Further Results on Virtual Holonomic Constraints

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Abstract: This paper continues recent work by the authors on virtual holonomic constraints (VHCs) for Euler-Lagrange control systems with n degrees-of-freedom and m control inputs. The focus of the paper is on implicit constraints of the form $h(q) = 0$. Under suitable regularity conditions, the enforcement of $k \leq m$ constraints induces constrained dynamics that are described by a reduced-order control system of dimension $2(n-k)$ with $(m-k)$ control inputs. When $m = k = n-1$, conditions are given guaranteeing that the constrained dynamics are Euler-Lagrange. It is shown that the presence of dissipation may have unexpected consequences on the constrained dynamics, turning stable equilibria into unstable ones. Finally, VHCs are applied to the problem of constraining a spherical pendulum to lie on the upper half plane.

1. INTRODUCTION

A virtual holonomic constraint (VHC) is a relation of the form $h(q) = 0$ that can be made invariant through suitable feedback. The idea of VHC figures prominently in the work of Jessy Grizzle and collaborators on biped locomotion (see, e.g., Plestan et al. [2003], Westervelt et al. [2003, 2007], Chevallereau et al. [2008]), where it is used to express a desired walking gait. In the context of motion planning, Shiriaev and collaborators in Shiriaev et al. [2005, 2006], Freidovich et al. [2008], Shiriaev et al. [2010] proposed to use VHCs to search for periodic orbits, and employed the technique of transverse linearization to stabilize such orbits. Inspired by Grizzle's work, in Consolini and Maggiore [2010a,b, 2011a,b] we initiated a systematic investigation of VHCs for Euler-Lagrange control systems. In that work, we considered systems with n degrees-of-freedom, $n-1$ actuators, and we investigated the enforcement of $n-1$ constraints. The constraints in questions were expressed in the parametric form $q_1 = \phi_1(q_n), \ldots, q_{n-1} = \phi_{n-1}(q_n)$. This paper takes a step towards generalizing the framework presented in Consolini and Maggiore [2010a,b, 2011a,b]. We consider a class of Euler-Lagrange with n degrees-of-freedom, m actuators, and k VHCs in implicit form $h_i(q) = 0, i = 1, \ldots, k$. In Section 2, we give a precise definition of VHC, discuss a notion of regularity, and give necessary and sufficient conditions for regularity. In Section 3 we show that regular VHCs induce well-defined constrained dynamics. More specifically, by applying a suitable coordinate and feedback transformation, the reduced dynamics are given by a control system on a manifold with $2(n-k)$ states and $m-k$ controls, where k is the number of constraints. The special case $m = k = n-1$ is investigated in Section 4. Conditions are given under which the constrained dynamics are Euler-Lagrange. These conditions generalize an analogous result developed in Consolini and Maggiore [2010a] for constraints in parametric form. In Section 5 we investigate the effect of dissipation on the constrained dynamics when $m = k = n-1$. We discover that dissipation may turn stable equilibria of the constrained dynamics into unstable ones. In Section 6, we present the spherical pendulum example. This is a system with $n = 4$ degrees-of-freedom and $m = 2$ actuators. We find a VHC of order $k = 2$ with the property that the constrained pendulum does not fall over. Moreover, we show that the constrained dynamics are Euler-Lagrange by providing an explicit integral of motion.
Notation. If $h : M \to N$ is a smooth function of manifolds, and $p \in M$, dh_p denotes the differential of h at p. If $x \in \mathbb{R}$, and $T > 0$, we denote by $[x]_T$ the number x modulo T. We let S^1 denote the unit circle. S^1 has a natural additive group structure: if $p, q \in S^1$, we denote $p + q$ the point on S^1 whose angle is the sum of the angles of p and q. The inverse of q is denoted by $-q$, and the identity element is denoted by 0. For a matrix function $M(x)$, $L_v M(x)$ denotes the directional derivative of M along vector v.

2. VIRTUAL HOLONOMIC CONSTRAINTS

Throughout the paper, we consider an Euler-Lagrange control system with an n-dimensional configuration space $\mathcal{Q}$ and m controls $\tau \in \mathbb{R}^m$,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = B(q)\tau.$$

In the above, $B : \mathcal{Q} \to \mathbb{R}^{n \times m}$ is C^1 and it has full rank m for all $q \in \mathcal{Q}$. The Lagrangian function $L(q, \dot{q})$ is C^1 and it has the form $L(q, \dot{q}) = (1/2)\dot{q}^\top D(q)\dot{q} - P(q)$, where $D(q)$, the generalized mass matrix, is symmetric and positive definite for all $q \in \mathcal{Q}$. The above system can be rewritten in the standard form

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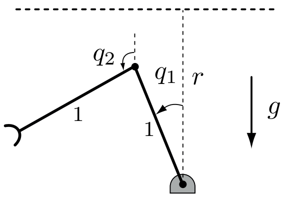

Fig. 1. A planar manipulator. Through appropriate feedback, we wish to constrain the end effector to lie on the horizontal line shown.

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla P(q) = B(q)\tau. \tag{1}$$

Throughout this paper we will assume that $\mathcal{Q}$ is a generalized cylinder, i.e., that $\mathcal{Q} = \{(q_1,\ldots,q_n) : q_i \in S^1 \text{ or } q_i \in \mathbb{R}, i = 1,\ldots,n\}$. This corresponds to the situation when the generalized coordinates are either displacements or angles. We will also assume that there exists a left annihilator of B on $\mathcal{Q}$, i.e., a smooth function $B^\perp : \mathcal{Q} \to \mathbb{R}^{(n-m)\times n}$ such that $B^\perp(q)B(q) = 0$ on $\mathcal{Q}$.

