

On the Swing-Up of the Pendubot Using Virtual Holonomic Constrains

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Abstract: In this paper we design a swing-up controller making the pendubot transition from the low-high to the high-high equilibrium in such a way that during the swing-up phase the unactuated link does not fall over. To achieve this control objective, we design and stabilize a virtual holonomic constraint expressing a relationship between the angles of the actuated and unactuated links. Such relationship guarantees that the unactuated link does not fall over. Then, in order to stabilize the energy level corresponding to the high-high equilibrium while preserving the invariance of the virtual constraint, we dynamically change the geometry of the constraint.

1. INTRODUCTION

The pendubot system, depicted in Figure 1, is a planar robot composed of two links, a shoulder and an elbow. Only the shoulder is actuated. The pendubot was introduced in Spong and Block [1995] as a benchmark problem to study challenging nonlinear control problems. Since then, several researchers have tested their design techniques on this system, including Fantoni et al. [2000], Freidovich et al. [2008], Albahkali et al. [2009]. The typical control specification for this system is the swing-up, whereby system trajectories are made to transition to the highest energy equilibrium.

The configuration variables of the pendubot are $q = (\theta_1, \theta_2)$. Assuming, for simplicity of exposition, that the masses and lengths of the two links are equal and unitary, and neglecting friction, the pendubot model reads as

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla P(q) = B\tau, \quad (1)$$

where

$$D(q) = \begin{bmatrix} 2 & \cos(\theta_1 - \theta_2) \\ \cos(\theta_1 - \theta_2) & 1 \end{bmatrix}$$

$$C(q, \dot{q}) = \begin{bmatrix} 0 & \sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ -\sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$P(q) = 2g \cos \theta_1 + g \cos \theta_2, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

When $\tau = 0$, the pendubot has four equilibria, depicted in Figure 1. The equilibria lie on different level sets of the energy. Roughly speaking, swinging up the pendubot means designing a control law which makes the system transition from one of the equilibria to another one with at least one of the two links in the high position. The swing-up problem that is typically investigated by researchers is the transition from low-low to high-high. For the solution to this problem, we point out, in particular,

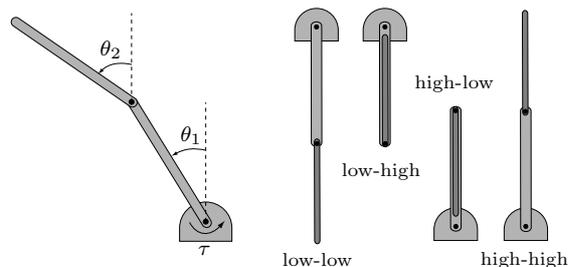


Fig. 1. The pendubot and its equilibrium configurations.

the work in Fantoni et al. [2000], where an energy-based controller is introduced with a complete stability analysis. Recently, Freidovich et al. [2008] used virtual holonomic constraints to determine periodic orbits of the pendubot in which the elbow oscillates without performing complete revolutions. Then, using a technique of transverse linearization, they stabilized the periodic orbits in question.

In this paper we investigate the following

Low-high to high-high swing-up problem. Design a feedback law yielding the following two properties:

- (1) **Swing-up:** For any neighborhood U of the high-high equilibrium, there exists a punctured neighborhood V of the low-high equilibrium such that for each initial condition in V , the solution enters U in finite time.
- (2) **Boundedness:** For any initial condition in V , the solution has the property that $\theta_2(t) \in (-\pi, \pi)$ for all $t \geq 0$. In other words, the unactuated link does not fall over.

The swing-up property guarantees that as long as the pendubot is initialized in a neighborhood V of the low-high equilibrium (but not initialized at the equilibrium itself), the pendubot reaches an arbitrarily small neighborhood U of the high-high equilibrium in finite time. As is customarily done in the literature, once the solution has reached

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U , one may use a linear feedback to stabilize the high-high equilibrium. In and by itself, the swing-up problem is not difficult, as it can be solved using the passivity tools of Fantoni *et al.*. What makes our problem challenging is the second requirement. This demands that the second link, which is initialized near its upward position, remains near the upward position without falling over.

The technique used in this paper to solve this problem is to find a desired relationship, $\theta_2 = \phi(\theta_1)$, between the configuration variables which expresses what the elbow angle θ_2 should be as a function of the shoulder angle θ_1 . This relationship, referred to as a virtual holonomic constraint, is designed following the theory of Consolini and Maggiore [2010a] in such a way that it can be asymptotically enforced via feedback, and it has the following properties: (1) $\phi(0) = \phi(\pi) = 0$, so that the elbow is high when the shoulder is either low or high; (2) the image of the function ϕ is contained in the interval $(-\pi, \pi)$, so that the elbow doesn't fall over as the shoulder revolves. Stabilizing this virtual constraint guarantees that the boundedness requirement of the swing-up problem is met. In order to meet the swing-up objective, one needs to design a controller that, while preserving the invariance of the virtual constraint, stabilizes the energy level set associated to the high-high equilibrium. This is a problem that was not investigated in Consolini and Maggiore [2010a]. In this paper, we present a general technique to solve this stabilization problem which completes the theory in Consolini and Maggiore [2010a] and, when applied to the pendubot, solves the swing-up problem.

