

Path Following Controller for a Quadrotor Helicopter

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Abstract—A path following controller is presented for a quadrotor helicopter model. The controller relies on input dynamic extension and feedback linearization. The controller allows the designer to specify the speed profile of the quadrotor on the path and its yaw angle as a function of the displacement.

I. INTRODUCTION

Applications for search and rescue, exploration, security and inspection are often dangerous or time consuming for a human operator to perform. One solution is to have a model sized quadrotor helicopter do the task. That is, command a quadrotor to follow a predefined path while meeting additional specifications regarding its motion such as speed or orientation. For instance, the quadrotor may orient itself to point an on-board camera or tool in a desired direction as a function of displacement along the path. Two methods to accomplish this are trajectory tracking and path following while secondary problems may deal with disturbance rejection (e.g., wind in outdoor environments), unmodelled dynamics (e.g., aerodynamic effects) and parameter uncertainties.

In the trajectory tracking case, the quadrotor tracks a time parameterized reference signal that moves along the path while simultaneously specifying yaw angle. The majority of the literature for quadrotor trajectory tracking uses a hierarchical method called backstepping. For instance, in [1] and [2], the controller is split into an outer translational control loop and an inner rotational control loop. In the outer loop, the reference signal is the desired displacement along the path and the control input is the thrust. There are also additional virtual inputs that together with the desired yaw angle, become the reference signals for the inner rotational loop. In this inner loop, the remaining control inputs representing body torques are assigned. In [1] a model predictive controller is used for the translational subsystem whereas a robust nonlinear \mathcal{H}^∞ controller is used for the rotational subsystem. In [2] a sliding-mode controller is used for both subsystems where neural networks are used for disturbance rejection. In [3], the approach has three control steps. The first step uses the thrust input and yawing torque to control the quadrotor height and yaw angle respectively. In the second step, the pitching torque is used to control y -position and pitch angle. In this step, a nested saturation control is used to bound the pitching torque. The third step is similar to the second, where the rolling torque is used to control x -position and roll angle. In [4], the authors use a feedback linearization approach where the path and yaw specifications are described through the system outputs and disturbance rejection is provided through an adaptive estimator. Other relevant work regarding quadrotor control

can be found in [5] and [6] which both present attitude, position and trajectory tracking controllers implemented on a testbed to obtain experimental results. Quaternion-based attitude control is presented in [7] while quadrotor maneuvers are explored in [8] and [9]. Specifically, [8] has a quadrotor fly through openings and perch on angled surfaces while [9] presents a learning strategy for flips.

The main issue with the trajectory tracking method is that if the quadrotor loses track of the reference signal, it may leave the path and collide with off-course objects. For this reason, we will focus on the path following approach.

The goal of the path following approach is to stabilize a set of permitted trajectories where the quadrotor is on the path and meets additional specifications regarding its speed and orientation as a function of its displacement. The advantage of the path following method comes from its invariance properties where if the quadrotor begins on a permitted trajectory, it will remain on it for all time. However, even if the quadrotor is not exactly on a permitted trajectory or has been perturbed slightly off of one, it will at least converge towards nearby points on the path rather than chasing a distant reference signal. This contrasts the trajectory tracking approach where this invariance property does not hold.

The development in this paper uses a feedback linearization method inspired by the work of Chris Nielsen in [10]. As in [4], we rely on an input dynamic extension to design a path following controller. Besides solving the path following problem, our control design allows one to specify the speed profile and yaw angle of the quadrotor along the path.

Notation. Throughout the paper, we will use the shorthand $c_\theta := \cos \theta$, $s_\theta := \sin \theta$, $t_\theta := \tan \theta$, $se_\theta := \sec \theta$. By df_x we will denote the Jacobian of a function f at x . If f is a real-valued function, the gradient of f with respect to the vector $x = \text{col}(x_1 \dots x_n)$ will be denoted by $\partial_{(x_1, \dots, x_n)} f$, or more concisely $\partial_x f$. Note that $\partial_x f = df_x^\top$ for a real-valued f . Finally, $v \cdot w$ denotes the Euclidean inner product between vectors v and $w \in \mathbb{R}^3$.

II. BACKGROUND

The quadrotor model developed in this section is standard. For instance, see Mokhtari, Benallegue, and Orlov in [4]. A quadrotor helicopter consists of four rotors connected to a rigid frame, as shown in Figure 1. Denote the inertial frame by \mathcal{I} , the body frame attached to the quadrotor centre of mass by \mathcal{B} and the distance from the centre of mass to the rotors by d . Corresponding to each rotor is a thrust force f_i acting along the z_b axis and a reaction torque τ_{r_i} that opposes the direction of rotation. To produce a thrust f_i in the negative z_b (i.e., upward) direction, the rotors on the x_b

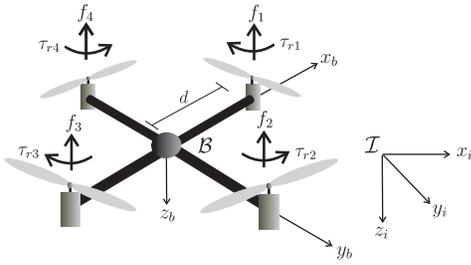


Fig. 1: Quadrotor system.

axis rotate in the clockwise direction and the rotors on the y_b axis rotate in the counter-clockwise direction.