Definition 1. A **virtual holonomic constraint (VHC) of order** k for system (1) is a relation $h(q) = 0$, where $h : \mathcal{Q} \to \mathbb{R}^k$ is a C^1 function which has a regular value at 0, and is such that letting

$$\Gamma = \{(q,\dot{q}) : h(q) = 0, dh_q\dot{q} = 0\}, \tag{2}$$

there exists a smooth feedback $\tau(q,\dot{q})$ defined on Γ such that Γ is positively invariant for the closed-loop system. The set Γ is called the **constraint manifold** associated with $h(q) = 0$.

The requirement of existence of a smooth feedback turning Γ into a positively invariant set for the closed-loop system is referred to as *controlled invariance of* Γ. If Γ were not controlled invariant, then $h(q) = 0$ would not be a feasible VHC. The next example illustrates the definition above.

Example 2.1. Consider the planar manipulator arm in Figure 1. For this system we have $q = (q_1, q_2) \in S^1 \times S^1$. Assuming that all kinematic and dynamic parameters are unitary, we have $P(q) = 2g\cos q_1 + g\cos q_2$, and

$$D(q) = \begin{bmatrix} 2 & \cos(q_1 - q_2) \\ \cos(q_1 - q_2) & 1 \end{bmatrix}$$
$$C(q,\dot{q}) = \begin{bmatrix} 0 & \sin(q_1 - q_2)\dot{q}_2 \\ -\sin(q_1 - q_2)\dot{q}_1 & 0 \end{bmatrix}.$$

Suppose we wish to emulate via feedback the presence of a physical surface, also displayed in Figure 1, by constraining the end effector of the robot to lie on the surface. The surface in question is a horizontal line situated above the robot base at a distance $r = 1.5$ from the base. Letting $h(q) = \cos q_1 + \cos q_2 - r$, we ask whether or not the constraint manifold associated with $h(q)$ is controlled invariant. Assuming first that the robot is fully actuated, i.e., $B(q) = I_2$, it is easy to see that Γ is controlled invariant. Indeed, letting $e = h(q)$, we have $\ddot{e} = \mu(q,\dot{q}) + [-\sin q_1 \quad -\sin q_2]D^{-1}(q)\tau$, and the feedback $\tau(q,\dot{q}) = \mu(q,\dot{q})D(q)/(\sin^2 q_1 + \sin^2 q_2)[\sin q_1 \quad \sin q_2]^\top$ is smooth on Γ and such that $\ddot{e}|_\Gamma = 0$. This implies that Γ is invariant for the closed-loop system. Now suppose that only the first link is actuated, so that $B = [1 \ \ 0]^\top$. In this case we have $\ddot{e} = \mu(q,\dot{q}) + \tau \cdot \cos q_2 \sin(q_2 - q_1)/(2 - \cos^2(q_1 - q_2))$. One can check that there exists $\bar{q} = (\bar{q}_1, \bar{q}_2)$ such that $h(\bar{q}) = 0$ and $\bar{q}_1 = \bar{q}_2$, so that the coefficient of the control input vanishes at $\bar{q}$. Now pick $q(0) = \bar{q}$ and $\dot{q}(0) = 0$. With this choice we have $(q(0), \dot{q}(0)) \in \Gamma$. Moreover, one can check that $\ddot{e}(0) = \mu(\bar{q}, 0) \neq 0$. In conclusion, no matter how the torque is chosen, the solution with initial condition $(q(0), \dot{q}(0)) = (\bar{q}, 0) \in \Gamma$ leaves Γ and violates the relation $h(q) = 0$. We see, therefore, that Γ is not controlled invariant and the relation $h(q) = 0$ is not feasible for the system, in that it cannot be made invariant via feedback. In conclusion, the requirement of controlled invariance in the definition of constraint manifold embodies the notion of feasibility of the VHC. $\triangle$

Definition 2. A relation $h(q) = 0$, with $h : \mathcal{Q} \to \mathbb{R}^k$ a C^1 function and $k \leq m$, is said to be a **regular VHC of order** k if system (1) with output $e = h(q)$ has a well-defined vector relative degree $\{2,\cdots,2\}$ everywhere on $\Gamma = \{(q,\dot{q}) : h(q) = 0, dh_q\dot{q} = 0\}$.

As the definition implies, regular VHCs are indeed VHCs in that the associated constraint manifold is controlled invariant. It is precisely the zero dynamics manifold associated with the output $e = h(q)$. Moreover, one can use input-output feedback linearization to stabilize the set Γ, thus enforcing the VHC $h(q) = 0$. The next result gives necessary and sufficient conditions for regularity of VHCs.

Proposition 2.2. Let $h : \mathcal{Q} \to \mathbb{R}^k$ be C^1, with $k \leq m$. Suppose that 0 is a regular value of h. Then, $h(q) = 0$ is a regular VHC of order k if and only if

$$(\forall q \in h^{-1}(0)) \ \dim\left[T_q h^{-1}(0) \cap \mathrm{Im}(D^{-1}(q)B(q)\right] = m - k.$$

The obvious proof is omitted. The condition for regularity above has a simple intuitive explanation. For the relation $h(q) = 0$ to be a regular VHC, it is necessary and sufficient that at each $q \in h^{-1}(0)$, k of the m acceleration directions imparted by the control input be transversal to the tangent space of $h^{-1}(0)$ at q.

Remark 2.3. If system (1) is fully actuated, i.e., $m = n$, then $\mathrm{Im}(D^{-1}(q)B(q)) = T_q\mathcal{Q}$ and the condition of the proposition above is automatically satisfied. Thus, for fully actuated systems, *any* relation $h(q) = 0$ such that h is smooth and 0 is a regular value of h, is a regular VHC.