This paper is organized as follows. The virtual holonomic constraint theory of Consolini and Maggiore [2010a] is reviewed in Section 2. In Section 3 we present an approach to stabilize a level set of the energy for the reduced system describing the motion on the virtual constrained manifold. In Section 4 the two approaches are applied to the pendubot.

2. VIRTUAL HOLONOMIC CONSTRAINTS

In this section we review the theory of Consolini and Maggiore [2010a].

Consider an Euler-Lagrange system of the form (1) with n degrees-of-freedom and $n - 1$ controls and a **virtual holonomic constraint (VHC)** of the form

$$\text{col}(q_1, \dots, q_{n-1}) = \text{col}(\phi_1(q_n), \dots, \phi_{n-1}(q_n)) = \phi(q_n),$$

where $q_n \in S^1$ is an angular configuration variable parametrizing the constraint. More generally, one could define a VHC to be an implicit relation $h(q) = 0$ but the explicit description above is sufficient and convenient for our purposes. Throughout this paper, we let $\hat{\phi}(q_n) = \text{col}(\phi(q_n), q_n)$, so that we can conveniently express the constraint as $q = \hat{\phi}(q_n)$. A VHC $q = \hat{\phi}(q_n)$ is **feasible** if the set

$$\Gamma = \{(q, \dot{q}) : \text{col}(q_1, \dots, q_{n-1}) = \phi(q_n), \\ \text{col}(\dot{q}_1, \dots, \dot{q}_{n-1}) = \phi'(q_n)\dot{q}_n\},$$

is controlled invariant, i.e., if it can be made invariant by a suitable feedback $\tau(q, \dot{q})$. We call the set Γ the **constraint manifold**. Γ is a two-dimensional embedded submanifold of the state space, and being parametrized by (q_n, \dot{q}_n) , it

is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$. The controlled invariance of the constraint manifold expresses the fact that whenever the configuration variable $q(0)$ is initialized on the constraint, and its initial velocity $\dot{q}(0)$ is tangent to the constraint, a suitable feedback makes the resulting solution $q(t)$ satisfy the constraint for all t .

Proposition 2.1. (Consolini and Maggiore [2010a]). A VHC $q = \hat{\phi}(q_n)$ is feasible if

$$(\forall q_n \in S^1) \text{Im}(D(\hat{\phi}(q_n))\hat{\phi}'(q_n)) \cap \text{Im}(B) = \{0\}$$

or, equivalently, if

$$B^\perp D(\hat{\phi}(q_n))\hat{\phi}'(q_n) \neq 0, \quad (2)$$

where B^\perp is a nonzero row vector such that $B^\perp B = 0$. Moreover, the output function $e = \text{col}(q_1, \dots, q_{n-1}) - \phi(q_n)$ yields a vector relative degree $\{2, \dots, 2\}$ on Γ , and therefore the constraint manifold Γ is locally exponentially stabilizable.

A VHC $q = \hat{\phi}(q_n)$ satisfying (2) will be called **regular**. Hence, regular VHC's are feasible. The mechanical interpretation of the regularity property is this. The generalized momentum of a solution $(q(t), \dot{q}(t))$ satisfying the virtual constraint at all time is $D(\hat{\phi}(q_n))\hat{\phi}'(q_n)\dot{q}_n$. The regularity condition implies that, on Γ , it is always possible to choose τ such that the generalized momentum is compatible with motion on Γ .

There is a systematic way to *generate* regular holonomic constraints as solutions of a scalar ordinary differential equation. The idea is to select $n - 2$ of the $n - 1$ required functions in $\phi(q_n)$, for instance $\phi_2(q_n), \dots, \phi_{n-1}(q_n)$, and find a function $\phi_1(q_n)$ satisfying the equation $B^\perp D(\hat{\phi}(q_n))\hat{\phi}'(q_n) = \delta(q_n)$, where $\delta(q_n)$ is a nonzero function $S^1 \rightarrow \mathbb{R} \setminus \{0\}$ to be assigned. The latter equation can be rewritten as

$$f_1(\phi_1, q_n) \frac{d\phi_1}{dq_n} + f_2(\phi_1, q_n) = \delta(q_n). \quad (3)$$

The above is a T -periodic ordinary differential equation for ϕ_1 , where T is the period of the angular variable q_n . If, for a given $\delta(q_n) : S^1 \rightarrow \mathbb{R} \setminus \{0\}$, (3) has a T -periodic solution $\phi_1(q_n)$, then this function together with the functions $\phi_2(q_n), \dots, \phi_{n-1}(q_n)$ forms a regular holonomic constraint. For this reason, we call (3) a **virtual constraint generator (VCG)**. The issue then becomes whether, for a given initial condition, it is possible to choose $\delta \neq 0$ such that the solution of (3) is T -periodic. The answer to this question for the case when the ODE (3) has no singularities is contained in the next

Proposition 2.2. (Consolini and Maggiore [2010a]).

Consider equation (3) and suppose that $f_1 \neq 0$. Fix an initial condition $\phi_1(q_{n0}) = \phi_0$. There exists a C^1 function $\delta(q_n) : S^1 \rightarrow \mathbb{R} \setminus \{0\}$ such that the solution $\phi_1(q_n)$ is T -periodic if and only if the solution when $\delta = 0$ is not T -periodic, and in this case $\delta(q_n)$ can be chosen as follows. Choose a C^1 function $\mu(q_n) : S^1 \rightarrow \mathbb{R} \setminus \{0\}$ and let $\delta(q_n) = \epsilon\mu(q_n)$. Then, there exists a unique $\epsilon \neq 0$ such that the solution of (3) is T -periodic.