The control inputs to the system are u_1, u_2, u_3, u_4 where u_1 is the total thrust along the z_b axis and u_2, u_3, u_4 are the torques about the x_b, y_b, z_b axes respectively. The physical inputs are the motor torques applied to the rotors denoted by τ_1, τ_2, τ_3 , and τ_4 . Using the development from Castillo, Lozano, and Dzul in [3], the rotor dynamics are given by $I_{rz} \dot{\Omega}_{ri} = -b\Omega_{ri}^2 + \tau_i$ where I_{rz} is the rotor moment of inertia about the rotor z -axis, Ω_{ri} is the angular speed of rotor i and b is a coefficient of friction due to aerodynamic drag on the rotor. There is also an algebraic relationship between the rotor thrust and rotor speed given by, $f_i = \gamma \Omega_{ri}^2$ where γ is a parameter that can be experimentally measured. If we assume steady-state rotor dynamics such that $\dot{\Omega}_{ri} = 0$ then $f_i = (\gamma/b)\tau_i = c\tau_i$ where $c = \gamma/b$ is the algebraic scaling factor between the rotor thrust and the applied motor torque. Using this fact, it is readily seen that the relationship between the control inputs and the motor torques is given by

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} c & c & c & c \\ 0 & -cd & 0 & cd \\ cd & 0 & -cd & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix} = M\tau.$$

In the above, the total thrust is equal to the summation of the four rotor thrusts; the torque about the x_b axis is proportional to the differential thrust of the two rotors on the y_b axis, $f_4 - f_2$; the torque about the y_b axis is proportional to the differential thrust of the two rotors on the x_b axis, $f_1 - f_3$; and the torque about the z_b axis is equal to the summation of the four reaction torques τ_{ri} which are equal and opposite to the applied motor torques τ_i . Therefore given u , one can determine the required motor torques by $\tau = M^{-1}u$. Let the states of the quadrotor system be given by,

$$\begin{aligned} X &= \text{col}(x, y, z): \text{position in } \mathcal{I} \\ \Phi &= \text{col}(\psi, \theta, \phi): \text{orientation (yaw, pitch, and roll)} \\ V &= \text{col}(u, v, w): \text{velocity in } \mathcal{I} \\ \omega &= \text{col}(p, q, r): \text{angular velocity in } \mathcal{B}. \end{aligned}$$

We now model the system dynamics neglecting gyroscopic effects and disturbances. Using Newton's equation, the translational dynamics of the quadrotor system are given by,

$$\begin{aligned} \dot{X} &= V \\ m\dot{V} &= \sum F_{ext} = mge_3 - u_1 Re_3 \end{aligned} \quad (1)$$

where $\sum F_{ext}$ is the sum of external forces composed of a gravity force and thrust force, m is the quadrotor mass, g is the acceleration due to gravity, $e_3 = [0 \ 0 \ 1]^T$ and R is the rotation matrix from \mathcal{B} to \mathcal{I} given by,

$$R = \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & s_\theta s_\phi s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}.$$

Using Newton-Euler's equation, the rotational dynamics of the quadrotor system are given by,

$$\begin{aligned} \dot{\Phi} &= \begin{bmatrix} 0 & s_\phi s e_\theta & c_\phi s e_\theta \\ 0 & c_\phi & -s_\phi \\ 1 & s_\phi t_\theta & c_\phi t_\theta \end{bmatrix} \omega \\ J\dot{\omega} + \omega \times J\omega &= \sum T_{ext} = [u_2 \ u_3 \ u_4]^T \end{aligned} \quad (2)$$

where $\sum T_{ext}$ is the vector of external torques in \mathcal{B} and J is the symmetric inertia matrix in \mathcal{B} with diagonal elements I_x, I_y and I_z . The state vector is taken as $\chi = \text{col}(X, \Phi, V, \omega) \in \mathcal{X} := \mathbb{R}^{12}$ and the system model can be represented in the control-affine form,

$$\dot{\chi} = f(\chi) + g(\chi)u, \quad y = h(\chi)$$

with the definitions of f, g derived from equations (1) and (2).

III. PROBLEM STATEMENT

The objective of the path following problem is to control the quadrotor helicopter to follow a predefined path in \mathbb{R}^3 while attaining a desired velocity and yaw angle as it travels along this path. This is stated more precisely below.

Path Following Problem (PFP): Consider the system given by equations (1) and (2). Given a Jordan curve \mathcal{C} in \mathbb{R}^3 , find a continuous feedback $u = \text{col}(u_1, \dots, u_4)$ such that under appropriate initial conditions, the closed-loop system meets the following goals:

- G1 The quadrotor asymptotically converges to \mathcal{C}
- G2 The velocity asymptotically converges to a specified value dependent on the quadrotor displacement along \mathcal{C}
- G3 The yaw angle asymptotically converges to a specified value dependent on the quadrotor displacement along \mathcal{C} .

The path, velocity and yaw specifications are formulated in terms of system outputs that we would like to asymptotically zero out.

Definition 3.1: A **path specification** is a smooth function $\text{col}(h_1(X), h_2(X)), \mathbb{R}^3 \rightarrow \mathbb{R}^2$, such that the path $\mathcal{C} = \{X : h_1(X) = h_2(X) = 0\}$ is a Jordan curve in \mathbb{R}^3 and $\partial_X h_1, \partial_X h_2$ are linearly independent on \mathcal{C} (i.e., zero is a regular value of $\text{col}(h_1, h_2)$).

The enforcement of the path specification meets goal G1 in PFP. Next, we would like to define velocity and yaw specifications.

Definition 3.2: A **velocity specification** is a smooth function $\mathcal{X} \rightarrow \mathbb{R}$ defined as $h_3(\chi) = (\partial_X h_1 \times \partial_X h_2) / \|\partial_X h_1 \times \partial_X h_2\| \cdot V - \alpha(X)$ where

the smooth function $\alpha(X) \neq 0$ specifies the velocity along \mathcal{C} .

For each point $X \in \mathcal{C}$ and each velocity vector $V \in \mathbb{R}^3$, the vector $(\partial_X h_1 \times \partial_X h_2) / \|\partial_X h_1 \times \partial_X h_2\| \cdot V$ represents the component of V tangent to the path \mathcal{C} at X . Thus, $h_3(\chi)$ expresses the requirement that when $X \in \mathcal{C}$ the component of the quadrotor's velocity tangential to \mathcal{C} be equal to a desired value given by $\alpha(X)$. The magnitude of $\alpha(X)$ is the desired speed, and its sign indicates the desired direction of travel along \mathcal{C} (clockwise or counter-clockwise). Hence, the enforcement of the velocity specification meets goal G2 in PFP.