3. CONSTRAINED DYNAMICS

Throughout the rest of this paper, if $h(q) = 0$ is a regular VHC, we will denote by $A(q)$ the decoupling matrix associated to the output $e = h(q)$, $A(q) := dh_q D^{-1}(q)B(q)$. By the regularity property, this $k \times m$ matrix has full rank k for all $q \in h^{-1}(0)$. The next proposition shows that for a regular VHC of order k, one can define a regular feedback transformation partitioning the control inputs into two components transversal and tangential to $h^{-1}(0)$, in such a way that k control inputs can be used to enforce the VHC, while the remaining $m-k$ control inputs can be used to affect the dynamics on the constraint manifold. In particular, the motion of the system on the constraint manifold is described by a well-defined reduced-order control system with $n-k$ states and $m-k$ control inputs.

Proposition 3.1. Let $h(q) = 0$ be a regular VHC of order k for system (1), and suppose there exists a smooth function $N : \mathcal{Q} \to \mathbb{R}^{m\times m-k}$ such that $\mathrm{Im}(N(q)) = \ker(A(q))$ for all $q \in h^{-1}(0)$. Let $A^\dagger(q)$ be a right-inverse of $A(q)$ (e.g., $A^\dagger(q) = A(q)^\top(A(q)A(q)^\top)^{-1}$), and consider the feedback transformation

$$\tau = A^\dagger(q)u^{\pitchfork} + N(q)u^{\shortparallel}, \tag{3}$$

where $(u^{\pitchfork}, u^{\shortparallel}) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ are new control inputs. Then, the relation $h(q) = 0$ is a regular VHC of order k for system (1), (3) with input $u^{\pitchfork}$. Moreover, the motion on Γ is governed by a well-defined control system with input $u^{\shortparallel}$. Specifically, if (W, ψ) is a coordinate chart of $h^{-1}(0)$, then in local coordinates $(s, \dot{s}) = (\psi(q), d\psi_q \dot{q}) \in \mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$ the motion on Γ has the form

$$\hat{D}(s)\ddot{s} + \hat{C}(s,\dot{s})\dot{s} + \hat{g}(s) = \begin{bmatrix} 0_{(n-m)\times 1} \\ u^{\shortparallel} \end{bmatrix}, \qquad (4)$$

where, letting $\sigma := \psi^{-1}$,

$$\hat{D}(s) = \begin{bmatrix} B^{\perp} \\ N^{\dagger}B^{\dagger} \end{bmatrix} D\Big|_{q=\sigma(s)} d\sigma_s,$$

$$\hat{C}(s,\dot{s}) = \begin{bmatrix} B^{\perp} \\ N^{\dagger}B^{\dagger} \end{bmatrix} \left(C d\sigma_s + D[(d/dt)d\sigma_s]\right)\dot{s}\Big|_{q=\sigma(s)},$$

$$\hat{g}(s) = \begin{bmatrix} B^{\perp} \\ N^{\dagger}B^{\dagger} \end{bmatrix} \nabla P\Big|_{q=\sigma(s)},$$

$N^{\dagger}$ and $B^{\dagger}$ are left inverses of N and B, and $\hat{D}(s)$ is everywhere invertible.

Remark 3.2. Note that although system (4) "looks like" system (1), it is not written in a canonical form, in that $\hat{D}(s)$ is not symmetric, and $\hat{C}(s,\dot{s})$ does not contain the Christoffel symbols of $\hat{D}$. More importantly, (4) is not necessarily Euler-Lagrange (see Consolini and Maggiore [2010b] for an example).

Remark 3.3. If system (1) is fully actuated, then $n - m = 0$, and the constrained motion in (4) is feedback equivalent to the trivial Euler-Lagrange system $\ddot{s} = 0$. The feedback transformation in question is given by $u^{\shortparallel} = \hat{C}(s,\dot{s})\dot{s} + \hat{g}(s)$.

Proof. From the proof of Proposition 2.2 we have that $\ddot{e} = \mu(q,\dot{q}) + A(q)\tau = \mu(q,\dot{q}) + A(q)A^{\dagger}(q)u^{\pitchfork} + A(q)N(q)u^{\shortparallel} = \mu(q,\dot{q}) + u^{\pitchfork}$, hence system (1), (3) with input $u^{\pitchfork}$ and output $e = h(q)$ has vector relative degree $\{2, \cdots, 2\}$, proving that the VHC $h(q) = 0$ is regular also for the system with input $u^{\pitchfork}$. Next, multiplying (1) on the left by $B^{\perp}(q)$, we get

$$B^{\perp}D\ddot{q} + B^{\perp}(C\dot{q} + \nabla P) = 0_{(n-m)\times 1}. \qquad (5)$$

Feedback transformation (3) is regular because $\operatorname{rank}[A^{\dagger} \quad N] = m$ since $\operatorname{Im} N = \ker A$. Substituting (3) into (1) and premultiplying both sides of the equation by $N^{\dagger}B^{\dagger}$, we obtain

$$\begin{aligned} N^{\dagger}B^{\dagger}D\ddot{q} + N^{\dagger}B^{\dagger}(C\dot{q} + \nabla P) &= N^{\dagger}B^{\dagger}Bu \\ &= N^{\dagger}(A^{\dagger}u^{\pitchfork} + Nu^{\shortparallel}) \qquad (6) \\ &= u^{\shortparallel}. \end{aligned}$$