Once a regular VHC has been found, the motion on the virtual constraint manifold is found by left-multiplying both sides of (1) by B^\perp , letting $q = \hat{\phi}(q_n)$, $\dot{q} = \hat{\phi}'(q_n)\dot{q}_n$, $\ddot{q} = \hat{\phi}''(q_n)\dot{q}_n^2 + \hat{\phi}'''(q_n)\dot{q}_n^2$, and using the fact that

$B^\perp D(\hat{\phi}(q_n))\hat{\phi}'(q_n) = \delta(q_n) \neq 0$. Doing so, one obtains

$$\dot{q}_n = \Psi_1(q_n) + \Psi_2(q_n)\dot{q}_n^2, \quad (4)$$

for suitable C^1 functions $\Psi_i : S^1 \rightarrow \mathbb{R}$, $i = 1, 2$. As pointed out earlier, Γ is parametrized by (q_n, \dot{q}_n) , and so the system above describes the dynamics of the system on the virtual constraint manifold. Note that system (4) is unforced. This is because the original system (1) has degree of underactuation one, and all control directions are used to make Γ invariant. As shown in Consolini and Maggiore [2010b], the constrained dynamics (4) are not, in general, Euler-Lagrange or Hamiltonian. However, under the following conditions they are in fact Euler-Lagrange:

- C1 $D(q)$, $P(q)$, and $B(q)$ in the original system (1) are even functions.
- C2 $\phi_2(q_n), \dots, \phi_{n-1}(q_n)$ are chosen to be odd functions.
- C3 In Proposition 2.2, the initial condition is chosen to be $\phi_1(0) = 0$, and $\mu(q_n)$ to be an even function.

Throughout the next section we will assume that conditions C1-C3 above hold. Under these conditions, the Lagrangian function is $\mathcal{L}(q_n, \dot{q}_n) = \frac{1}{2}M(q_n)\dot{q}_n^2 - V(q_n)$, where

$$\begin{aligned} M(q_n) &= \exp \left\{ -2 \int_0^{q_n} \Psi_2(\tau) d\tau \right\} \\ V(q_n) &= - \int_0^{q_n} \Psi_1(\mu) M(\mu) d\mu. \end{aligned} \quad (5)$$

The total energy of the system evolving on the constraint manifold is

$$E(q_n, \dot{q}_n) = \frac{1}{2}M(q_n)\dot{q}_n^2 + V(q_n). \quad (6)$$

The theory reviewed above provides answers to these questions:

- (1) When is a VHC feasible?
- (2) How to stabilize the virtual constraint manifold?
- (3) How to systematically select regular (and hence feasible and stabilizable) VHC's?
- (4) When are the constrained dynamics Euler-Lagrange?

The question left open in Consolini and Maggiore [2010a] is *how to stabilize a desired level set of the energy $E(q_n, \dot{q}_n)$ corresponding to a desired motion on Γ ?* The answer to this question is elusive because system (4) has no control! Some modification is required in order to solve the problem.

3. ENERGY LEVEL STABILIZATION ON CONSTRAINT MANIFOLD

Suppose we have found a regular VHC, $q = \hat{\phi}(q_n)$, chosen according to the procedure reviewed in the previous section, and such that the motion on the constraint manifold in equation (4) is Euler-Lagrange with energy $E(q_n, \dot{q}_n) = (1/2)M(q_n)\dot{q}_n^2 + V(q_n)$. The objective now is, for a given constant $E_0 \in \mathbb{R}$, to stabilize a connected component of the set $\gamma = \{(q, \dot{q}) \in \Gamma : E(q_n, \dot{q}_n) = E_0\}$, for some $E_0 \in \mathbb{R}$, where $E(q_n, \dot{q}_n) = \frac{1}{2}M(q_n)\dot{q}_n^2 + V(q_n)$ is the energy function of the reduced system, with M and V defined in (5). γ is the union of a finite number of phase curves of (4). Let $\underline{V} = \min_{q_n \in S^1} V(q_n)$ and $\bar{V} = \max_{q_n \in S^1} V(q_n)$. Then, for all $E_0 > \bar{V}$, γ is the union of two closed curves parametrized by q_n with opposite orientations: $\dot{q}_n = \pm \sqrt{(2/M)(E_0 - V)}$. Such motions correspond to complete

revolutions of the angular variable q_n , and therefore we call them **rotations**. For all $E_0 \in [\underline{V}, \bar{V}]$, if $V'(q_n) \neq 0$ for all $q_n \in V^{-1}(E_0)$, then γ is the union of a finite number of closed phase curves homeomorphic to the circle $q_n^2 + \dot{q}_n^2 = 1$. These solutions correspond to motions where q_n oscillates without performing complete revolutions, and therefore we call them **oscillations**. Finally, if $V'(q_n) = 0$ for some $q_n \in V^{-1}(E_0)$, then γ is the union of a finite number of closed phase curves, some of which contain equilibria. Henceforth, for notational simplicity, we replace γ above with the connected component we wish to stabilize.

