Robust Output Feedback Stabilization, Compressors Surge and Stall Example

by

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For My Father

Abstract

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This thesis introduces an approach to output feedback stabilization of SISO nonlinear systems which are not uniformly completely observable and are affected by disturbances. This approach is applied to the output feedback control of axial flow compressors in jet engines represented by a three state model developed by Moore and Greitzer. The extension of this approach to the case when constant unknown parameters affect the plant is addressed and its limitations are discussed.

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If the facts don't fit the theory, change the facts.

-Albert Einstein

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Chapter 1

Introduction

Two of the fundamental instability phenomena in axial flow compressors are surge and rotating stall. Rotating stall develops when there is a region of stagnant flow rotating around the circumference of the compressor which causes undesired vibrations in the blades and reduces pressure rise of the compressor and surge is an axisymmetric oscillation of the flow through the compressor that can cause damage and undesired vibrations in other components of the system [23]. Moore and Greitzer in [26] developed a model for axial flow compressors which is frequently used in the control literature.

The Moore-Greitzer model has three states which are rotating stall, mass flow and pressure rise (see Section 3.1.5). One of the applications of axial air flow compressors is in jet engines. Commercial jet engines include temperature, pressure and rotor speed sensors, but most often do not include sensors for mass flow, hence making it desirable to estimate. This, together with the fact that sensors for stall are not commercially available, makes the problem of controlling stall and surge inherently an output feedback control problem, whereby the only measurable variable is the pressure rise. One of the features of Moore-Greitzer three state model (MG3) is that when pressure rise is the only measurable state, the model becomes unobservable whenever there is no flow through the compressor; in other words it is observable on an open region of the state space, rather than everywhere, or it is not *uniformly completely observable* (Non-UCO). This is one of the reasons that most of the researchers have only focused on the development of state feedback controllers for this model which may not be implementable since they assume that all of the states of this model are measurable. To the best of our knowledge, output feedback control solutions using only pressure rise as the measurable state in MG3 do not rely on the estimation of the entire state of the system, and, except in [23], no attempt has been made to solve this problem. In [23], the authors apply the observer that they have developed in [16] to estimate the entire state based on the pressure rise as the only measurable state.

A missing feature of the work in [23] is the absence of any external disturbances or uncertainties in the MG3 model they use, which is the motivation of this thesis. We first tried to apply the observer developed in [16] to an MG3 model subject to some general external disturbances. In our path we realized that the technique we were developing for this particular MG3 problem could be applied to a class of non-UCO nonlinear systems whose observability maps have a particular specification (see Assumption A1) , and are subject to some slow varying or constant external disturbances. This observation led us to develop a more general theory for this class of systems which also includes the MG3 model. This theory is illustrated in Chapter 2. As the next natural extension, we considered the case when the MG3 model is affected by constant unknown disturbances. Our attempts in this direction were not successful and we realized that the observability mapping in MG3 and systems with uncertainties similar to the MG3 model has a feature (see Section 4.4) that prevents us from using the observer in [16] with its current format. This line of work needs further investigation which is beyond the scope of this thesis. As a result we have not developed a general theory for non-UCO systems with uncertainties, but again we show that the observer in [16] can be applied to a class of nonlinear non-UCO systems whose observability mappings have a particular feature (see Section 4.4).

1.1 Literature Review

In what follows we present a literature review both on nonlinear output feedback stabilization and the surge and rotating stall control problem and later we outline the organization of this thesis. We emphasize that the MG3 application part of this thesis is purely theoretical and is therefore to be seen as an application of the theory developed in Chapter 2.