The last equality follows from the property $AN = 0$ which implies that $N^{\dagger}A^{\dagger} = 0$. Since $h(q) = 0$ is a regular VHC, it follows that dh_q has full rank k for all $q \in h^{-1}(0)$, and thus $h^{-1}(0)$ is an embedded submanifold of $\mathcal{Q}$. Let (W, ψ) be a coordinate chart of $h^{-1}(0)$ and let $s = \psi(q)$, $s \in \mathbb{R}^{n-k}$. Letting $\sigma := \psi^{-1}$, we have that, on W, q can be parametrized as $q = \sigma(s)$, so that $\dot{q} = d\sigma_s \dot{s}$, and $\ddot{q} = d\sigma_s \ddot{s} + [(d/dt)d\sigma_s]\dot{s}$. After substituting these expressions in (5) and (6), we obtain system (4). If we show that the matrix $\hat{D}$ is nonsingular at all $s \in \psi(W)$, then (4) is a well-defined control system on $\psi(W) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{2(n-k)}$ characterizing the motion of $(s, \dot{s})$, and therefore the motion of $(q, \dot{q})$ on $\Gamma \cap \{(q,\dot{q}) : q \in W\}$. To prove that $\hat{D}$ is invertible, we will show that $\operatorname{Im}(Dd\sigma_s) \cap \ker(\operatorname{col}(B^{\perp}, N^{\dagger}B^{\dagger})) = 0$. To this end, suppose $v \in \mathbb{R}^{n-k}$ is such that $B^{\perp}v = 0$ and $N^{\dagger}B^{\dagger}v = 0$. The first identity implies that $v \in \operatorname{Im}(B)$, or $v = Bv'$. Substituting this in the second identity, we obtain $N^{\dagger}v' = 0$. Now suppose also that $v = Bv' \in \operatorname{Im}(Dd\sigma_s)$, so that $D^{-1}Bv' \in \operatorname{Im}(d\sigma_s) = T_{\sigma(s)}h^{-1}(0) = \ker(dh_{\sigma(s)})$. Thus, $dh_{\sigma(s)}D^{-1}Bv' = 0$, or $A(\sigma(s))v' = 0$. By definition of N, $\ker A = \operatorname{Im}(N)$, so $Av' = 0$ implies $v' = Nv''$. We have obtained that $N^{\dagger}v' = 0$ and $v' = Nv''$, so $N^{\dagger}Nv'' = 0$, implying that v'', and hence v' and v, are zero. $\square$

Remark 3.4. Of particular interest is a special type of VHC in parametric form $\hat{q} = \phi(\bar{q})$, where $\hat{q} = \begin{bmatrix} q_1 & \cdots & q_k \end{bmatrix}^{\top}$ and $\bar{q} = \begin{bmatrix} q_{k+1} & \cdots & q_n \end{bmatrix}^{\top}$. In this case, $h(q) = \hat{q} - \phi(\bar{q})$. Letting $\hat{\phi}(\bar{q}) := \begin{bmatrix} \phi(\bar{q})^{\top} & \bar{q}^{\top} \end{bmatrix}^{\top}$, one can parametrize q on $h^{-1}(0)$ as $q = \hat{\phi}(\bar{q})$. The necessary and sufficient condition for regularity in Proposition 2.2 becomes $(\forall \bar{q})$ $\operatorname{rank}\left[B^{\perp}(\hat{\phi}(\bar{q}))D(\hat{\phi}(\bar{q}))d\hat{\phi}_{\bar{q}}\right] = n - m$. For this type of constraint, after feedback transformation (3), the motion on Γ is described by system (4) with $s = \bar{q}$ and $\psi^{-1}(s) = \hat{\phi}(\bar{q})$. In this case, (4) describes the constrained motion globally on Γ, rather than just on a coordinate chart.

Example 3.5. We return to the manipulator in Example 2.1. Assume that $B = I_2$ and consider again the regular VHC $h(q) = \cos q_1 + \cos q_2 - r$. Following Proposition 3.1, we let $\tau = -D/(\sin^2 q_1 + \sin^2 q_2)[\sin q_1 \quad \sin q_2]^{\top} u^{\pitchfork} + D[\sin q_2 \quad -\sin q_1]^{\top} u^{\shortparallel}$. The input $u^{\pitchfork}$ is used to enforce the VHC. Indeed, using the fact that $\ddot{e} = \mu(q,\dot{q}) + u^{\pitchfork}$, if we let $u^{\pitchfork}(q,\dot{q}) = -\mu(q,\dot{q}) - K_1 e - K_2 \dot{e}$ then Γ is asymptotically stable for the closed-loop system. The input $u^{\shortparallel}$ has no effect on the e dynamics, and it can be used to achieve desired control specifications on the constraint manifold Γ in (2). The set $h^{-1}(0)$ is a closed curve on the torus $\mathcal{Q} = S^1 \times S^1$, so the state space of the constrained system is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$. One can choose various coordinate representations for the constrained dynamics. For instance, we could parametrize a chart of $h^{-1}(0)$ with q_1. Let $W = \{(q_1, q_2) \in h^{-1}(0) : \sin q_2 > 0\}$ and $\psi : W \to (-\pi/3, \pi/3)$ be defined as $\psi(q_1, q_2) = \hat{q}_1$, with $\hat{q}_1$ = representation of $q_1 \in (-\pi/3, \pi/3)$. Then,

$$\sigma(\hat{q}_1) = \left([\hat{q}_1]_{2\pi}, \arctan\left(\sqrt{1 - (r - \cos\hat{q}_1)^2}/(r - \cos\hat{q}_1)\right)\right).$$

The pair (W, ψ) defines a coordinate chart on $h^{-1}(0)$ parametrizing the motion on the VHC when the angle of the second link is in the interval $(0, \pi)$ modulo 2π. In an analogous way one can define a second coordinate chart so as to obtain an atlas for $h^{-1}(0)$. According to Proposition 3.1, the constrained dynamics on the chosen chart have the form $\hat{D}(\hat{q}_1, \dot{\hat{q}}_1)\ddot{\hat{q}}_1 = \alpha_1(\hat{q}_1, \dot{\hat{q}}_1) + u^{\shortparallel}$. An alternative, and more intuitive parametrization is the displacement of the end effector, $x = \sin q_1 + \sin q_2$. In this case, we could let $W = \{(q_1, q_2) \in h^{-1}(0) : \sin(q_1 - q_2) > 0\}$, and $\psi : W \to (-\sqrt{7}/2, \sqrt{7}/2)$, $\psi(q_1, q_2) = x = \sin q_1 + \sin q_2$. Then,

$$\sigma(x) = \left(\arctan(x/r) + D/2, \arctan(x/r) - D/2\right),$$

where $D = \arctan\left(\sqrt{1 - c^2}/c\right)$ and $c = (x^2 + r^2 - 2)/2$. The constrained dynamics with this parametrization have the form $\hat{D}(x, \dot{x})\ddot{x} = \alpha_2(x, \dot{x}) + u^{\shortparallel}$. It is easy to achieve control specifications for $(x, \dot{x})$. For instance, one could design $u^{\shortparallel}$

to make the end effector converge to a specific point on the horizontal line $\{h(q) = 0\}$. From a control design perspective, it is convenient and computational easier to express $\ddot{x}$ in terms of $(q, \dot{q})$, so that the feedback u'' can be expressed in original coordinates. △