Since the reduced dynamics in (4) have no control input, it is impossible to stabilize γ while at the same time preserving the invariance of the constraint manifold Γ . To obviate the problem just described, in this section we present an approach to stabilize the closed orbit γ that relies on dynamically changing the geometry of the VHC while preserving its invariance. The idea is to make the VHC depend on a parameter s . When $s = 0$, we have the original VHC. Variations of s affect the dynamics on the constraint manifold. The objective, then, is to control s so that the desired orbit γ is stabilized while, at the same time, driving $s(t)$ to zero.

Consider a one-parameter family of VHC's $\phi^s(q_n)$, where $s \in \mathcal{S}$ is the variable parametrizing the family, and \mathcal{S} is diffeomorphic to S^1 . We will use a base point in \mathcal{S} which we will denote 0. Denote, as before, $\hat{\phi}^s(q_n) = \text{col}(\phi^s(q_n), q_n)$. Assume that $(q_n, s) \mapsto \phi^s(q_n)$ is a C^1 function which satisfies the properties:

- (a) For each $s \in \mathcal{S}$, $q = \hat{\phi}^s(q_n)$ is a regular VHC, i.e., $B^\perp(\hat{\phi}^s(q_n))D(\hat{\phi}^s(q_n))\partial_{q_n}\hat{\phi}^s \neq 0$ for all $(q_n, s) \in [\mathbb{R}]_{T_n} \times \mathcal{S}$.
- (b) For all $q_n \in [\mathbb{R}]_{T_n}$, $\phi^0(q_n) = \phi(q_n)$, where $\phi(q_n)$ is the odd VHC we wish to enforce by feedback.

There are various ways to generate families of VHC's satisfying properties (a) and (b) above. Here we present two methods.

Method 1. Consider the nonsingular VCG in (3) with initial condition $\phi_{i_1}(0) = s$ and $\delta(\phi_{i_1}, q_n) = \epsilon\mu(\phi_{i_1}, q_n)$, where $\phi_{i_2}(q_n), \dots, \phi_{i_{n-1}}(q_n)$ are odd and T_n -periodic, and $\mu(\phi_{i_1}, q_n)$ is bounded away from zero and even. By Proposition 2.2, there exists a unique value of $\epsilon \neq 0$ (dependent on s) such that the resulting solution of (3) is T_n -periodic. Denoting by $\phi_{i_1}^s(q_n)$ this solution, we obtain a vector function

$$\phi^s(q_n) = \sigma(\phi_{i_1}^s(q_n), \phi_{i_2}(q_n), \dots, \phi_{i_{n-1}}(q_n)).$$

By construction, $\phi^s(q_n)$ satisfies property (a). It also satisfies property (b) since ϕ^0 is odd. An analogous methodology to get $\phi^s(q_n)$ can be applied when the VCG is singular.

Method 2. Given an odd and regular VHC $\text{col}(q_1, \dots, q_{n-1}) = \phi(q_n)$, we set $\phi^s(q_n) = \phi(q_n) + Ls$, where $L \in \mathbb{R}^{n-1}$ is a vector to be designed. Since $\phi(q_n)$ is regular, we have $B^\perp(\hat{\phi}(q_n))D(\hat{\phi}(q_n))\hat{\phi}'(q_n) \neq 0$ for all $q_n \in [\mathbb{R}]_{T_n}$. Thus, there exists $\bar{s} > 0$ such that

$$B^\perp(\hat{\phi}(q_n) + Ls)D(\hat{\phi}(q_n) + Ls)\hat{\phi}'(q_n) \neq 0$$

for all $q_n \in [\mathbb{R}]_{T_n}$ and all $s \in (-\bar{s}, \bar{s})$. The family thus obtained satisfies the required properties (a) and (b).

Varying the parameter s in the VHC $q = \hat{\phi}^s(q_n)$ corresponds to changing the geometry of the VHC. The idea is to let s be the state of a dynamic compensator

$$\dot{s} = v, \quad (7)$$

where v is a scalar control input, and design τ in (1) to stabilize the family of VHC's $q = \hat{\phi}^s(q_n)$. By so doing, the reduced dynamics on the constraint manifold will be four-dimensional, with state $(q_n, \dot{q}_n, s, \dot{s})$. We will then design the control input v in (7) to stabilize the closed orbit γ .

Consider the augmented system (1), (7). With some abuse of notation, denote

$$\begin{aligned} \Gamma &= \{(q, \dot{q}, s, \dot{s}) : q = \hat{\phi}(q_n), \dot{q} = \hat{\phi}'(q_n)\dot{q}_n, s = \dot{s} = 0\}, \\ \gamma &= \{(q, \dot{q}, s, \dot{s}) \in \Gamma : E(q_n, \dot{q}_n) = E_0, s = \dot{s} = 0\}, \end{aligned}$$

and define

$$\begin{aligned} \bar{\Gamma} &= \{(q, \dot{q}, s, \dot{s}) : q = \hat{\phi}^s(q_n), \dot{q} = [\partial_{q_n} \hat{\phi}^s(q_n)]\dot{q}_n \\ &\quad + [\partial_s \hat{\phi}^s(q_n)]\dot{s}\}, \end{aligned} \quad (8)$$