1.1.1 Output Feedback Control

In 1992 H. Khalil and F. Esfandiari published their work [1] on nonlinear output feedback area where they introduced a new technique for designing robust output feedback controllers for input-output linearizable systems. Their technique has two features. First a high-gain observer that estimates the derivatives of the output and second, a bounded state feedback control, obtained by saturating a continuous state feedback function outside of a compact set, and hence protecting the state of the plant from peaking when the high gain observer estimates are used. This technique has been adopted by several researchers in their work (see e.g. [2, 3, 4, 5, 6, 7]) to solve various problems in output feedback control. In most of these papers the authors consider input-output feedback linearizable systems. One of the remarkable works in this line is [9], which proves a separation principle for a rather general class of nonlinear systems.

In most of the existing and mentioned works in the literature on output feedback control, the controller is designed in two steps. First, a globally bounded state feedback control is designed. Second, a *high-gain, fast* observer, recovers the performance achieved by the state feedback. Most of these works have two common features which are, the assumption of uniformly complete observability, and using the vector $[y, \dot{y}, \ldots, y^{(n_y)}, u, \dot{u}, \ldots, u^{(n_u)}]^\top$ as feedback, for some integers n_y, n_u, where y and u denote the system output and input, respectively. The latter feature implies that, for systems which are not input-output feedback linearizable, the explicit inverse of the observability mapping has to be known. Both of the mentioned features have the disadvantage that in some situations systems may not possess either or both of these two properties.

In [16], which this thesis is based on, the authors achieve a separation principle for systems that are not observable on some regions of the state space and the input space, or in short, systems that are not uniformly completely observable (non-UCO). The interesting feature of their work is that for implementation of the controller, they do not need the knowledge of the explicit inverse of the observability mapping. As we will show, these two properties are essential for our work on the MG3 model.

1.1.2 Surge and Stall

In 1986 Moore and Greitzer developed a three-state model (MG3) for axial flow compressors which became a benchmark in the control literature for several researchers (e.g., [24, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]) to design stabilizing controllers for stall and surge. As we mentioned earlier, most of the existing results, in some cases robust ones (e.g. see [24]), assume that all the states are measurable and the focus is on the development of state feedback controllers, which may not be implementable, and in the area of output feedback control, where compressor pressure rise is the only output, not all the system states are estimated (see e.g. Sections 12.6 and 12.7 in [40]). For a rather complete discussion on the literature and existing models of axial compressors and methods for controlling surge and stall phenomena, one can refer to [39]. The work in [23] is the only one which addresses the output feedback control problem for MG3 by estimating the entire state of the system. However, a limitation of [23] is that no disturbances affect the model.

As in all physical systems, disturbances occur in compression systems. Authors in [17] and [18] study the effect of circumferential inlet flow distortion on the stability properties.

In [19] and [20] mass flow and pressure disturbances are studied. Authors in [21] study the effects of inlet temperature distortion and find that these effects can be as significant as those of inlet pressure distortion. As in [19], the effect of pressure and flow disturbances are considered in this thesis (see Section 3.1.6).

1.2 Organization of the Thesis

In Chapter 2 we introduce the class of SISO systems affected by a particular kind of disturbance which the technique in [16] can be applied to, and we develop the theory which underlies our technique. In Chapter 3, after a brief discussion on the Moore-Greitzer model for surge and stall in axial flow compressors, we apply our technique to stabilize the MG3 model in the presence of disturbances. In Chapter 4 we discuss our attempts to apply the technique in [16] to the MG3 model when there is uncertainty in the compressor model, and after that we discuss the class of systems with uncertainty that this output feedback stabilization technique can be applied to.

Chapter 2

Disturbed Non-UCO Systems, the Theory

In this chapter we develop an output feedback control theory for a class of SISO, non-UCO, nonlinear systems that are affected by some disturbances. First we describe the problem formulation and our assumptions. Later we describe our nonlinear observer structure and prove its properties. Finally we prove the closed loop stability when using the observer estimates at the control input. Parts of this chapter rely on the work in [16].