4. VHCS OF ORDER $n-1$

In this section we investigate the special case $k = n - 1$, so that the set $h^{-1}(0)$ is a one-dimensional embedded submanifold of $\mathcal{Q}$. In this case, each connected component of $h^{-1}(0)$ is a regular curve without self-intersections diffeomorphic to either $\mathbb{R}$ or S^1. If system (1) is fully actuated, then by Proposition 3.1 the reduced dynamics have one control input, and they are feedback equivalent to the trivial Euler-Lagrange system $\ddot{s} = 0$. A more interesting situation occurs when system (1) has degree of underactuation one, in which case $m = n - 1$ and the reduced dynamics have no control input. A natural question that arises in this context is: *under what conditions are the reduced dynamics Euler-Lagrange?* The next proposition answers this question.

Proposition 4.1. Let $h(q) = 0$ be a regular VHC of order $n-1$ for system (1). Assume that $m = n - 1$ and that $h^{-1}(0)$ is a connected set. Let $\sigma : \Theta \to \mathcal{Q}$, with Θ either $\mathbb{R}$ or $[\mathbb{R}]_T$, be a regular parametrization of $h^{-1}(0)$. Then, the constrained dynamics on the set Γ in (2) have the form

$$\ddot{s} = \Psi_1(s) + \Psi_2(s)\dot{s}^2. \tag{7}$$

where

$$\begin{aligned} \Psi_1(s) &= -\left.\frac{B^\perp \nabla P}{B^\perp D\sigma'}\right|_{q=\sigma(s)} \\ \Psi_2(s) &= -\left.\frac{B^\perp D\sigma'' + \sum_{i=1}^n B_i^\perp \sigma'^\top Q_i \sigma'}{B^\perp D\sigma'}\right|_{q=\sigma(s)}, \end{aligned} \tag{8}$$

and where $B_i^\perp$ is the i-th component of $B^\perp$ and $(Q_i)_{jk} = (1/2)\big(\partial_{q_k} D_{ij} + \partial_{q_j} D_{ik} - \partial_{q_i} D_{kj}\big)$. If $h^{-1}(0) \simeq \mathbb{R}$, then the reduced dynamics are always Euler-Lagrange with Lagrangian $L(s, \dot{s}) = (1/2)M(s)\dot{s}^2 + V(s)$, where

$$\begin{aligned} M(s) &= \exp\Big\{-2\int_0^s \Psi_2(\tau)d\tau\Big\} \\ V(s) &= -\int_0^s \Psi_1(\mu)M(\mu)d\mu. \end{aligned} \tag{9}$$

If $h^{-1}(0) \simeq S^1$, then the reduced dynamics are Euler-Lagrange if there exists a regular parametrization of $h^{-1}(0)$, $\sigma : [\mathbb{R}]_T \to \mathcal{Q}$ such that $M(s)$ and $V(s)$ in (9) are T-periodic.

Proof. Let $\sigma : \Theta \to \mathcal{Q}$ be a regular parametrization of $h^{-1}(0)$ so that $\sigma : \Theta \to h^{-1}(0)$ is a diffeomorphism. Using this parametrization, the fact that the motion on Γ has the form (7) follows from (4), the fact that $m - k = 0$, and the definition of C. If $\Theta = \mathbb{R}$, one can readily check that the function $L(s, \dot{s}) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L(s, \dot{s}) = (1/2)M(s)\dot{s}^2 - V(s)$ is a Lagrangian for system (7), so that (7) is Euler-Lagrange. On the other hand, if $\Theta = [\mathbb{R}]_T$ for some $T > 0$, then $L(s, \dot{s})$ is a well-defined function $[\mathbb{R}]_T \times \mathbb{R} \to \mathbb{R}$ if and only if the functions $M(s)$, $V(s)$ in (9) are T-periodic. □

Remark 4.2. The above proposition is a straightforward adaptation of the analysis in Consolini and Maggiore [2010a] concerning VHCs in parametric form. Following a result in [Consolini and Maggiore, 2010a, Lemma 4.1], when $h^{-1}(0) \simeq S^1$ a sufficient condition for $M(s)$ and $V(s)$ in (9) to be T-periodic is that $\sigma(s)$ is and odd function, while $D(q)$, $P(q)$, and $B(q)$ in (1) are even functions. We also refer the reader to the work in Shiriaev et al. [2005, 2006], which presented for the first time an integral of motion for systems of the form (7) which depends on initial conditions, but appears to be equivalent to total energy $(1/2)M(s)\dot{s}^2 + V(s)$.

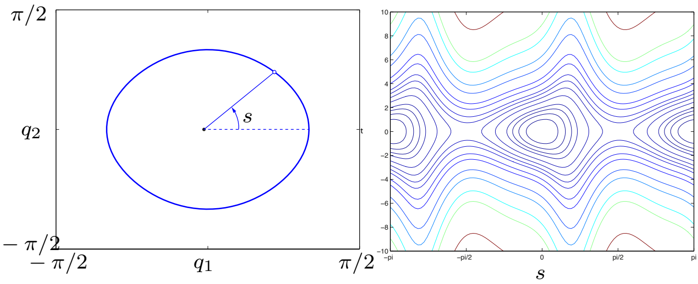

Fig. 2. Left: The set $h^{-1}(0)$ and its parametrization. Right: The phase portrait of the constrained motion on Γ.