Since $\phi^0 = \phi$, we have that $\Gamma = \bar{\Gamma} \cap \{(q, \dot{q}, s, \dot{s}) : s = \dot{s} = 0\}$, and so we can also write $\gamma = \{(q, \dot{q}, s, \dot{s}) \in \bar{\Gamma} : E(q_n, \dot{q}_n) = E_0, s = \dot{s} = 0\}$. The control objective is to design feedbacks $\tau(q, \dot{q}, s, \dot{s})$ and $v(q_n, \dot{q}_n, s, \dot{s})$ for the augmented system (1), (7) that simultaneously stabilize the two nested sets $\gamma \subset \bar{\Gamma}$. In particular, the feedbacks in question should render $\bar{\Gamma}$ an invariant set for the closed-loop system.

Our design is based on two steps. First, we design a feedback $\tau(q, \dot{q}, s, \dot{s}, v)$ which stabilizes $\bar{\Gamma}$ for any choice of v . Then, we design a feedback $v(q_n, \dot{q}_n, s, \dot{s})$ that makes γ asymptotically stable relative to $\bar{\Gamma}$, i.e., asymptotically stable for the reduced closed-loop dynamics on $\bar{\Gamma}$. Once the two steps above are completed, owing to the fact that γ is a closed curve and hence a compact set, the Seibert-Florio reduction principle for asymptotic stability of compact sets (see Seibert and Florio [1995], El-Hawwary and Maggiore [2009]) guarantees that γ is asymptotically stable for the closed-loop system, and so both control objectives are simultaneously met.

Step 1. Defining the output $e = \text{col}(q_1, \dots, q_{n-1}) - \phi^s(q_n)$, and taking derivatives of e along the vector field of the augmented system (1), (7), we have

$$\ddot{e}|_{e=0, \dot{e}=0} = (\star) + [I_{n-1} \quad -\partial_{q_n} \phi^s] D^{-1}(\hat{\phi}^s(q_n)) B(\hat{\phi}^s(q_n)) \tau,$$

where the term (\star) is a suitable C^1 function of (q, \dot{q}, s, \dot{s}) and v . Since, by property (a) of the function $\phi^s(q_n)$, $B^\perp(\hat{\phi}^s(q_n)) D(\hat{\phi}^s(q_n)) \partial_{q_n} \hat{\phi}^s \neq 0$, it follows that the matrix $[I_{n-1} \quad -\partial_{q_n} \phi^s] D^{-1}(\hat{\phi}^s(q_n)) B(\hat{\phi}^s(q_n))$ is nonsingular. Therefore, viewing v as a parameter, the augmented system (1), (7) with input τ and output e has vector relative degree $\{2, \dots, 2\}$. Owing to this property, assuming that v has been selected to guarantee that the (s, \dot{s}) subsystem does not have finite escape times, the input-output feedback linearizing controller

$$\begin{aligned} \tau(q, \dot{q}, s, \dot{s}, v) &= \{[I_{n-1} \quad -\partial_{q_n} \phi^s] D^{-1} B\}^{-1} \{-k_1 e - k_2 \dot{e} \\ &\quad + [I_{n-1} \quad -\partial_{q_n} \phi^s] D^{-1} (C\dot{q} + \nabla P) \\ &\quad + (\partial_{q_n}^2 \phi^s) \dot{q}_n^2 + (2\partial_{q_n} \partial_s \phi^s) \dot{q}_n \dot{s} + (\partial_s^2 \phi^s) \dot{s}^2 + (\partial_s \phi^s) v\}, \end{aligned} \quad (9)$$

where $e = \text{col}(q_1, \dots, q_{n-1}) - \phi^s(q_n)$, $\dot{e} = \text{col}(\dot{q}_1, \dots, \dot{q}_{n-1}) - (\partial_{q_n} \phi^s) \dot{q}_n - (\partial_s \phi^s) \dot{s}$, and $k_1, k_2 > 0$, exponentially stabilizes the constraint manifold $\bar{\Gamma}$ in (8).

Step 2. By left-multiplying (1) by B^\perp and by evaluating the resulting equation on $\bar{\Gamma}$, we obtain the equations describing the motion of the augmented system (1), (7) on $\bar{\Gamma}$ when τ is chosen as in (9):

$$\begin{aligned} \ddot{q}_n &= \Psi_1^s(q_n) + \Psi_2^s(q_n) \dot{q}_n^2 + \Psi_3^s(q_n) \dot{q}_n \dot{s} + \Psi_4^s(q_n) \dot{s}^2 + \Psi_5^s(q_n) v \\ \dot{s} &= v, \end{aligned} \quad (10)$$

where $\Psi_i^s(q_n)$, $i = 1, \dots, 5$, are suitable C^1 functions and Ψ_1^s, Ψ_2^s have the property that

$$\Psi_1^0 = \Psi_1, \quad \Psi_2^0 = \Psi_2, \quad (11)$$

where Ψ_1, Ψ_2 are the functions in (4) characterizing the reduced motion on Γ . Now the objective is to design a feedback $v(q_n, \dot{q}_n, s, \dot{s})$ for (10) that stabilizes the closed orbit $\bar{\gamma} = \{(q_n, \dot{q}_n, s, \dot{s}) : E(q_n, \dot{q}_n) = E_0, s = \dot{s} = 0\}$. We first consider the case when $\bar{\gamma}$ is a rotation of q_n , i.e., $E_0 > \bar{V}$. In this case², $\bar{\gamma} = \{(q_n, \dot{q}_n, s, \dot{s}) : \dot{q}_n = \sqrt{(2/M(q_n))(E_0 - V(q_n))}, s = \dot{s} = 0\}$, where, once again, we consider the connected component of interest.

