$. Then, there exists $K^\star \in \big(0, (v/2r)\big)$ such that for all $K \in (0, K^\star)$ and all $k_i \in (0, v/2)$, $K_i > 0$, $i \in \mathbf{n}$, the feedback laws

$$\begin{aligned} w_1^i &= \frac{I + mR^2}{R}\left[\dot{u}_1^i(\chi) - K_i(x_5^i - u_1^i(\chi))\right] \\ w_2^i &= J\left[\dot{u}_2^i(\chi) - K_i(x_4^i - u_2^i(\chi))\right],\ i \in \mathbf{n}, \end{aligned}$$

where, for $i \in \mathbf{n}$,

$$\begin{aligned} u_1^i(\chi) &= v - k_i \sin(L^i(x_3 - \alpha)) \\ u_2^i(\chi) &= \frac{u_1^i(\chi)}{r} + K\phi_i(y_i)y_i,\ y_i = \begin{bmatrix}\cos x_3^i & \sin x_3^i\end{bmatrix} L^i_{(2)} c(\chi), \end{aligned} \tag{17}$$

and $c(\chi) = \begin{bmatrix} c^1(x^1)^{\top} & \cdots & c^n(x^n)^{\top}\end{bmatrix}^{\top}$,

$$c^i(x^i) = \begin{bmatrix} x_1^i - r\sin x_3^i & x_2^i + r\cos x_3^i\end{bmatrix}^{\top},$$

solve CFSP, yielding the following properties:

(a) The set Γ_1 in (4) is globally asymptotically stable, and the sets Γ_2, Γ_3 in (5), (6) are asymptotically stable for the closed-loop system. Additionally, if $\mathcal{G}$ is undirected, Γ_2 is globally asymptotically stable.

(b) For any initial condition in a neighbourhood of Γ_2, the unicycles converge to a common stationary circle of radius r, whose centre depends on the initial condition. Their linear speed on the circle is $x_5^i = v$, $i \in \mathbf{n}$. If $\mathcal{G}$ is undirected, then this property holds for any initial condition.

Remark 6. The feedbacks in the theorem above are distributed in the sense described in part (iii) of the statement of CFSP. As we pointed out in Remarks 2 and 5, in order to compute $u_1^i(\chi)$ and $u_2^i(\chi)$, unicycle i needs its relative displacement (measured in its own local frame) and relative heading with respect to unicycles that are visible to it. Then, in order to compute the control values w_1^i, w_2^i, unicycle i needs to measure its own angular and linear speeds x_4^i, x_5^i, and it must compute $\dot{u}_1^i(\chi), \dot{u}_2^i(\chi)$. It is easy to see that this latter computation requires the additional measurements of the relative angular speeds with respect to unicycles that are visible to unicycle i according to $\mathcal{G}$. In conclusion, the feedback in the theorem above meets requirement (iii) of CFSP.

Remark 7. We conjecture that the proposed feedback *globally* asymptotically stabilizes Γ_2 even when $\mathcal{G}$ is a directed graph. In Remark 3 we have provided two avenues of investigation to generalize the result of Proposition V.2 to the case of directed graphs. Such a generalization would automatically yield global asymptotic stability of Γ_2 in the above theorem.

Remark 8. The solution of CFSP presented in Theorem IV.5 has a simple intuitive explanation. An inner velocity feedback loop, the feedbacks $w_1^i(\chi)$ and $w_2^i(\chi)$, makes the linear and angular speeds of the unicycle track references $u_1^i(\chi(t))$, $u_2^i(\chi(t))$. An outer feedback loop computes the reference signals $u_1^i(\chi(t))$, $u_2^i(\chi(t))$ as follows. The linear velocity feedback u_1^i is the sum of the desired steady-state speed v plus a correction term that computes the average relative heading of unicycle i with respect to its neighbours in the sensor digraph. This average is compared to the average of the desired angle separations between the unicycle headings. Thus, u_1^i makes unicycle i speed up or slow down in such a way that its average relative heading angle with respect to neighbouring unicycles in the sensor digraph coincides with the desired average angle separation. On the circle, this guarantees that unicycle i meets its own formation specification. The $\sin(\cdot)$ function in u_1^i guarantees that the correction term is computed up to angle differences of multiples of 2π. The angular speed feedback u_2^i has two terms. The first term, $u_1^i(\chi)/r$, makes the unicycle move around a circle of radius r. The second term controls the centre of rotation c^i in such a way that it asymptotically approaches the centres of rotation of other unicycles. The functions $\phi_i(\cdot) \in \mathsf{Sat}(\mathbb{R})$ guarantee that the angular speed of unicycle i is always positive, so the unicycle is guaranteed to travel around the circle in the counter-clockwise direction. It also plays a role in the global convergence of the centres of rotation. In conclusion, the linear speeds u_1^i are used to control the formation spacings, while the angular speeds u_2^i are used to steer the unicycles to a common circle. All of the above is done in a distributed fashion.