Example 4.3. Consider again the manipulator of Figure 1. This time, let $B(q) = \begin{bmatrix}\sin q_1 & \sin q_2\end{bmatrix}^\top$. The relation $h(q) = \cos q_1 + \cos q_2 - r$, $r = 3/2$, is a regular VHC in that $A(q) = dh_q D^{-1}(q)B(q) = -\cos^2 q_2/(\cos^2(q_1 - q_2) - 2) - 1$, and this function vanishes when $q_1 = q_2 = [0]_{2\pi}$, which is not on Γ. Note that $B(q)$ is not full-rank everywhere, but it is full-rank in a neighbourhood of Γ. The set $h^{-1}(0)$ is a closed curve, displayed in Figure 2. We see from the figure that $h^{-1}(0)$ can be parametrized by the argument of the complex number $q_1 + iq_2$. In other words, we can let $\sigma : [\mathbb{R}]_{2\pi} \to \mathcal{Q}$ be defined as $\sigma(s) = \rho(s)\big[\cos s \;\; \sin s\big]^\top$, for a suitable smooth function $\rho : [\mathbb{R}]_{2\pi} \to (0, \infty)$. With this parametrization, and letting $B^\perp = [-\sin q_2 \;\; \sin q_1]$, one can show that the functions $M(s)$ and $V(s)$ are 2π-periodic. The phase portrait of the reduced motion on Γ is displayed in Figure 2. As expected, the constrained motion is Euler-Lagrange. △

The following result, taken from Consolini and Maggiore [2010a], characterizes the qualitative properties of the constrained motion.

Proposition 4.4. (Consolini and Maggiore [2010a]). Suppose $h(q) = 0$ is a regular VHC of order $n-1$, and that $m = n - 1$. Suppose the assumptions of Proposition 4.1 hold, so that the constrained dynamics in (7) are Euler-Lagrange with mass M and potential V given in (9). Then, the equilibrium configurations are the extrema of $V(s)$, i.e., points $(s^\star, 0)$ such that $\Psi_1(s^\star) = 0$, or equivalently $\nabla P(\sigma(s^\star)) \in \operatorname{Im} B(\sigma(s^\star))$. Isolated local minima of V correspond to stable equilibria, while isolated local maxima of V correspond to unstable equilibria. In particular, the stability type is characterized by the sign of $[B^\perp D\sigma'(d/ds)(B^\perp \nabla P)]_{\sigma(s^\star)}$ (positive $\implies$ stable, negative $\implies$ unstable).

Remark 3. The criterion for stability of equilibria in Proposition 4.4 can be shown to be equivalent to a condition found in Theorem 3 of Shiriaev et al. [2006].

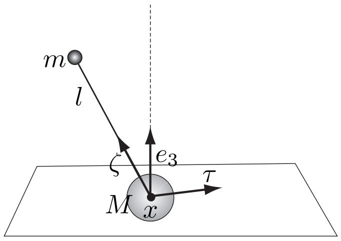

Fig. 3. Spherical pendulum.

5. EFFECT OF DISSIPATION

Consider again a regular VHC of order $k = n-1$, and assume there are $m = n-1$ controls. Suppose system (1) is affected by dissipation as follows

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + K(q)\dot{q} + \nabla P(q) = B(q)\tau, \qquad (10)$$

where $K(q) = K(q)^\top$ is positive semidefinite for all $q \in \mathcal{Q}$. The dissipation has no effect on the regularity of the VHC. However, it has an impact on the nature of the constrained dynamics. As we will see in this section, the dissipation has implications that are sometimes counter-intuitive.

Proposition 5.1. Consider system (10) and a regular VHC $h(q) = 0$ with a parametrization $\sigma : \Theta \to \mathcal{Q}$ satisfying the assumptions of Proposition 4.1. Then, the reduced dynamics have the form $\ddot{s} = \Psi_1(s) + \Psi_2(s)\dot{s}^2 + \Psi_3(s)\dot{s}$, where $\Psi_1(s)$, $\Psi_2(s)$ are given in (8) and $\Psi_3(s) = -\left(B^\perp K\sigma'\right) / \left(B^\perp D\sigma'\right)\Big|_{q=\sigma(s)}$. Let $(s,\dot{s}) = (s^\star, 0)$ be a stable equilibrium of the system without dissipation such that $V(s^\star)$ is an isolated local minimum of $V(s)$. If $\Psi_3(s^\star) < 0$ then $(s,\dot{s}) = (s^\star, 0)$ is an asymptotically stable equilibrium of (10). If $\Psi_3(s^\star) > 0$ then $(s,\dot{s}) = (s^\star, 0)$ is unstable.

The proof of this proposition is straightforward. It relies on the application of the LaSalle invariance principle with the energy function $E = (1/2)M(s)\dot{s}^2 + V(s)$. The result above is surprising: the addition of dissipation in an Euler-Lagrange system may turn a stable equilibrium into one that is unstable.