2.1 Problem Formulation

Consider the following system,

$$\begin{aligned} \dot{x} &= f(x,u) + \delta(t) \\ y &= h(x,u) \end{aligned} \tag{2.1}$$

where $x(\cdot), \delta(\cdot) \in \mathbb{R}^n, \delta(t)$ is a bounded disturbance with bounded time derivatives and $\sup_t \|\delta(t)\| = d$, $u, y \in \mathbb{R}$, f and h are known smooth functions and $f(0,0) = 0$. In this chapter we show that the closed-loop output feedback system depicted in the Figure 2.1 is uniformly ultimately bounded. $\Upsilon(\hat{x}^P, z, y)$ represents the observer (2.49), $0 \leq n_u \leq n$

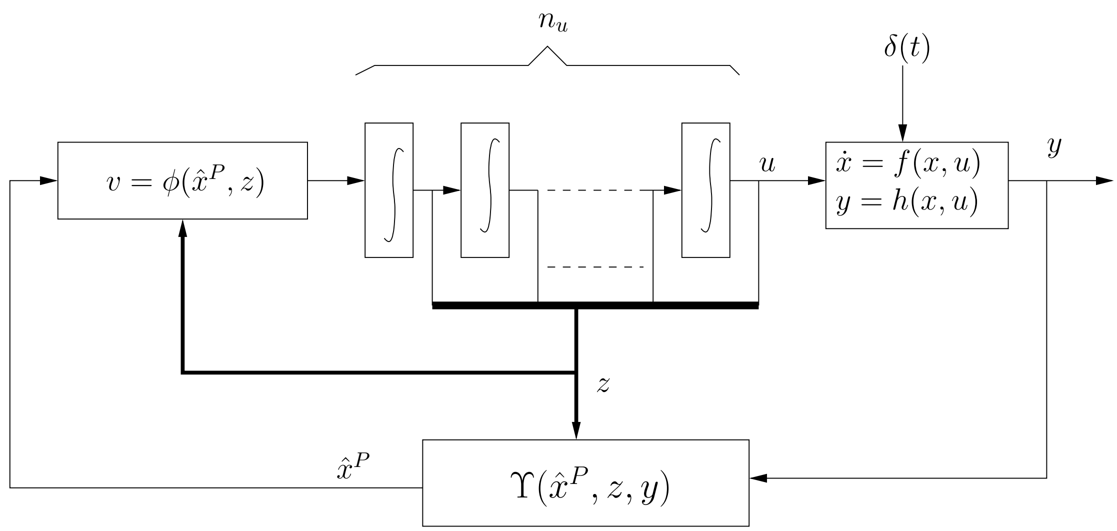

Figure 2.1: Block diagram.

is an integer and $v = \phi(\hat{x}^P, z)$ is the control law which is the input to a chain of n_u integrators. We also find an estimation of the domain of attraction of closed-loop system and a bound on the disturbance $\delta(t)$.

We start by defining the following observability mapping H,

$$y_e \triangleq \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(x,u) \\ \phi_1(x,z) \\ \vdots \\ \phi_{n-1}(x,z) \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_1(\bar{\delta}) \\ \vdots \\ \theta_{n-1}(\bar{\delta}) \end{bmatrix} = H_1(x,z) + H_2(\bar{\delta}) = H(x,z,\bar{\delta}) \qquad (2.2)$$

(with a slight abuse of the notation, notice that y_e is a vector in $\mathbb{R}^n$ and not the $e-th$ element of y) where $z = (u, \dot{u}, \ddot{u}, \ldots, u^{(n_u-1)})$, $\bar{\delta} = (\delta, \dot{\delta}, \ldots, \delta^{(n_\delta)})$, n_u and n_δ are integers $(0 \leq n_u \leq n, 0 \leq n_\delta \leq n-1)$, and

$$\dot{y} = y_{e,2} = \frac{\partial h}{\partial x} f(x,u) + \frac{\partial h}{\partial u}\dot{u} + \frac{\partial h}{\partial x}\delta(t) \tag{2.3}$$