Remark 9. The proposed solution *simultaneously* stabilizes the three sets $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3$. One of the consequences of this feature is this. Suppose that the unicycles are in steady-state, traveling in formation around a common circle of radius r. If one changes the formation vector α on the fly to reconfigure the formation, and if the unicycles are within the domain of attraction of the new set Γ_3 resulting from the change in α, one is guaranteed that the unicycles will converge to the new formation without leaving the common circle. To make sure that the unicycles remain in the domain of attraction of Γ_3 as α is changed, it suffices to change α in small increments, or sufficiently slowly.

Remark 10. The work in [15] solves a circular formation stabilization problem for kinematic unicycles which is similar to CFSP, with two differences: in [15], one cannot arbitrarily specify the ordering and spacing of unicycles on the circle. Moreover, the computation of the control law in [15] requires all-to-all communication between unicycles. The control law in [15] is derived using a potential function approach. A spacing potential is used to make unicycles converge to a common circle. A phase potential is used to induce a number of different symmetric formations on the circle. All of this is achieved by only controlling the angular speed of the unicycles (their linear speed is assumed to be one). In contrast to the work in [15], we use the linear speeds of the unicycles to stabilize the formation, controlling both the distances and ordering of unicycles in a distributed manner. A consequence of the assumption, in [15], that the unicycles have unit speed is that when using phase potentials, the set where the centres of the unicycles coincide (Γ_2 in this paper) is not invariant for the closed-loop system, and hence it is unstable. In [16], the all-to-all communication requirement is relaxed through the introduction of so-called consensus filters (an idea which originated in [17]). Such filters require each unicycle to broadcast the state of its own dynamic controller to all of its neighbouring unicycles. In contrast, the control laws proposed in Theorem V.5 are static and do not require any communication between unicycles (besides the measurement of relative variables).

Proof of Theorem V.5: By our choice of $\phi(\cdot)$, the functions $u_1(\chi), u_2(\chi)$ are uniformly bounded on $\mathcal{X}$, and therefore by Proposition V.1, all solutions are globally defined, and Γ_1 is globally asymptotically stable for the closed-loop system. Moreover, for all $i \in \mathbf{n}$, $\inf_{\chi \in \mathcal{X}} u_1^i(\chi) > v/2$, and so by Proposition V.3 there exists $K^\star > 0$ such that for all $K \in (0, K^\star)$ Γ_2 is asymptotically stable relative to Γ_1 for the closed-loop system. Finally, by Proposition V.4, Γ_3 is asymptotically stable relative to Γ_2 for the closed-loop system. Now we apply Corollary IV.6: if the closed-loop system is LUB near Γ_3, then part (a) of the theorem statement follows.

To prove that the closed-loop system is LUB near Γ_3, it suffices to show that there exists $M > 0$ such that for all $\chi(0)$ in a neighbourhood of Γ_2 (and hence for all initial conditions in a neighbourhood of Γ_3), it holds that $\|\chi(t) - \chi(0)\| < M$. Consider the diffeomorphism we used in Section V-B, $\chi = (x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$, $y^i = \begin{bmatrix} c^i(x^i)^\top & x_3^i & e^i(\chi)^\top \end{bmatrix}^\top$. Since $x_3^i \in S^1$, a compact set, to prove the property above it is sufficient to show that $\|c(\chi(t)) - c(\chi(0))\|$ and $\|e(\chi(t))) - e(\chi(0))\|$ have a bound which is uniform for all $\chi(0)$ on a neighborhood of Γ_2. The analysis in Section V-A readily implies that $e(t)$ satisfies the required property, so we only need to focus on the boundedness of $c(\chi)$. To this end, we write the centre dynamics for the closed-loop system

$$\dot{c}^i = -r \begin{bmatrix} \cos x_3^i \\ \sin x_3^i \end{bmatrix} \begin{bmatrix} 1 & -1/r \end{bmatrix} \begin{bmatrix} x_4^i \\ x_5^i \end{bmatrix}.$$