Definition 3.3: A **yaw specification** is a smooth function $\mathcal{X} \rightarrow \mathbb{R}$ defined as $h_4(\chi) = \psi - \beta(X)$.

In the above, the desired yaw angle at a point X along \mathcal{C} is given by $\beta(X)$. The enforcement of the yaw specification meets goal G3 in PFP.

Let $s := \text{col}(s_1, \dots, s_4)$ and define $\pi : \mathcal{X} \rightarrow \mathbb{R}^3$ as $\pi(X, \Phi, V, \omega) = X$. Defining the output function $h : \mathcal{X} \rightarrow \mathbb{R}^4$ as

$$s = h(\chi) := \begin{bmatrix} h_1(\pi(\chi)) \\ h_2(\pi(\chi)) \\ \frac{\partial_X h_1 \times \partial_X h_2}{\|\partial_X h_1 \times \partial_X h_2\|} \cdot V - \alpha(X) \\ \psi - \beta(X) \end{bmatrix}, \quad (3)$$

the objective of PFP can be restated as that of designing a feedback such that, for a suitable set of initial conditions, $h(\chi(t)) \rightarrow 0$ along solutions of the closed-loop system. In other words, we want the set $\Gamma = h^{-1}(0)$ to be attractive for the closed-loop system. Attractivity alone, however, is not desirable in path following applications because any slight perturbation of the quadrotor off of Γ may result in system behaviour where the quadrotor initially diverges significantly from the path before converging back to it. Rather, we would like Γ to be asymptotically stable. This, however, is not possible since Γ is not controlled invariant, i.e., it cannot be made invariant by any choice of feedback. To illustrate, if the system is initialized on Γ but the velocity vector $V(0)$ is not tangent to \mathcal{C} , the quadrotor will leave \mathcal{C} , and hence Γ , no matter what feedback we choose. In light of this observation, we will stabilize the maximal controlled invariant subset of Γ which we call the **path following manifold** introduced in [10]. The path following manifold is simply the zero dynamics manifold of system (1), (2) with output (3), and it will be characterized in the next section.

IV. SOLUTION OF PFP

The simplest way to stabilize the zero dynamics of a nonlinear control system is to use, when feasible, input-output feedback linearization [11]. In the context of PFP, in order to apply this technique we need to find conditions under which the output $h(\chi)$ in (3) yields a well-defined vector relative degree. We begin by finding conditions under which a vector relative degree is well-defined. Then, we design a controller based on input-output feedback linearization, and we prove that it solves PFP.

A. Investigation of vector relative degree conditions

Taking time derivatives of the four outputs in (3) along the vector fields of the control system (1)-(2), one can check that the control inputs first appear in $d^2 s_1/dt^2$, $d^2 s_2/dt^2$, ds_3/dt , and $d^2 s_4/dt^2$ suggesting that if vector relative degree were well-defined it would be given by $\{2, 2, 1, 2\}$. However, one can check that the corresponding decoupling matrix D has the form

$$D = \begin{bmatrix} \star & 0 & 0 & 0 \\ \star & 0 & 0 & 0 \\ \star & 0 & 0 & 0 \\ \star & 0 & \star & \star \end{bmatrix},$$

so that D is always singular, implying that the vector relative degree $\{2, 2, 1, 2\}$ is not well-defined anywhere. Interestingly, the same problem arises when one wants to address the tracking problem, a fact pointed out in [4], although in this case the output function is completely different. Here, as in [4], we achieve a well-defined vector relative degree through the technique of input dynamic extension [11]. Specifically, we add two integrators at the thrust input u_1 , while leaving the remaining inputs unchanged. That is, we let

$$u_1 = \zeta, \quad \ddot{\zeta} = \bar{u}_1, \quad u_2 = \bar{u}_2, \quad u_3 = \bar{u}_3, \quad u_4 = \bar{u}_4 \quad (4)$$

where $\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4$ are the new control inputs. The quadrotor model with dynamic compensation is given by equations (1), (2), and (4) with the augmented state vector defined as $\bar{\chi} = \text{col}(X, \Phi, V, \omega, \zeta, \dot{\zeta}) \in \bar{\mathcal{X}} := \mathcal{X} \times \mathbb{R}^2$. The system model can be represented in the control-affine form,

$$\dot{\bar{\chi}} = \bar{f}(\bar{\chi}) + \bar{g}(\bar{\chi}) \bar{u}, \quad \bar{y} = \bar{h}(\bar{\chi}),$$

with the definitions of \bar{f}, \bar{g} derived from equations (1), (2), and (4). Letting $\bar{\pi} : \bar{\mathcal{X}} \rightarrow \mathcal{X}$ be the projection $(X, \Phi, V, \omega, \zeta, \dot{\zeta}) \mapsto (X, \Phi, V, \omega)$, the new output function $\bar{h} : \bar{\mathcal{X}} \rightarrow \mathbb{R}^4$ is simply given by

$$\bar{h}(\bar{\chi}) = h \circ \bar{\pi}(\bar{\chi}), \quad (5)$$

with h given in (3).

To solve PFP for the augmented system, we want to stabilize the maximal controlled invariant subset of $\bar{\Gamma} = \bar{h}^{-1}(0)$. The following lemma outlines the conditions under which the vector relative degree is well-defined for the augmented system.

Lemma 4.1: The augmented quadrotor system given by equations (1), (2) and (4) with output (5) has a well-defined vector relative degree $\{r_1, r_2, r_3, r_4\} := \{4, 4, 3, 2\}$ at a point $\bar{\chi} = (X, (\psi, \theta, \phi), V, \omega, \zeta, \dot{\zeta})$ if and only if $\zeta \neq 0$ and $\phi \neq \pm \frac{\pi}{2}$.