$$= \phi_1(x,z) + \theta_1(x,\bar{\delta})$$

$$\ddot{y} = y_{e,3} = \frac{\partial \phi_1(x,z)}{\partial x} f(x,u) + \frac{\partial \phi_1(x,z)}{\partial z}\dot{z} + \frac{\partial \phi_1(x,z)}{\partial x}\delta(t) + \frac{\partial \theta_1(\bar{\delta})}{\partial \bar{\delta}}\dot{\bar{\delta}}(t) \tag{2.4}$$

$$= \phi_2(x,z) + \theta_2(x,z,\bar{\delta})$$

$$\vdots$$

$$y^{(i-1)} = y_{e,i} = \frac{\partial \phi_{i-2}(x,z)}{\partial x} f(x,u) + \frac{\partial \phi_{i-2}(x,z)}{\partial z}\dot{z} + \frac{\partial \phi_{i-2}(x,z)}{\partial x}\delta(t) + \frac{\partial \theta_{i-2}(\bar{\delta})}{\partial \bar{\delta}}\dot{\bar{\delta}}(t) \tag{2.5}$$

$$= \phi_{i-1}(x,z) + \theta_{i-1}(x,z,\bar{\delta})$$

$$\vdots$$

where

$$\frac{\partial \phi_i(x,z)}{\partial z}\dot{z} \triangleq \sum_{k=1}^{n_i} \frac{\partial \phi_i}{\partial z_k} z_{k+1} \tag{2.6}$$

Remark 1: In general θ_i's are functions of x, z and $\bar{\delta}$, but in our case we assume that θ_i's are functions of only $\bar{\delta}$ to get the form of mapping (2.2); see assumption A1.

The map (2.2) is parameterized by a vector z containing the derivatives of the control input u, which for our purpose should be available for feedback. To this end, we augment the system dynamics with n_u integrators at the input side, which corresponds to using a compensator of order n_u, so that the state of the compensator gives precisely the desired vector z. System (2.1) can be rewritten as follows,

$$\dot{x} = f(x, z_1) + \delta(t),\ \dot{z}_1 = z_2,\ \ldots,\ \dot{z}_{n_u} = v. \tag{2.7}$$

Define the extended state variable $X = [x^\top, z^\top]^\top \in \mathbb{R}^{n+n_u}$, and the associated *extended* *system*

$$\dot{X} = f_e(X) + g_e v + \delta_e(t)$$
$$y = h_e(X) \tag{2.8}$$

where $f_e(X) = [f^\top(x, z_1),\ z_2, \ldots, z_{n_u}, 0]^\top$, $g_e = [0, \ldots, 1]^\top$, $\delta_e(t) = [\delta(t), 0, \cdots, 0]^\top$, and $h_e(X) = h(x, z_1)$. Now, we are ready to state our assumptions.

Assumption A1(Observability): Assume that $\forall t \geq 0$, $\bar{\delta}(t) \in \Delta$, where Δ is a compact set. Suppose $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ which is an open set contains the origin, and $F_1 : \mathcal{O} \to \mathbb{R}^n \times \mathbb{R}^{n_u}$ and $F_2 : \mathcal{O} \to \mathbb{R}^n \times \mathbb{R}^{n_u}$ are two smooth functions such that the mapping $(x, z) \mapsto (y_e, z)$ has the following structure,

$$Y = \begin{bmatrix} y_e \\ z \end{bmatrix} = F(x, z, \bar{\delta}) \triangleq F_1(x, z) + F_2(\bar{\delta}) = \begin{bmatrix} H_1(x, z) \\ z \end{bmatrix} + \begin{bmatrix} H_2(\bar{\delta}) \\ 0 \end{bmatrix}. \tag{2.9}$$