Using the fact that $\begin{bmatrix} x_4^i & x_5^i \end{bmatrix}^\top = e^i + \begin{bmatrix} u_2^i(\chi) & u_1^i(\chi) \end{bmatrix}^\top$, and substituting $u_1^i(\chi)$ and $u_2^i(\chi)$ from (17), we obtain

$$\dot{c} = -rK\phi_{(2)}(S(x_3)L_{(2)}c)R(x_3)L_{(2)}c - rS(x_3)^\top \beta e, \tag{18}$$

where $\beta = \text{blockdiag}([1 \ -1/r], \cdots, [1 \ -1/r])$ and the matrices $R(x_3), S(x_3)$ were defined in Section V-B. As we did in Section V-B, we can view (18) as a time-varying system, where the time variation is brought about by the signals $x_3(t)$ and $e(t)$, and apply averaging theory. Recall that $\dot{x}_3 = x_4^i = u_2^i + e_1^i$, and that u_2^i was chosen in such a way that there exists $\mu_1 > 0$ such that $u_2^i > \mu_1 > 0$. Since Γ_1 is stable and, on it, $e^i = 0$, there exists a positively invariant neighbourhood U of Γ_1 on which $\dot{x}_3 \geq \mu_1/2$. As shown in Section V-B, this inequality implies that the matrix $\phi(\cdot)R(x_3(t))$ has a well-defined positive definite average $\bar{R}$, whose eigenvalues are bounded away from zero uniformly over all $x_3(0)$ and all $e(\chi(0))$. Moreover, for the average of the second term in (18) we have that there exists $m > 0$ such that

$$\begin{aligned} &\left\| -r \lim_{T\to\infty} \frac{1}{T} \int_0^T S(x_3(\tau))^\top \beta e(\tau) d\tau \right\| \\ &\quad \leq r \lim_{T\to\infty} \frac{1}{T} \int_0^T \|S(x_3(\tau))\| \|\beta e(\tau)\| d\tau \\ &\quad \leq m \lim_{T\to\infty} \frac{1}{T} \int_0^T \|\beta e(\tau)\| d\tau = 0. \end{aligned}$$

In the above, we have used the fact that the norm of $S(x_3(t))$ is bounded because its entries are globally bounded functions, and moreover the average of $\|\beta e(t)\|$ is zero because $e(t) \to 0$ exponentially, since Γ_1 is exponentially stable and globally asymptotically stable. Putting everything together, we have that the averaged system associated with (18) is

$$\dot{c}_{\text{avg}} = -rK\bar{R}(L_{(2)}c_{\text{avg}})L_{(2)}c_{\text{avg}},$$

where for each fixed c_{avg}, $\bar{R}(L_{(2)}c_{\text{avg}})$ is a positive definite matrix. This system coincides with the one in (13), and therefore the analysis in Section V-B shows that, if $K^\star > 0$ is chosen sufficiently small, then for all initial conditions in U there exists $\bar{c} \in \mathbb{R}^2$ such that for all $i \in \mathbf{n}$, $c^i(x^i(t)) \to \bar{c}$ exponentially as $t \to \infty$. Since $\ker L = \text{span}\{\mathbf{1}\}$, we have $\ker\{L_{(2)}\} = \{c : c^1 = \cdots = c^n\}$, and so we can equivalently

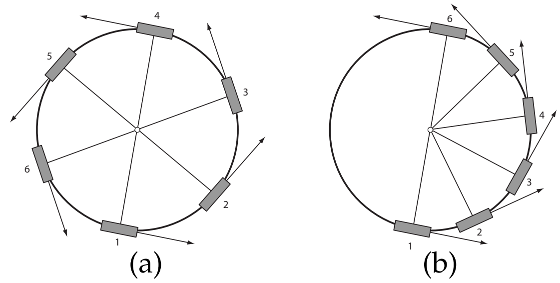

Fig. 6. Desired unicycle formations in the simulations.