Denote $r(q_n, E) = \sqrt{2/M(q_n)(E - V(q_n))}$, and consider the coordinate transformation

$(q_n, \dot{q}_n, s, \dot{s}) \mapsto x = (\theta, \rho)$, $\theta = q_n$, $\rho = (E(q_n, \dot{q}_n) - E_0, s, \dot{s})$, which is a diffeomorphism in a neighborhood of $\bar{\gamma}$. System (10) in new coordinates reads as

$$\begin{aligned} \dot{\theta} &= r(\theta, E_0 + \rho_1) \\ \dot{\rho}_1 &= M\dot{q}_n \left[(\Psi_1^s - \Psi_1^0) + (\Psi_2^s - \Psi_2^0) \dot{q}_n^2 + \Psi_3^s \dot{q}_n \rho_3 \right. \\ &\quad \left. + \Psi_4^s \rho_3^2 + \Psi_5^s v \right]_{q_n=\theta, \dot{q}_n=r(\theta, E_0+\rho_1)} \\ \dot{\rho}_2 &= \rho_3 \\ \dot{\rho}_3 &= v. \end{aligned} \quad (12)$$

In deriving the expression for $\dot{\rho}_1$ in (12) we have used the identities in (11) and $M'(q_n) = -2M(q_n)\Psi_2(q_n)$, $V'(q_n) = -\Psi_1(q_n)M(q_n)$. We are now to design v to stabilize the periodic orbit $\{(\theta, \rho) : \rho = 0\}$ of this system. Our approach relies on the classical theory of stability of periodic orbits exposed in Chapter VI of Hale [1980], and is based on the following considerations. We can concisely rewrite (12) as

$$\begin{aligned} \dot{\theta} &= \vartheta(\theta, \rho) \\ \dot{\rho} &= f(\theta, \rho) + g(\theta, \rho)v. \end{aligned} \quad (13)$$

Let $v(\theta, \rho)$ be a C^1 function such that (13) with feedback $v(\theta, \rho)$ has an exponentially stable periodic orbit $\{(\theta, \rho) : \rho = 0\}$. Consider the *linear variational system*

$$\frac{dx}{d\theta} = \begin{bmatrix} \partial_\theta \vartheta & \partial_\rho \vartheta \\ \partial_\theta f + (\partial_\theta g)v + g(\partial_\theta v) & \partial_\rho f + (\partial_\rho g)v + g(\partial_\rho v) \end{bmatrix} \Big|_{\rho=0} x.$$

The characteristic multipliers of this linear time-periodic system characterize the stability of the periodic orbit. In particular, $\{(\theta, \rho) : \rho = 0\}$ is exponentially stable if and only if $n - 1$ characteristic multipliers of the variational system have modulus < 1 . The remaining characteristic multiplier is always equal to one. It turns out that due to the fact that v makes $\bar{\gamma}$ invariant, it holds that $f(\theta, 0) = v(\theta, 0) = 0$, and therefore $\partial_\theta f(\theta, 0) = \partial_\theta v(\theta, 0) = 0$, so that the variational system takes the form

² Here we assume that the connected component $\bar{\gamma}$ we wish to stabilize is the positive root of the equation $E(q_n, \dot{q}_n) = E_0$. The analysis for the negative root is identical.

$$\frac{dx}{d\theta} = \left[\begin{array}{cc} \partial_\theta \vartheta & \partial_\rho \vartheta \\ 0 & \partial_\rho f + g(\partial_\rho v) \end{array} \right] \Bigg|_{\rho=0} x.$$

Since, in (θ, ρ) coordinates, $\bar{\gamma}$ is the set $\{\rho = 0\}$, the stability of the motion transversal to $\bar{\gamma}$ is characterized by the subsystem

$$\frac{d\rho}{d\theta} = (\partial_\rho f(\theta, 0))\rho + g(\theta, 0)(\partial_\rho v(\theta, 0))\rho. \quad (14)$$

In particular, the feedback $v(\theta, \rho)$ exponentially stabilizes $\bar{\gamma}$ for (10) if and only if the origin of the above linear periodic system is exponentially stable, i.e., if its $n - 1$ characteristic multipliers have modulus < 1 . Returning to the synthesis problem, we see that in order to exponentially stabilize $\bar{\gamma}$ it is necessary and sufficient to design a feedback $v = K(\theta)\rho$ stabilizing the origin of the linear periodic control system

$$\frac{dy}{d\theta} = A(\theta)y + b(\theta)v, \quad (15)$$

where $A(\theta) = (\partial_\rho f(\theta, 0))$ and $b(\theta) = g(\theta, 0)$. In Banaszuk and Hauser [1995], this subsystem is called the **transverse linearization of (13) along $\bar{\gamma}$** . To determine the existence of an exponentially stabilizing feedback for (15), and to synthesize such a feedback, one may use the notion of uniform controllability in Silverman [1966]. Let $Q = [b(\theta) \ \mathcal{A}b(\theta) \ \mathcal{A}^2b(\theta)]$, where \mathcal{A} is the differential operator $h(\theta) \mapsto h'(\theta) - A(\theta)h(\theta)$. If, and only if, the matrix Q is invertible, then there exist time-dependent coordinate and feedback transformations mapping system (15) to a linear time-invariant controllable system, for which an exponentially stabilizing feedback is easily designed.