Proof: The determinant of the decoupling matrix D of the augmented system is given by

$$\det(D) = -\frac{d^3 \zeta^2 (\partial_\psi h_4) c_\phi}{m^3 I_x I_y I_z c_\theta} (\partial_X h_1 \times \partial_X h_2) \cdot \partial_V h_3.$$

It follows from the definitions of h_1, h_2, h_3 that $\partial_X h_1, \partial_X h_2$ and $\partial_V h_3$ form a linearly independent set. This is because $\partial_X h_1$ and $\partial_X h_2$ are linearly independent, $\|\partial_V h_3\| = 1 \neq 0$,

and $\partial_V h_3 = (\partial_X h_1 \times \partial_X h_2) / \|\partial_X h_1 \times \partial_X h_2\|$ is perpendicular to $\partial_X h_1$ and $\partial_X h_2$. It follows from the definition of h_4 that $\partial_\psi h_4 = 1 \neq 0$. Therefore $\det(D) \neq 0$ if and only if $\zeta \neq 0$ and $\phi \neq \pm \frac{\pi}{2}$.

It follows that the system has well-defined vector relative degree at any point $\bar{\chi}$ where the conditions $\zeta \neq 0$ and $\phi \neq \pm \frac{\pi}{2}$ are satisfied because the decoupling matrix D has full rank and $L_{\bar{g}_j} L_{\bar{f}}^k \bar{h}_i = 0$ for $i, j \in \{1, \dots, 4\}, k \in \{0, \dots, r_i - 2\}$. ■

Note also that, in addition to the conditions of the lemma above, we must impose $\theta \neq \pm \frac{\pi}{2}$ as this presents a singularity in the model inherited by the singularity in the Euler angle representation. Consider the set given by

$$\bar{\Gamma}^* = \{\bar{\chi} : L_{\bar{f}}^j \bar{h}_i = 0, j = 0, \dots, r_i - 1, i = 1, \dots, 4\}, \quad (6)$$

where $\{r_1, r_2, r_3, r_4\} := \{4, 4, 3, 2\}$. If the augmented quadrotor system has a well-defined vector relative degree $\{4, 4, 3, 2\}$ at each point $\bar{\chi} \in \bar{\Gamma}^*$, then it follows that $\bar{\Gamma}^*$ is the maximal controlled invariant subset of $\bar{\Gamma}$, which is precisely the set we wish to stabilize. In light of Lemma 4.1, in order to have a well-defined vector relative degree on $\bar{\Gamma}^*$ we need to determine whether, on $\bar{\Gamma}^*$, $\zeta \neq 0$, $\phi \neq \pm \pi/2$, and $\theta \neq \pm \pi/2$. This is the subject of the next proposition.

Proposition 4.2: For the augmented quadrotor system given by equations (1), (2), and (4), with output (5), there exists $K^* > 0$ such that for all $K \in (0, K^*)$, if the velocity specification is chosen so that $\max_{X \in \mathcal{C}} |\alpha(X)| \leq K$ and $\max_{X \in \mathcal{C}} \|d\alpha_X(X)\| \leq K$ the system has a well-defined vector relative degree $\{r_1, r_2, r_3, r_4\} = \{4, 4, 3, 2\}$ on $\bar{\Gamma}^*$.

Proof: Let $K_1 = \max_{X \in \mathcal{C}} |\alpha(X)|$ and $K_2 = \max_{X \in \mathcal{C}} \|d\alpha_X(X)\|$. The constants K_1 and K_2 exist and are finite because α is smooth and \mathcal{C} is compact. From Lemma 4.1, the system has well-defined vector relative degree and no Euler angle singularities are encountered on $\bar{\Gamma}^*$ if and only if,

- (i) ζ is bounded away from 0 on $\bar{\Gamma}^*$.
- (ii) ϕ, θ are bounded away from $\pm \frac{\pi}{2}$ on $\bar{\Gamma}^*$.

By definition, on $\bar{\Gamma}^*$ we have $\bar{h}_1(\bar{\chi}) = \bar{h}_2(\bar{\chi}) = 0$ and $L_{\bar{f}} \bar{h}_1(\bar{\chi}) = L_{\bar{f}} \bar{h}_2(\bar{\chi}) = 0$ or, recalling that $\bar{\chi} = (X, \Phi, V, \omega, \zeta, \dot{\zeta})$, $h_1(X) = h_2(X) = 0$ and $\partial_X h_1(X) \cdot V = \partial_X h_2(X) \cdot V = 0$. The latter two identities imply that, on $\bar{\Gamma}^*$, V is orthogonal to the vectors $\partial_X h_1(X)$ and $\partial_X h_2(X)$. We also have $\bar{h}_3(\bar{\chi}) = 0$, or $(\partial_X h_1 \times \partial_X h_2) / \|\partial_X h_1 \times \partial_X h_2\| \cdot V = \alpha(X)$. Since V is orthogonal to both $\partial_X h_1$ and $\partial_X h_2$, the above identity implies that on $\bar{\Gamma}^*$,

$$V = \mu(X)\alpha(X), \quad \mu(X) := \frac{\partial_X h_1 \times \partial_X h_2}{\|\partial_X h_1 \times \partial_X h_2\|}. \quad (7)$$