Assume further that H_1 is a diffeomorphism with respect to its first argument over $\mathcal{O}$ so that, $\forall Y \in \mathcal{Y} \triangleq \{Y \in \mathbb{R}^{n+n_u} | \, Y = F_1(x, z) + F_2(\bar{\delta}), (x, z) \in \mathcal{O}, \bar{\delta} \in \Delta\}$,

$$\begin{bmatrix} x \\ z \end{bmatrix} = F^{-1}(Y, \bar{\delta}) \triangleq F_1^{-1}(Y - F_2(\bar{\delta})) = \begin{bmatrix} H_1^{-1}(y_e - H_2(\bar{\delta}), z) \\ z \end{bmatrix}. \tag{2.10}$$

In other words, we assume that as long as $(x, z) \in \mathcal{O}$, having the output derivatives y_e and knowing the disturbances in $\bar{\delta}$, one can calculate the state x. We later show that how to deal with the unknown vector $\bar{\delta}$.

Assumption A2(Input to State Stability): There exists a smooth function $\bar{u}(x)$ such that system

$$\dot{x} = f(x, \bar{u}(x)) + \delta(t) \tag{2.11}$$

viewed as a system with state x and input $\delta(t)$ is input-to-state-stable.

Considering $\dot{x} = f(x, \bar{u}(x))$ as an unforced system, this assumption implies that a bounded disturbance $\delta(t)$ yields bounded state trajectories, so in particular it does not drive the unforced system to instability. We will use this assumption in Section 2.4 to prove closed-loop stability when using the observer states at the control input.

Remark 2: Assumption A2 implies that there exists a smooth function $v(X) = \phi(x, z)$ such that the augmented system (2.7) is also input-to-state-stable. The proof is a special case of Lemma 5.4 (ISS-Backstepping Lemma) and Corollary 5.5 in [12] when backstepping and no adaptation are used. For the sake of illustration we will show a proof for a system with two integrators (i.e. $n_u = 2$) in the next chapter.

It is obvious that when $\delta(t) = 0$, systems (2.11) and (2.7) are asymptotically stable with the $\bar{u}(x)$ and $v(X)$, respectively.

Remark 3: Notice that if in some neighborhood of the origin, $[\partial f/\partial x]$ and $[\partial f_e/\partial X]$ are bounded and only local properties are needed, then it will be enough to find a smooth function $\bar{u}(x)$ such that the origin of the system $\dot{x} = f(x, \bar{u}(x))$ is asymptotically stable. This implies that when $\delta(t) = 0$, the origin of the extended system (2.7) is locally stabilzing by a function of X, $\bar{v}(X)$. A proof of the local stabilizability property for (2.7) may be found, e.g., in [10], and its global counterpart can be found in, e.g., Theorem 9.2.3 in [11] or Corollary 2.10 in [12].

When $\delta(t) \neq 0$, once local asymptotic stability is established, by Lemma 5.4 in [14], system (2.7) with $v = \phi(x, z)$ as the state feedback controller, viewed as a system with state (x, z) and input δ, is locally input-to-state-stable.

2.2 Nonlinear Observer

Assumption A2 allows us to design a stabilizing state feedback control law $v = \phi(x, z)$. In order to perform output feedback control x should be replaced by its estimate.

Many researchers have considered undisturbed systems ($\delta(t) = 0$) in which the knowledge of H_1^{-1} in (2.2) is needed (e.g. [7, 8, 9]). In that case, with knowing $x = H_1^{-1}(y_e, z)$ explicitly, one can estimate $\hat{x} = H_1^{-1}(\hat{y}_e, z)$ with the estimation of $n-1$ derivatives of y, since the vector z, containing the states of the controller, is known. Next, to estimate the derivatives of y, a high gain observer can be employed.

Since, even if H_1^{-1} exists, it may not be possible to calculate it explicitly, we use the observer in [16] where instead of estimating the derivatives of y and using $H_1(\cdot,\cdot)$, x is estimated directly.