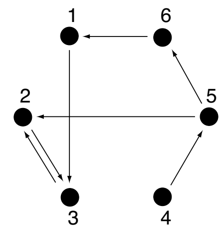

Fig. 7. Sensor digraph used in the simulations. See Figure 2 for the relationship between this digraph and the sensor measurements available to each unicycle.

say that $L_{(2)}c(\chi(t)) \to 0$ exponentially as $t \to \infty$. Recall also that for any initial condition, $e(t) \to 0$ exponentially. Moreover, for both $L_{(2)}c(\chi(t))$ and $e(t)$ the rate of exponential convergence is uniform on neighborhoods $\{\chi : \|L_{(2)}c(\chi)\| < k, \|e(\chi)\| < k\}$, with $k > 0$. Such sets contain Γ_2 in their interior. Going back to the centre dynamics in (18), it holds that $\|\dot{c}\| \leq k_1\|L_{(2)}c(\chi(t))\| + k_2\|e(t)\|$ for suitable $k_1, k_2 > 0$, implying that the bound of the norm of $\|c(\chi(t)) - c(\chi(0))\|$ is uniform over neighborhoods of Γ_2 of the form $\{\chi : \|L_{(2)}c(\chi)\| < k, \|e(\chi)\| < k\}$, proving that the closed-loop system is LUB near Γ_2, and hence also near $\Gamma_3 \subset \Gamma_2$. By Corollary IV.6, Γ_2 and Γ_3 are asymptotically stable. When $\mathcal{G}$ is undirected, a straightforward Lyapunov analysis based on the function $W(\chi) = V_1(\chi) + V_2(\chi)$, where $V_1(\chi)$, $V_2(\chi)$ are defined in Sections V-A and V-B, can be used to show that all trajectories of the closed-loop system enter U in finite time, so that the considerations above hold globally. ∎

VI. Simulation results

Figure 8 presents simulation results for six dynamic unicycles for the two formations in Figure 6: (a) The unicycles are uniformly distributed on the circle in a counter-clockwise cyclic order, with formation vector $\alpha_1 = \begin{bmatrix} 0 & \frac{2\pi}{6} & \frac{4\pi}{6} & \frac{6\pi}{6} & \frac{8\pi}{6} & \frac{10\pi}{6} \end{bmatrix}^\top$. (b) The unicycles are uniformly distributed on half of the circle in a counter-clockwise cyclic order, with formation vector $\alpha_1 = \begin{bmatrix} 0 & \frac{2\pi}{10} & \frac{4\pi}{10} & \frac{6\pi}{10} & \frac{8\pi}{10} & \frac{10\pi}{10} \end{bmatrix}^\top$.

In both cases, the sensor digraph is depicted in Figure 7. In the simulations we set $r = 1$, $v = 1$, $R = 1$, $J = 1$, $I = 1$ and $m = 1$. The controller parameters are $K_i = 1, k_i = 0.14$, $\phi_i(y) = 0.9/[(1 + \|y\|)]$, for all $i \in \{1, \ldots, 6\}$, and $K = 0.49$. Empirically, we observed that increasing K by some amount gives better convergence of the centres of rotation. However, beyond a threshold value of K, performance degrades and solutions even become unbounded. To illustrate, the left-hand side of Figure 9 shows the simulation results for formation (a) when $K = 1.1$. Notice how the transient performance improved with respect to the left-hand side of Figure 8. An analogous result is obtained for formation (b). On the other hand, the right-hand side of Figure 9 shows that for larger K ($K = 5$ in that case) solutions become unbounded. This behavior applies to both formations (a) and (b), and it confirms the theoretical prediction of Theorem V.5.