In particular, $\dot{V} = \dot{\mu}\alpha + \mu\dot{\alpha} = d\mu_X(\mu\alpha^2) + \mu d\alpha_X(\mu\alpha)$ where $\|\mu\| \leq 1$ and therefore, $\|\dot{V}\| \leq \|d\mu_X\|K_1^2 + K_1K_2$. Since $d\mu_X$ is continuous and $X \in \mathcal{C}$, a compact set, it follows that, on $\bar{\Gamma}^*$, $\|\dot{V}\| \leq K_1(K_1C + K_2)$, for some $C > 0$. On $\bar{\Gamma}^*$ we have

$$\|\dot{V}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + \zeta \begin{bmatrix} -\frac{1}{m}(c_\phi c_\psi s_\theta + s_\phi s_\psi) \\ -\frac{1}{m}(c_\phi s_\theta s_\psi + c_\psi s_\phi) \\ -\frac{1}{m}(c_\theta c_\phi) \end{bmatrix} \right\| \leq K_1(K_1C + K_2)$$

from which we have that $|\zeta| \geq m(g - K_1(K_1C + K_2))$, and thus for all $K_1(K_1C + K_2) < g$, $|\zeta| > 0$, implying that property (i) above is satisfied. We also have that $|\zeta| \leq m(g + K_1(K_1C + K_2))$. In other words, ζ is bounded from above on $\bar{\Gamma}^*$. Using the fact that $|\dot{w}| = |g - \zeta/m(c_\theta c_\phi)| \leq \|\dot{V}\| \leq K_1(K_1C + K_2)$, we have

$$|c_\theta c_\phi| \geq \frac{g - K_1(K_1C + K_2)}{\max \left| \frac{\zeta}{m} \right|} = \frac{g - K_1(K_1C + K_2)}{g + K_1(K_1C + K_2)}.$$

It thus follows that if $K_1(K_1C + K_2) < g$, $|c_\theta c_\phi| > 0$, and thus θ and ϕ are bounded away from $\pm \frac{\pi}{2}$ on $\bar{\Gamma}^*$, implying that property (ii) above holds. In conclusion, setting $K^* = \sqrt{g/(1+C)}$, for all $K \in (0, K^*)$ the augmented quadrotor system has a well-defined vector relative degree on $\bar{\Gamma}^*$. ■ Proposition 4.2 states that if the velocity specification is chosen so that the desired tangential velocity $\alpha(X)$ and $d\alpha_X(X)$ have a sufficiently small upper bound K , then the augmented quadrotor system has a well-defined vector relative degree on $\bar{\Gamma}^*$, and therefore this set is the path following manifold we wish to stabilize to solve PFP. The path following manifold is one-dimensional since $n - (r_1 + r_2 + r_3 + r_4) = 14 - (4 + 4 + 3 + 2) = 1$ where $n = 14$ is the number of states for the augmented system.

Remark 4.3: Recall the function $V = \mu(X)\alpha(X)$ in (7) which expresses the velocity of the quadrotor in terms of its displacement X when $\bar{\chi} \in \bar{\Gamma}^*$, and its time derivative $\dot{V} = d\mu_X(\mu\alpha^2) + \mu d\alpha_X(\mu\alpha)$. If one computes $K_1(K_1C + K_2) := \max_{X \in \mathcal{C}} \|\dot{V}\|$, then the proof of Proposition 4.2 provides the upper bounds K_1 and K_2 satisfying $K_1(K_1C + K_2) < g$. Since \mathcal{C} is a closed curve, the computation of \mathcal{C} , and hence of K_1 and K_2 , can be easily carried out numerically.

B. Controller Design

From now on we will assume that the velocity specification has been chosen so that $\max_{X \in \mathcal{C}} |\alpha(X)| < K^*$ and $\max_{X \in \mathcal{C}} \|d\alpha_X(X)\| < K^*$. The design of an input-output feedback linearization controller is standard. Let $b(\bar{\chi}) := \left[L_{\bar{f}}^{r_1} \bar{h}_1(\bar{\chi}) \quad \dots \quad L_{\bar{f}}^{r_4} \bar{h}_4(\bar{\chi}) \right]^T$ and define the feedback transformation

$$\bar{u} = D^{-1}(\bar{\chi})(-b(\bar{\chi}) + v), \quad (8)$$

where $v = \text{col}(v_1, \dots, v_4)$ is the new control input, and where $D(\bar{\chi})$ is the decoupling matrix with entries $D_{ij} = L_{\bar{g}_j} L_{\bar{f}}^{r_i-1} \bar{h}_i$, $i, j \in \{1, \dots, 4\}$. Let $\xi^i = \left[\bar{h}_i(\bar{\chi}) \quad \dots \quad L_{\bar{f}}^{r_i-1} \bar{h}_i(\bar{\chi}) \right]^T$, $i = 1, \dots, 4$, and choose v_i as

$$v_i = -k_{r_i-1} \xi_{r_i}^i - \dots - k_0 \xi_1^i, \quad i = 1, \dots, 4 \quad (9)$$

such that $s^r + k_{r_i-1} s^{r_i-1} + \dots + k_0$ has roots in the open left-half plane. The resulting control system is illustrated in Figure 2. Now the problem is whether the controller just designed indeed stabilizes the path following manifold $\bar{\Gamma}^*$. In view of the fact that

$$\dot{\xi}^i = \begin{bmatrix} 0 & I_{r_i-1} \\ 0_{1 \times r_i-1} & 0 \end{bmatrix} \xi^i + \begin{bmatrix} 0_{r_i-1 \times 1} \\ 1 \end{bmatrix} v_i, \quad (10)$$

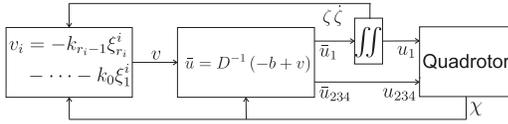


Fig. 2: Control system block diagram.