The observer has the form[1]

$$\begin{aligned} \dot{\hat{x}} &= \hat{f}(\hat{x}, z, y) = f(\hat{x}, z_1) + \underbrace{\left[\frac{\partial H_1(\hat{x}, z)}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L[y(t) - \hat{y}(t)]}_{Q} \\ \hat{y}(t) &= h(\hat{x}, z_1) \end{aligned} \tag{2.12}$$

where L is an $n \times 1$ vector, $\mathcal{E} = \operatorname{diag}[\rho, \rho^2, \ldots, \rho^n]$, and $\rho \in (0, 1]$ is a fixed design constant.

Notice that (2.12) does not require any knowledge of H_1^{-1} and has the advantage of operating in x-coordinates. Assumption A1 implies that the Jacobian of the mapping H_1 with respect to x is invertible, and hence the inverse of $\partial H_1(\hat{x}, z)/\partial \hat{x}$ in (2.12) is well-defined.

[1]Throughout this section we assume A1 to hold globally, since we are interested in the ideal convergence properties of the state estimates. In the next section we will show how to modify the observer equation in order to achieve the same convergence properties when A1 holds over the set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$.

We show the properties of the observer (2.12) in in new coordinates using transformation (2.2). We start by expressing system (2.1) in the new coordinates. By definition we have

$$y_e \triangleq \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(x,u) \\ \phi_1(x,z) \\ \vdots \\ \phi_{n-1}(x,z) \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_1(\bar{\delta}) \\ \vdots \\ \theta_{n-1}(\bar{\delta}) \end{bmatrix} \tag{2.13}$$

Using the definition of ϕ_{n-1} in (2.6) we have

$$\begin{aligned} y^{(n)} = \dot{y}_{e,n} =& \frac{\partial \phi_{n-1}(x,z)}{\partial x} f(x,u) + \frac{\partial \phi_{n-1}(x,z)}{\partial x}\delta(t) \\ &+ \sum_{k=1}^{n_u-1} \frac{\partial \phi_{n-1}(x,z)}{\partial z_k} z_{k+1} + \frac{\partial \phi_{n-1}(x,z)}{\partial z_n} v \\ &+ \frac{\partial \theta_{n-1}(\bar{\delta})}{\partial \bar{\delta}} \dot{\bar{\delta}} \end{aligned}$$

Using the following definitions for α and β

$$\begin{aligned} \left(\frac{\partial \phi_{n-1}(x,z)}{\partial x} f(x,u) + \sum_{k=1}^{n_u-1} \frac{\partial \phi_{n-1}(x,z)}{\partial z_k} z_{k+1} \right)\Bigg|_{(x,z)=F^{-1}(Y,\bar{\delta})} &= \alpha(y_e,z) \\ \frac{\partial \phi_{n-1}(x,z)}{\partial z_n} v \Bigg|_{(x,z)=F^{-1}(Y,\bar{\delta})} &= \beta(y_e,z)v \end{aligned} \tag{2.14}$$

we have

$$\dot{y}_{e,n} = \alpha(y_e,z) + \beta(y_e,z)v + \frac{\partial \phi_{n-1}(x,z)}{\partial x}\delta(t) + \frac{\partial \theta_{n-1}(\bar{\delta})}{\partial \bar{\delta}} \dot{\bar{\delta}}.$$

Therefore, system (2.1) in the new coordinates is

$$\dot{y}_e = A_c y_e + B_c \left[\alpha(y_e,z) + \beta(y_e,z)v + \frac{\partial \phi_{n-1}(x,z)}{\partial x}\delta(t) + \frac{\partial \theta_{n-1}(\bar{\delta})}{\partial \bar{\delta}} \dot{\bar{\delta}} \right]. \tag{2.15}$$