VII. Conclusions

We have presented a solution to the distributed circular formation stabilization problem. We took a hierarchical point of view, posing the problem as one of simultaneous stabilization of three nested

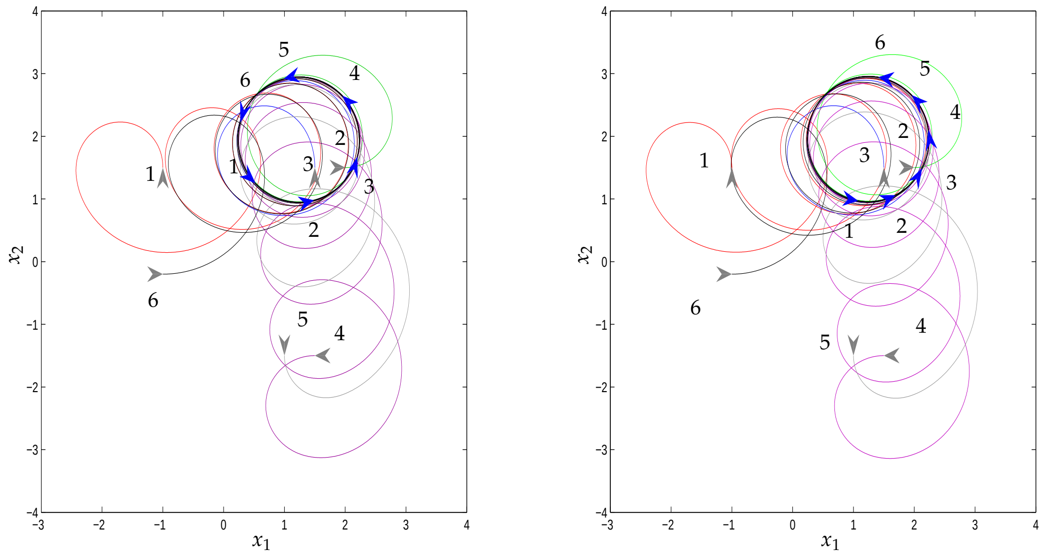

Fig. 8. Simulation results for formations (a) and (b).

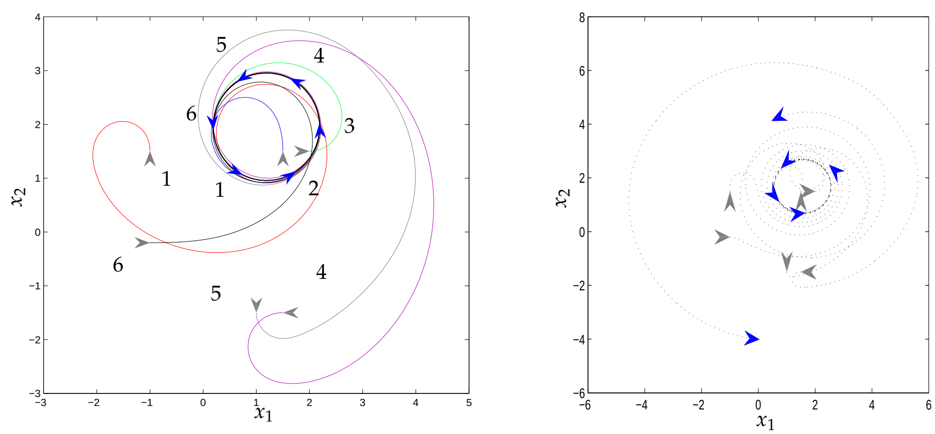

Fig. 9. On the left-hand side, increasing K up to a point improves transient performance. On the right-hand side, larger K causes solutions to grow unbounded.

sets $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3$, and we performed the stabilization in three decoupled steps of lower complexity. The principle that allowed the decoupling in question is a reduction theorem for asymptotic stability of closed sets. We believe that the same design philosophy can be applied to other formation stabilization problems, and to different vehicle models, such as satellites and miniature flying vehicles.

Our analysis in this paper rests upon the assumption that the sensor digraph is static. The extension of our results to the case when the digraph is time-varying and uniformly connected is straightforward, but not particularly insightful. Indeed, from a practical viewpoint the case of interest is not when the sensor digraph is time-varying, but rather when it is state-dependent. In this context, a question that arises is this: suppose that each vehicle can only sense vehicles within a certain range. Does the control law we have proposed guarantee that if the sensor digraph is initially connected, it will remain uniformly connected for all time?

In this work we have assumed that the vehicles have identical dynamics. While this is a common assumption in the literature, it limits the applicability of our results, and it would be worthwhile to extend these results to handle the case of heterogeneous vehicle networks.

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