The input-output linearizing feedback guarantees that solutions of the closed-loop system originating in a neighbourhood of $\bar{\Gamma}^*$ are such that $\xi^i(\bar{\chi}(t)) \rightarrow 0$, $i = 1, \dots, 4$, provided that there are no finite escape times. Besides having to show that the closed-loop system has no finite escape times in a neighbourhood of $\bar{\Gamma}^*$, there is another issue that requires some analysis. Although $\bar{\Gamma}^* = \{\bar{\chi} : \xi^i(\bar{\chi}) = 0, i = 1, \dots, 4\}$, the fact that $\xi^i(\bar{\chi}(t)) \rightarrow 0$, $i = 1, \dots, 4$, does not imply, in general, that $\bar{\chi}(t) \rightarrow \bar{\Gamma}^*$. To illustrate, consider the function $\xi(\bar{\chi}) = \bar{\chi}/(1 + \bar{\chi}^2)$, and suppose that $\bar{\chi}(t) = t$. Then, $\xi(\bar{\chi}) \rightarrow 0$ but $\bar{\chi}(t)$ does not tend to $\{\bar{\chi} : \xi(\bar{\chi}) = 0\}$.

In order to address the two issues described above we define a diffeomorphism valid in a neighbourhood of $\bar{\Gamma}^*$ which maps the system into a standard normal form. In order to do that, we need some preliminary definitions.

Let L denote the length of the curve \mathcal{C} , and denote by S^1 the set of real numbers modulo L (this set is diffeomorphic to the unit circle). Fix a point o on \mathcal{C} , and define the map $\Lambda : \mathcal{C} \rightarrow S^1$ as $X \mapsto \eta$, where η is the arc length of the portion of \mathcal{C} from o to X found moving in the counter-clockwise direction. Since \mathcal{C} is a Jordan curve, the function Λ is a diffeomorphism $\mathcal{C} \rightarrow S^1$. Let $p(X)$ be the function mapping a point $X \in \mathbb{R}^3$ to the closest point on \mathcal{C} . Note that $p|_{\mathcal{C}}$ is the identity map. On some neighbourhood U of \mathcal{C} the function $p : U \rightarrow \mathcal{C}$ is well-defined and smooth. Finally, recall the definition of $\bar{\pi}$ and π

$$\bar{\chi} \in \bar{\mathcal{X}} \xrightarrow{\bar{\pi}} \chi \in \mathcal{X} \xrightarrow{\pi} X \in \mathbb{R}^3$$

and define a function $\lambda(\bar{\chi})$ as $\lambda(\bar{\chi}) := \Lambda \circ p \circ \pi \circ \bar{\pi}(\bar{\chi})$. By construction, λ is a smooth function in a neighbourhood of $\{\bar{h}_1(\bar{\chi}) = \bar{h}_2(\bar{\chi}) = 0\}$, and hence in a neighbourhood of $\bar{\Gamma}^*$. Now define the coordinate transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}^T = \sigma(\bar{\chi}) := \begin{bmatrix} \xi^1(\bar{\chi}) \dots \xi^4(\bar{\chi}) \\ \lambda(\bar{\chi}) \end{bmatrix}^T \quad (11)$$

Lemma 4.4: Under the conditions of Proposition 4.2, the function $\sigma(\bar{\chi}) : \bar{\mathcal{X}} \rightarrow \mathbb{R}^{13} \times S^1$ is a diffeomorphism of a neighbourhood of $\bar{\Gamma}^*$ onto its image, and $\bar{\Gamma}^*$ is diffeomorphic to S^1 .

Proof: According to the generalized inverse function theorem in [12], we must show that,

- (i) for all $\bar{\chi} \in \bar{\Gamma}^*$, $d\sigma_{\bar{\chi}}$ is an isomorphism
- (ii) $\sigma|_{\bar{\Gamma}^*} : \bar{\Gamma}^* \rightarrow S^1$ is a diffeomorphism.

For each $\bar{\chi} \in \bar{\Gamma}^*$, $d\sigma_{\bar{\chi}}$ has determinant,

$$\det(d\sigma_{\bar{\chi}}) = -\frac{\zeta^4 (\partial_{\psi} \bar{h}_4)^2 c_{\phi}^2}{m^6 c_{\theta}} \left[(\partial_{\bar{\chi}} \bar{h}_1 \times \partial_{\bar{\chi}} \bar{h}_2) \cdot \partial_{\bar{\chi}} \bar{h}_3 \right]^3 \cdot \left[(\partial_{\bar{\chi}} \bar{h}_1 \times \partial_{\bar{\chi}} \bar{h}_2) \cdot \partial_{\bar{\chi}} \lambda \right].$$

Under the conditions of proposition 4.2, $\zeta \neq 0$ and $\theta, \phi \neq \pm \frac{\pi}{2}$ on $\bar{\Gamma}^*$. Therefore, $\det(d\sigma_{\bar{\chi}}) \neq 0$ if and only if $\partial_{\bar{\chi}} \lambda$ is

linearly independent from $\partial_{\bar{\chi}} \bar{h}_1$ and $\partial_{\bar{\chi}} \bar{h}_2$ or, what is the same, if $\{\partial_X h_1, \partial_X h_2, \partial_X \Lambda\}$ is a linearly independent set, where $X = \pi \circ \bar{\pi}(\bar{\chi})$. That this is indeed the case on $\bar{\Gamma}^*$ follows from the fact that Λ is the displacement along \mathcal{C} and hence, $\partial_X \Lambda$ is tangent to \mathcal{C} and perpendicular to $\partial_X h_1$ and $\partial_X h_2$. To show (ii) consider the restriction of σ to $\bar{\Gamma}^*$, $\sigma|_{\bar{\Gamma}^*} = \text{col}(0, \dots, 0, \lambda|_{\bar{\Gamma}^*})$. The map $\lambda|_{\bar{\Gamma}^*}$ is smooth and surjective because it is the composition of four smooth surjective maps, $\bar{\pi} : \bar{\mathcal{X}} \rightarrow \mathcal{X}$, $\pi : \mathcal{X} \rightarrow \mathbb{R}^3$, $p : \mathbb{R}^3 \rightarrow \mathcal{C}$, and $\Lambda : \mathcal{C} \rightarrow S^1$. Therefore $\sigma|_{\bar{\Gamma}^*}$ is also smooth and surjective. Since $\Lambda : \mathcal{C} \rightarrow S^1$ is a diffeomorphism, proving injectivity of $\sigma|_{\bar{\Gamma}^*}$ is equivalent to proving injectivity of $(p \circ \pi \circ \bar{\pi})|_{\bar{\Gamma}^*} : \bar{\Gamma}^* \rightarrow \mathcal{C}$. Thus, we must show that given a point $X \in \mathcal{C}$, we can construct $\bar{\chi} \in \bar{\Gamma}^*$ uniquely such that $p \circ \pi \circ \bar{\pi}(\bar{\chi}) = X$. Since on $\bar{\Gamma}^*$ we have $\bar{h}_i(\bar{\chi}) = 0$, $i = 1, \dots, 4$, given a point $X \in \mathcal{C}$ we know that $V = \mu(X)\alpha(X)$, where μ was defined in (7), and $\psi = \beta(X)$. Moreover, $\dot{V} = d\mu_X(\mu\alpha^2) + \mu d\alpha_X(\mu\alpha)$. From the identity