Transforming observer (2.12) to the new coordinates

$$\hat{y}_e = \left[\hat{y}, \dot{\hat{y}}, \ldots, \hat{y}^{(n-1)}\right]^\top = H_1(\hat{x}, z) + H_2(\bar{\delta}) = H(\hat{x}, z, \bar{\delta}) \tag{2.16}$$

we have

$$\begin{aligned}
\dot{\hat{y}}_{e,1} = \frac{\partial h(\hat{x},u)}{\partial \hat{x}}\dot{\hat{x}} &= \frac{\partial h(\hat{x},u)}{\partial \hat{x}} f(\hat{x},u) + \frac{\partial h(\hat{x},u)}{\partial \hat{x}} Q + \frac{\partial h(\hat{x},u)}{\partial u}\dot{u} \\
&= \phi_1(\hat{x},u) + \theta_1(\bar{\delta}) - \theta_1(\bar{\delta}) + \frac{\partial h(\hat{x},u)}{\partial \hat{x}} Q \\
&= \hat{y}_{e,2} + \frac{\partial h(\hat{x},u)}{\partial \hat{x}} Q \underbrace{-\theta_1(\bar{\delta})}_{\hat{\theta}_1(\bar{\delta})}.
\end{aligned}$$

For $i = 2, \ldots, n-1$ we have

$$\begin{aligned}
\dot{\hat{y}}_{e,2} =& \frac{\partial \phi_1(\hat{x},z)}{\partial \hat{x}}\dot{\hat{x}} + \frac{\partial \phi_1(\hat{x},z)}{\partial z}\dot{z} + \frac{\partial \phi_1(\hat{x},z)}{\partial \hat{x}} Q + \frac{\partial \theta_1(\bar{\delta})}{\partial \bar{\delta}}\dot{\bar{\delta}} \\
=& \frac{\partial \phi_1(\hat{x},z)}{\partial \hat{x}}\dot{\hat{x}} + \frac{\partial \phi_1(\hat{x},z)}{\partial z}\dot{z} - (\theta_2(\bar{\delta}) - \frac{\partial \theta_1(\bar{\delta})}{\partial \bar{\delta}}\dot{\bar{\delta}}) \\
&+ (\theta_2(\bar{\delta}) - \frac{\partial \theta_1(\bar{\delta})}{\partial \bar{\delta}}\dot{\bar{\delta}}) + \frac{\partial \theta_1(\bar{\delta})}{\partial \bar{\delta}}\dot{\bar{\delta}} + \frac{\partial \phi_1(\hat{x},z)}{\partial \hat{x}} Q \\
=& \hat{y}_{e,3} + \frac{\partial \phi_1(\hat{x},z)}{\partial \hat{x}} Q + \hat{\theta}_2(\bar{\delta}) \\
&\vdots \\
\dot{\hat{y}}_{e,i} =& \hat{y}_{e,i+1} + \frac{\partial \phi_{i-1}(\hat{x},z)}{\partial \hat{x}} Q + \hat{\theta}_i(\bar{\delta}) \\
&\vdots
\end{aligned}$$

where

$$\hat{y}_{e,i+1} = \frac{\partial \phi_i(\hat{x},z)}{\partial \hat{x}}\dot{\hat{x}} + \frac{\partial \phi_i(\hat{x},z)}{\partial z}\dot{z} + \theta_i(\bar{\delta})$$

and

$$\hat{\theta}_i(\bar{\delta}) = -(\theta_i(\bar{\delta}) - \frac{\partial \theta_{i-1}(\bar{\delta})}{\partial \bar{\delta}}\dot{\bar{\delta}}).$$

For $\dot{\hat{y}}_{e,n}$ we have

$$\begin{aligned}\dot{\hat{y}}_{e,n} =&\frac{\partial \phi_{n-1}(\hat{x},z)}{\partial \hat{x}} f(\hat{x},u) + \sum_{k=1}^{n_u-1} \frac{\partial \phi_{n-1}(\hat{x},z)}{\partial z_k} z_{k+1} + \frac{\partial \phi_{n-1}(\hat{x},z)}{\partial z_{n_u}} v \\ &+ \frac{\partial \phi_{n-1}(\hat{x},z)}{\partial \hat{x}} Q + \frac{\partial \theta_{n-1}(\bar{\delta})}{\partial \bar{\delta}} \dot{\bar{\delta}} \\ =&\alpha(\hat{y}_e,z) + \beta(\hat{y}_e,z)v + \frac{\partial \phi_{n-1}(\hat{x},z)}{\partial \hat{x}} Q + \frac{\partial \theta_{n-1}(\bar{\delta})}{\partial \bar{\delta}} \dot{\bar{\delta}}\end{aligned}$$