6. THE SPHERICAL PENDULUM

Consider a spherical inverted pendulum of mass m linked to a moving cart of mass M through a massless rod of length l. In Figure 3 the pendulum is represented as the smaller sphere and the cart as the bigger one. It is supposed that the cart moves on the (x_1, x_2)-plane and that the control force $\tau \in \mathbb{R}^3$, parallel to the (x_1, x_2)-plane, is applied on the center of mass x of M. The vector of the generalized coordinates is $q = (x, \zeta) \in \mathbb{R}^2 \times S^2$ (with $S^2 = \{\zeta \in \mathbb{R}^3 : \|\zeta\| = 1\}$), where x is the position of the center of mass of the moving base M and ζ is the orientation versor of the rod. Note that in recent literature there has been considerable interest in the study of the spherical pendulum (see for instance Shiriaev et al. [2004], Liu et al. [May 30 2007-June 1 2007], Chaturvedi and McClamroch [2007],Liu et al. [2008],Consolini and Tosques [2011]). Matrix $\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ represents the projection of $\mathbb{R}^3$ into $\mathbb{R}^2$ and $\mathcal{I} = \mathcal{P}^T$ represents the immersion of $\mathbb{R}^2$ on $\mathbb{R}^3$ along the first two coordinates. The Lagrangian is $L = T - U$, where the kinetic energy is given by $T = 1/2(m+M)\|\dot{x}\|^2 + 1/2ml^2\|\dot{\zeta}\| + ml\dot{\zeta}\cdot\mathcal{I}\dot{x}$ and the potential energy by $U = glm\zeta \cdot e_3$. Since $\zeta \in S^2$, $\dot{\zeta}$ is orthogonal to ζ, that is $\dot{\zeta} \in T_\zeta(S^2)$ (the tangent space to S^2 at ζ), and we can suppose that $\dot{\zeta} = \zeta \times \omega$, with $\omega \in \mathbb{R}^3$ and $\zeta \cdot \omega = 0$. The resulting dynamical system, obtained through the Euler-Lagrange equation, is given by

$$\begin{cases} (m+M)\ddot{x} + ml\mathcal{P}\ddot{\zeta} = f \\ l\dot{\omega} = \zeta \times \mathcal{I}\ddot{x} + g(\zeta \times e_3) \\ \dot{\zeta} = \zeta \times \omega\,. \end{cases} \qquad (11)$$

Being $m + M \neq 0$, in system (11) we consider the cart acceleration $\ddot{x}$ instead of the force τ as the control input.

Problem statement: For system (11), find a regular VHC of the form $h(q) = \zeta - f(x)$, where $f : \mathbb{R}^2 \to S^2$, such that $[0\ 0\ 0]f(x) > 0$. In other words, on the VHC, the pendulum must always be pointing upwards.

We first study the regularity of this VHC. Note that the configuration space of the spherical pendulum has not the structure assumed in section 2, however the concepts presented there can be readily extended to this case. This is not done here due to space limitations. We have that $\dot{h}(q) = \dot{\zeta} - f'(x)\dot{x}$, that is, by the third equation of (11) $\dot{h}(q) = \zeta \times \omega - f'(x)\dot{x}$. The derivative of this expression is given by $\ddot{h}(q) = \dot{\zeta} \times \omega + \zeta \times \dot{\omega} - (L_{\dot{x}} f'(x))\dot{x} - f'(x)\ddot{x}$, that is, by the second of (11), $\ddot{h}(q) = (\zeta \times \omega) \times \omega + \zeta \times \frac{1}{l}(\zeta \times \mathcal{I}\ddot{x} + g(\zeta \times e_3)) - (L_{\dot{x}} f'(x))\dot{x} - f'(x)\ddot{x}$, using the properties of triple product $(a \times b) \times c = -a(c \cdot b) + b(a \cdot c)$, $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ and the identities $\zeta \cdot \omega = 0$ and $\|\zeta\| = 1$, the expression becomes

$$\begin{aligned} \ddot{h}(q) = &-\zeta\|\omega^2\| + \frac{1}{l}(\zeta(\zeta \cdot \mathcal{I}\ddot{x}) - \mathcal{I}\ddot{x} \\ &+ g(\zeta(\zeta \cdot e_3) - e_3) - (L_{\dot{x}} f'(x))\dot{x} - f'(x)\ddot{x}\,. \end{aligned} \qquad (12)$$

Property 1. The VHC $h(q) = \zeta - f(x)$ is regular of order 2 if and only if the linear transformation $T(x) : \mathbb{R}^2 \to T_{f(x)}S^2$ defined as

$$T(x) = \frac{1}{l}(I - f(x)f(x)^T)\mathcal{I} + f'(x) \qquad (13)$$

is nonsingular $\forall x \in \mathbb{R}^2$.

The obvious proof is omitted. Let $R(\theta) \in SO(2)$ denote the rotation in $\mathbb{R}^2$ by angle θ, and let $R_3(\theta) \in SO(3)$ be the rotation on $\mathbb{R}^3$ about the z axis by angle θ.

Assumption 1. Function f satisfies (a) $R_3(\theta)f(x) = f(R(\theta)x), \forall \theta \in S^1, \forall x \in \mathbb{R}^2$, (b) $f(x)$ belongs to the plane spanned by x and e_3.

Note that (a) implies that $f(0) = e_3$ and (b) requires that the pendulum is always oriented on the vertical plane passing through the origin and the cart's position. To exploit the assumed rotational symmetry, we write x in polar coordinates $\xi = (\rho, \theta)^T$ as $x = p(\rho, \theta) = (\rho \sin\theta, \rho\cos\theta)^T$. Note that

$$\begin{aligned} p'(\xi) = \begin{pmatrix} \sin\theta & \rho\cos\theta \\ \cos\theta & -\rho\sin\theta \end{pmatrix},\ \dot{x} = p'(\xi)\dot{\xi}, \\ \ddot{x} = p'(\xi)\ddot{\xi} + (L_{\dot{\xi}} p'(\xi))\dot{\xi}\,. \end{aligned} \qquad (14)$$

The generic function f that satisfies assumption 1 is

$$f(p(\xi)) = \begin{pmatrix} \cos(\theta)\sin z(\rho) \\ \sin(\theta)\sin z(\rho) \\ \cos z(\rho) \end{pmatrix}, \qquad (15)$$

where z is an even function. We set $\phi(\xi) = f(p(\xi))$.