Returning to system (12), its transverse linearization along $\bar{\gamma}$ is

$$A(q_n) = \left[\begin{array}{ccc} 0 & Mr[\partial_s \Psi_1^s + \partial_s \Psi_2^s r^2] & Mr^2 \Psi_3^s \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] \Bigg|_{\substack{E=E_0 \\ s=\dot{s}=0}},$$

$$B(q_n) = \left[\begin{array}{c} Mr \Psi_5^s \\ 0 \\ 1 \end{array} \right] \Bigg|_{E=E_0, s=\dot{s}=0}$$

and a straightforward calculation reveals that this system is uniformly controllable if and only if $\Delta = \partial_{q_n}[-Mr^2 \Psi_3^s + \partial_{q_n}(Mr \Psi_5^s)] - Mr[\partial_s \Psi_1^s + \partial_s \Psi_2^s r^2] \Big|_{E=E_0, s=\dot{s}=0} \neq 0$ for all $q_n \in S^1$. In conclusion, if $\bar{\gamma}$ is a rotation of q_n and $\Delta \neq 0$, there exist functions $K_i(q_n)$, $i = 1, 2, 3$, such that the feedback

$$v(q_n, \dot{q}_n, s, \dot{s}) = K_1(q_n)[E(q_n, \dot{q}_n) - E_0] + K_2(q_n)s + K_3(q_n)\dot{s} \quad (16)$$

exponentially stabilizes $\bar{\gamma}$ for (12).

The analysis for the case when $\bar{\gamma}$ is an oscillation of q_n , which occurs when $E_0 \in (\underline{V}, \bar{V})$ and $V'(q_n) \neq 0$ for all $q_n \in V^{-1}(E_0)$, is analogous and is omitted for brevity. The only difference is that in this case the orbit $\bar{\gamma}$ can be parametrized by an angular variable θ , $\bar{\gamma} = \{(q_n, \dot{q}_n, s, \dot{s}) : (q_n, \dot{q}_n) = \sigma(\theta), \theta \in S^1, s = \dot{s} = 0\}$. One can use coordinates $(\theta, \rho, s, \dot{s})$, where ρ is defined as before, to transform system (10) into a new system which has a form analogous to that in (12), after which the analysis is identical.

As an alternative to the synthesis method illustrated above, we postulate that a feedback of the form

$$v = K_1[E(q_n, \dot{q}_n) - E_0]\dot{q}_n + K_2s + K_3\dot{s}, \quad (17)$$

stabilizes $\bar{\gamma}$ for suitable values of K_1, K_2, K_3 . Note that (17) is a special case of the feedback (16). The feasibility of such controller can be tested by substituting v in (10), computing the associated linear variational system by evaluating the Jacobian of the closed-loop vector field along the periodic orbit $\bar{\gamma}$, and checking whether or not three of the four characteristic multipliers of the variational system have modulus < 1 (the fourth one has always modulus $= 1$).

4. APPLICATION TO THE SWING-UP PROBLEM

The pendubot system satisfies condition C1 in Section 2. We look for a constraint $\theta_2 = \phi(\theta_1)$ with the following properties: $\phi(0) = \phi(\pi) = 0$, so that the second link is high when the first link is either low or high; the image $\phi(S^1) \subset (-\pi, \pi)$, so that the second link doesn't fall over as the first link revolves. For the pendubot we have

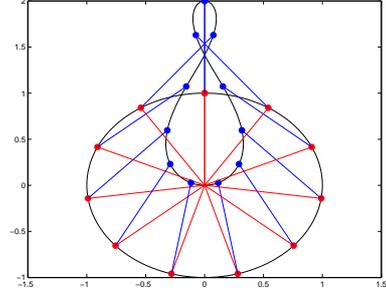


Fig. 2. Configurations of the double pendulum with the virtual constraint obtained by setting $\mu(\theta_1) = 1$.

$B^\perp = [0 \ 1]$ and the VCG is given by

$$\frac{d\phi}{d\theta_1} = -\cos(\theta_1 - \phi(\theta_1)) + \delta(\theta_1).$$

The solution with zero initial condition and with $\delta = 0$ is not 2π -periodic, so we can apply Proposition 2.2. We should select a 2π -periodic function $\mu(\theta_1) \neq 0$, set $\delta(\theta_1) = \epsilon\mu(\theta_1)$, and find the unique value of ϵ guaranteeing that the solution with zero initial condition is 2π -periodic. In order to meet condition C3, we must select $\mu(\theta_1)$ to be even. If we set $\mu = 1$, then we find $\epsilon = 1 - \sqrt{2}$ and the virtual constraint

$$\theta_2 = \phi(\theta_1) = \theta_1 + 2 \arctan[\tan(-\theta_1/2)(1 + \sqrt{2})] \quad (18)$$

depicted in Figure 2. As predicted by the theory of Consonini and Maggiore [2010a], the motion of the pendubot on the constraint manifold is Euler-Lagrange. The phase portrait of the dynamics on the constraint manifold is depicted in Figure 3. The level sets of E inside the shaded region of Figure 3 are oscillations, while the ones outside the shaded region correspond to rotations.