$$\dot{V} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -\zeta/m(c_{\phi}c_{\psi}s_{\theta} + s_{\phi}s_{\psi}) \\ -\zeta/m(c_{\phi}s_{\psi}s_{\theta} - s_{\phi}c_{\psi}) \\ g - \zeta/m(c_{\theta}c_{\phi}) \end{bmatrix},$$

letting $a(X) = -m(\dot{V} - \text{col}(0, 0, g))$, we can express ζ, θ, ϕ as smooth functions of X with $\zeta = \|a(X)\|$, $\phi = \sin^{-1}(s_{\psi}a_1(X) - c_{\psi}a_2(X))$ and $\theta = \sin^{-1}(c_{\psi}a_1(X) + s_{\psi}a_2(X)/c_{\phi})$ where $\zeta > 0$, $-\frac{\pi}{2} \leq \phi, \theta \leq \frac{\pi}{2}$ hold from Proposition 4.2. Hence, ζ, ϕ, θ are specified through smooth functions of X . Also, $\dot{\zeta} = d\zeta_X V = d\zeta_X \mu(X)\alpha(X)$. Finally, we observe the relationship between Φ and ω given by,

$$\dot{\Phi} = \begin{bmatrix} 0 & s_{\phi}s_{\theta} & c_{\phi}s_{\theta} \\ 0 & c_{\phi} & -s_{\phi} \\ 1 & s_{\phi}t_{\theta} & c_{\phi}t_{\theta} \end{bmatrix} \omega = Y\omega$$

and hence, $\omega = Y^{-1}\dot{\Phi} = Y^{-1}d\Phi_X \mu(X)\alpha(X)$. This is a smooth function of X because Y^{-1} , Φ , μ and α are smooth functions of X , proving that $\sigma|_{\bar{\Gamma}^*}$ is injective and its inverse is smooth. ■

The important feature of Lemma 4.4 is the fact that the map σ is proved to be a diffeomorphism in a neighbourhood of the entire set $\bar{\Gamma}^*$, rather than just in a neighbourhood of a point. We now present the main result of this paper.

Theorem 4.5: For the augmented quadrotor system given by equations (1), (2), and (4), with output (5), and with the feedback (8)-(9) there exists $K^* > 0$ such that for all $K \in (0, K^*)$, if the velocity specification is chosen so that $\max_{X \in \mathcal{C}} |\alpha(X)| \leq K$ and $\max_{X \in \mathcal{C}} \|d\alpha_X(X)\| \leq K$ the path following manifold $\bar{\Gamma}^*$ in (6) is exponentially stable for the closed loop system, and hence PFP is solved in a neighbourhood of $\bar{\Gamma}^*$.

Proof: By Proposition 4.2, the feedback (8)-(9) is well-defined in a neighbourhood of $\bar{\Gamma}^*$. By Lemma 4.4 the function $\sigma(\bar{\chi}) : \bar{\mathcal{X}} \rightarrow \mathbb{R}^{13} \times S^1$ is a diffeomorphism of a neighbourhood of $\bar{\Gamma}^*$ onto its image, and $\bar{\Gamma}^*$ is diffeomorphic to S^1 , and hence compact. In (ξ, η) coordinates, the set $\bar{\Gamma}^*$ is given by $\sigma(\bar{\Gamma}^*) = \{(\xi, \eta) : \xi = 0\}$. The ξ subsystem

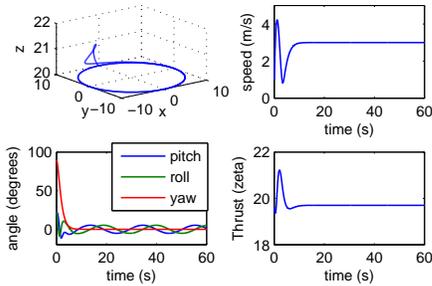


Fig. 3: Case 1: simulation results for $\alpha = 3$.