Property 2. The VHC $h(q) = \zeta - f(x)$ is regular of order 2 if and only if the linear transformation $T_p(x) : \mathbb{R} \times S^1 \to T_{f(x)}S^2$ defined as

$$T_p(\xi) = \frac{1}{l}(I - \phi(\xi)\phi(\xi)^T)\mathcal{I}p'(\xi) + \phi'(\xi) \tag{16}$$

is nonsingular $\forall \xi \in \mathbb{R}\backslash\{0\} \times S^1$.

The proof is omitted. Define the matrix

$$N(\xi) = \begin{pmatrix} -\rho^{-1}\sin(\theta) & \cos(\theta)\cos(z(\rho)) \\ \rho^{-1}\cos(\theta) & \cos(\theta)\cos(z(\rho)) \\ 0 & -\sin(z(\rho)) \end{pmatrix},$$

note that, for $\rho \neq 0$, Im $N(\xi) = T_{\phi(\xi)}S^2$, i.e. the image of $N(\xi)$ is orthogonal to $\phi(\xi)$. Left-multiplying (16) by $N(\xi)^T$ we obtain that condition (16) is equivalent to $\hat{\delta}(\xi) \triangleq \frac{1}{l}N^T(\xi)(\mathcal{I}p'(\xi) + \phi'(\xi))$ is nonsingular. Substituting the expressions for N and ϕ, we obtain $\hat{\delta}(\xi) = \text{diag}\left(-\frac{\sin z(\rho)}{\rho} - \frac{1}{l}, -f'(\rho) - \frac{\cos z(\rho)}{l}\right)$. Matrix $\hat{\delta}(\xi)$ is nonsingular if and only if

$$\frac{\sin z(\rho)}{\rho} + \frac{1}{l} \neq 0,\ z'(\rho) + \frac{\cos z(\rho)}{l} \neq 0\,. \tag{17}$$

To satisfy the second of (17), one possibility is to choose z as the solution of the differential equation $z'(\rho) + \frac{\cos(z(\rho))}{l} = \delta$, with the initial condition $f(0) = 0$, where $\delta \neq 0$ is a constant. If $|\delta l| \leq 1$, the solution is given by

$$z(\rho) = -2\arctan\left(\frac{\tanh\left(\frac{\rho\sqrt{1-\delta^2 l^2}}{2\,l}\right)(1-\delta\, l)}{\sqrt{1-\delta^2\, l^2}}\right). \tag{18}$$

If $0 < \delta l < 1$, then $z'(\rho) < 0$ and $-\frac{1}{l} < -\frac{1-\delta l}{l} < z'(\rho) < 0$, $\forall \rho \in \mathbb{R}$ and therefore the first of (17) is satisfied. Note that the constraint f with function z given by (18) solves the problem formulated in this section, since $|z(\rho)| < \frac{\pi}{2}$, $\forall \rho \in \mathbb{R}$. We now derive the reduced dynamics by setting $h(q) = \dot{h}(q) = \ddot{h}(q) = 0$ in (12) and using (14):

$$\begin{aligned} \ddot{\rho} &= \frac{l^{-1}g\sin z - z''\dot{\rho}^2 + \dot{\theta}^2\cos z\sin z + \dot{\theta}^2\rho\cos(z)}{l^{-1}\cos(z) + z'} \\ \ddot{\theta} &= -\frac{2\dot{\theta}\dot{\rho}(z'\cos z + 1)}{l^{-1}\rho + \sin(z)}\,. \end{aligned} \tag{19}$$

Note that equations (19) do not depend on θ as a consequence of the rotational symmetry of the virtual constraint. System (19) can be written more compactly as

$$\begin{aligned} \ddot{\rho} &= \psi_1(\rho) + \psi_2(\rho)\dot{\rho}^2 + \psi_3(\rho)\dot{\theta}^2 \\ \ddot{\theta} &= \psi_4(\rho)\dot{\theta}\dot{\rho}\,. \end{aligned} \tag{20}$$

System (20) possesses a first integral of motion of the form $H(\rho, \dot{\rho}, \dot{\theta}) = M(\rho)(\dot{\rho})^2 + N(\rho)(\dot{\theta})^2 + V(\rho)$. Indeed, $(d/dt)H(\rho, \dot{\rho}, \dot{\theta}) = (V' + 2M\psi_1)\dot{\rho} + (M' + 2M\psi_2)\dot{\rho}^3 + (N' + 2M\psi_3 + 2N\psi_4)\dot{\theta})^2\dot{\rho}$. Therefore, if M, N, V are the solution of $V' = -2M\psi_1$, $M' = -2M\psi_2$, $N' = -2(N\psi_4 + M\psi_3)$, it follows that $\frac{d}{dt}H(\rho, \dot{\rho}, \dot{\theta}) = 0$.

Simulation Results. We consider the VHC (15) with the function z as in (18) and parameters $l = 1$ m, $g = 9.8$ m/s^2, $\delta = 0.5$. Figure 4 shows the section of the virtual constraint on the plane $\{x_2 = 0\}$, and the solution of constrained dynamics (19) starting from initial condition $\rho(0) = 1$, $\dot{\rho}(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) = 1$. Note that the solution has been converted in the Cartesian coordinates x_1, x_2 (the cart position).

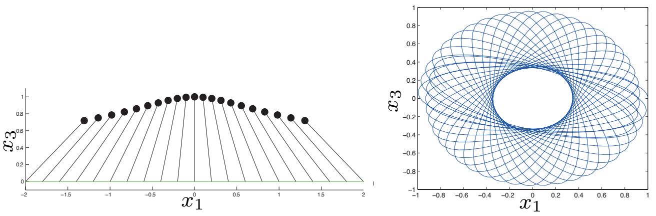

Fig. 4. Left: A section of VHC (15) on plane $\{x_2 = 0\}$. Right: Plot of constrained dynamics (x_1, x_2).

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