Swinging up the pendulum to the high-high equilibrium corresponds to stabilizing the level set of the energy bounding the shaded region, for which $E_0 = 0$. This level set is neither an oscillation nor a rotation and therefore, strictly speaking, one cannot use either of the coordinate transformations introduced in Section 3. To

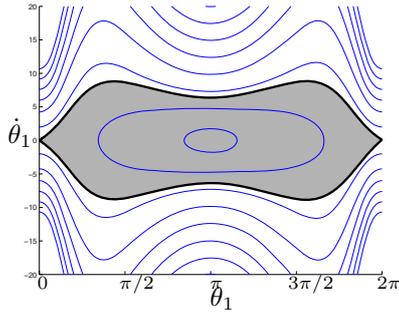


Fig. 3. Energy level sets for double pendulum on the VHC $\theta_2 = \theta_1 + 2 \arctan[\tan(-\theta_1/2)(1 + \sqrt{2})]$.

obviate this problem it suffices to perform the stability analysis using values of E_0 slightly greater or smaller than zero. Alternatively, instead of using uniform controllability to synthesize a feedback, one could use feedback (17) and check the characteristic multipliers of the resulting variational system. This is the route we follow here.

We return to the VCG with $\mu(\theta_1) = 1$, $\frac{d\phi}{d\theta_1} = -\cos(\theta_1 - \phi(\theta_1)) + \epsilon$. Given an initial condition $\phi(0) = \phi_0$, we are to find $\epsilon(\phi_0)$ such that the solution of the VCG is 2π -periodic. In this case all solutions are 2π -periodic when $\delta = \epsilon = 1 - \sqrt{2}$, and are given by $\phi(\theta_1, K) = \theta_1 + 2 \arctan[\tan(K - \theta_1/2)(1 + \sqrt{2})]$, where $K = \tan(\phi_0/2)/(1 + \sqrt{2})$. For convenience, we consider K , rather than ϕ_0 , as a state of our dynamic compensator, $\dot{K} = v$. To stabilize the VHC $\theta_2 = \phi(\theta_1, K)$, the physical input τ of the pendubot is designed to input-output linearize the system with output $e = \theta_2 - \phi(\theta_1, K)$ (which has relative degree 2). The input v of the dynamic compensator affects the shape of the virtual constraint manifold. Physically, varying K corresponds to changing the average value of the angle $\theta_2 = \phi(\theta_1)$ as θ_1 ranges over S^1 . We use the feedback v in (17) (replacing ϕ_0 with K) with $k_1 = -0.01$, $k_2 = -0.5$, $k_3 = -3$. We use $E_0 = 0$ to stabilize homoclinic orbit containing the high-high equilibrium. The swing-up controller switches to a linear stabilizing controller when $\|(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)\| < 0.1$. The corresponding characteristic multipliers are $\{1, 0.4884, 0.3704, 2.57 \cdot 10^{-4}\}$. Figure 4 shows the value of total energy $E(t)$ during the swing-up phase, while figure 5 shows the convergence of θ_2 to $\phi(\theta_1, K)$. Figure 6 shows the corresponding phase portrait in plane $(\theta_1, \dot{\theta}_1)$.

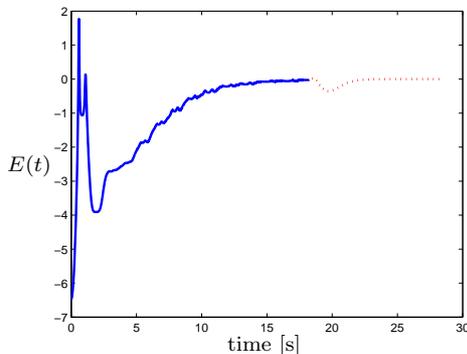


Fig. 4. Total energy $E(t)$.

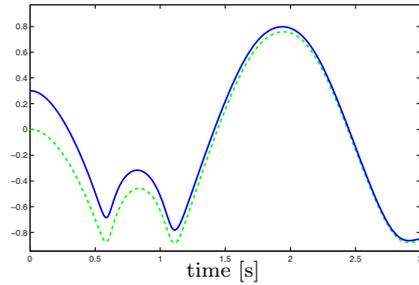


Fig. 5. Plots of θ_2 (dashed) and $\phi(\theta_1, K)$ (dotted).

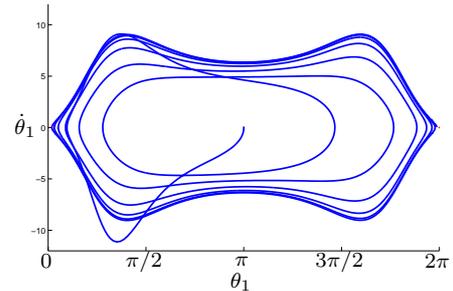


Fig. 6. Phase portrait of $(\theta_1, \dot{\theta}_1)$.

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