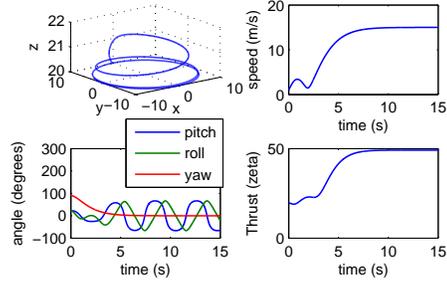


Fig. 4: Case 2: simulation results for $\alpha = 15$.

in (10) is LTI and the feedback (8)-(9) makes it exponentially stable. Thus, for all $\bar{\chi}(0)$ near $\bar{\Gamma}^*$ or, what is the same, for all $\xi(0)$ near 0, $\xi(t) \rightarrow 0$ exponentially as long as there are no finite escape times. There cannot be finite escape times because $\xi(t)$ is bounded and $\eta(t) \in S^1$, a compact set. Hence, $\sigma(\bar{\Gamma}^*)$ is exponentially stable, which implies that $\bar{\Gamma}^*$ is exponentially stable as well. ■

V. SIMULATION RESULTS

In this section, simulation results are presented for the quadrotor specified to travel at a constant speed along a circular path of radius r parallel to the $x - y$ plane, at a height of $z = 20m$ and yaw angle of 0° . That is, the specifications are given by $\bar{h}_1 = x^2 + y^2 - r^2$, $\bar{h}_2 = z - 20$, $\bar{h}_3 = uy/\sqrt{x^2 + y^2} - vx/\sqrt{x^2 + y^2} - \alpha$ and $\bar{h}_4 = \psi$. The initial conditions are taken as $\zeta = mg$, $\dot{\zeta} = 0$, $(x, y, z) = (0, 10, 20)m$, $(u, v, w) = (-1, 0, 0)m/s$ and $(\phi, \theta, \psi) = (\frac{\pi}{8}, \frac{\pi}{8}, \frac{\pi}{2})rad$. The parameters are chosen as in [4] to be $m = 2Kg$, $I_x, I_y, I_z = 1.2416N \cdot m/rad/s^2$, $d = 0.1m$ and $g = 9.81m/s^2$. On Γ^* , the relationship $V = \mu(X)\alpha(X)$ in (7) is given by $V = \alpha/\sqrt{x^2 + y^2} [y \ -x \ 0]^T$. Therefore $\dot{V} = \alpha^2/(x^2 + y^2) [-x \ -y \ 0]^T$ and $\max_C \|\dot{V}\| = \alpha^2/r$, which corresponds to the centripetal acceleration of the quadrotor moving around a circle at constant speed $|\alpha|$. Following Remark 4.3, we must choose α so that $\alpha^2/r < g$, or $|\alpha| < \sqrt{rg}$. Taking $r = 10m$, we must pick $|\alpha| < 9.9$. Two simulation cases are considered where the quadrotor travels around the path with a constant speed of $\alpha = 3m/s$ and $\alpha = 15m/s$ respectively. Therefore, the first case meets the conditions above but the second does not.

Figures 3 and 4 show simulation results for case 1 and case 2 respectively. In both cases, the quadrotor successfully converges to the path \mathcal{C} . Also, the velocity and yaw angle converge to the desired values. One difference between the two cases are the roll and pitch angles. In the first case, the two angles have a maximum magnitude of 5.25° while for the second, the maximum magnitude is 66.45° . Another difference is an increased thrust input for the second case. In neither case does the quadrotor hit a point where $\phi, \theta = \pm\pi/2$ or $\zeta = 0$ and therefore, the controller is well-defined for the specific initial conditions we have considered. Note, however, that in case 2 there is no guarantee that for any initial condition near $\bar{\Gamma}^*$ the solution does not cause singularities in the controller.

VI. CONCLUSIONS

We have presented a basic path following controller which relies on input dynamic extension and feedback linearization to solve the path following problem for a quadrotor helicopter. Our controller allows the designer to specify a speed profile on the path and the yaw angle of the quadrotor as a function of its displacement along the path. In future research we will investigate the solution of the same problem without employing input dynamic extension, and we will address issues of robustness against unmodelled effects such as aerodynamic drag forces and parametric uncertainties.

REFERENCES

- [1] G. V. Raffo, M. Ortega, and F. Rubio, "An integral predictive/nonlinear \mathcal{H}^∞ control structure for a quadrotor helicopter," *Automatica*, vol. 46, pp. 29–39, 2010.
- [2] A. Das, F. Lewis, and K. Subbarao, "Backstepping approach for controlling a quadrotor using lagrange form dynamics," *Journal of Intelligent Robot Systems*, vol. 56, pp. 127–151, 2009.
- [3] P. Castillo, R. Lozano, and A. Dzul, "Stabilization of a mini rotorcraft with four rotors," *IEEE Control Systems Magazine*, pp. 45–55, 2005.
- [4] A. Mokhtari, A. Benallegue, and Y. Orlov, "Exact linearization and sliding mode observer for a quadrotor unmanned arial vehicle," *International Journal of Robotics and Automation*, vol. 21, no. 1, pp. 39–49, 2006.
- [5] N. Michael, D. Mellinger, Q. Lindsey, and V. Kumar, "The grasp multiple micro-UAV testbed," *IEEE Robotics and Automation Magazine*, vol. 17, pp. 56–65, 2010.
- [6] G. Hoffmann, H. Huang, S. Waslander, and C. Tomlin, "Precision flight control for a multi-vehicle quadrotor helicopter testbed," *Control Engineering Practice*, vol. 19, pp. 1023–1036, 2011.
- [7] A. Tayebi and S. McGilvray, "Attitude stabilization of a VTOL quadrotor aircraft," *IEEE Transactions on Control Systems Technology*, vol. 14, no. 3, pp. 562–566, 2006.
- [8] D. Mellinger, N. Michael, and V. Kumar, "Trajectory generation and control for precise aggressive maneuvers with quadrotors," in *Proceedings of the International Symposium on Experimental Robotics*, 2010.
- [9] S. Lupashin, A. Schöllig, M. Sherback, and R. D'Andrea, "A simple learning strategy for high-speed quadcopter multi-flips," in *Proceedings of the 2010 IEEE International Conference on Robotics and Automation*, May 3-8 2010, pp. 1642–1648.
- [10] C. Nielsen, "Set stabilization using transverse feedback linearization," Ph.D. dissertation, University of Toronto, 2009.
- [11] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Springer Verlag, 1995.
- [12] V. Guillemin and A. Pollack, *Differential topology*. New Jersey: Prentice Hall, 1974.