In conclusion, the observer dynamics in y_e-coordinates is given by

$$\begin{aligned}\dot{\hat{y}}_e =&A_c \hat{y}_e + B_c \left[\alpha(\hat{x},z) + \beta(\hat{x},z)v + \frac{\partial \theta_{n-1}(\bar{\delta})}{\partial \bar{\delta}} \dot{\bar{\delta}} \right] \\ &+ \left[\frac{\partial H_1(\hat{x},z)}{\partial \hat{x}} \right] \left[\frac{\partial H_1(\hat{x},z)}{\partial \hat{x}} \right]^{-1} \mathcal{E}^{-1} L[y(t) - \hat{y}(t)] + G(\bar{\delta}) \\ =&A_c \hat{y}_e + B_c \left[\alpha(\hat{y}_e,z) + \beta(\hat{y}_e,z)v + \frac{\partial \theta_{n-1}(\bar{\delta})}{\partial \bar{\delta}} \dot{\bar{\delta}} \right] + \mathcal{E}^{-1} L[y(t) - \hat{y}(t)] + G(\bar{\delta})\end{aligned}$$

where (A_c, B_c) is in controllable canonical form and $G(\bar{\delta})$ is given by

$$G(\bar{\delta}) = \begin{bmatrix} \hat{\theta}_1(\bar{\delta}) \\ \hat{\theta}_2(\bar{\delta}) \\ \vdots \\ \hat{\theta}_{n-1}(\bar{\delta}) \\ 0 \end{bmatrix}.$$

Define $\tilde{y}_e = \hat{y}_e - y_e$ as the observer error whose dynamics are

$$\begin{aligned}\dot{\tilde{y}}_e =& (A_c - \mathcal{E}^{-1} L C_c)\tilde{y}_e \\ &+ B_c \left[\alpha(\hat{y}_e,z) + \beta(\hat{y}_e,z)v - \alpha(y_e,z) - \beta(y_e,z)v - \frac{\partial \phi_{n-1}(x,z)}{\partial x} \delta(t) \right] + G(\bar{\delta})\end{aligned} \tag{2.17}$$

where $C_c = [1, 0, \cdots, 0]_{1\times n}$. Equation (2.17) can be seen as a linear system with two inputs, namely

$$\alpha(\hat{y}_e,z) + \beta(\hat{y}_e,z)v - \alpha(y_e,z) - \beta(y_e,z)v - \frac{\partial \phi_{n-1}(x,z)}{\partial x} \delta(t) \tag{2.18}$$

and $G(\bar{\delta})$.

Theorem 1 *Consider system (2.8) and assume that A1 is satisfied for* $\mathcal{O} = \mathbb{R}^{n+n_u}$, *the state* X *belongs to a positively invariant, compact set* Ω, *and the following time signal is bounded as follows (see (2.18)),*

$$\|\alpha(\hat{y}_e(t), z(t)) + \beta(\hat{y}_e(t), z(t))v(t) - \alpha(y_e(t), z(t)) - \beta(y_e(t), z(t))v(t)\| \leq \gamma_1 \|\hat{y}_e(t) - y_e(t)\| \tag{2.19}$$

for some $\gamma_1 > 0$, *for all* $t \geq 0$, *and for all* $(\hat{x}(0), z(0)) \in \hat{\Omega}$ *(a compact set) where* α *and* β *are defined in (2.14). Choose* $L = [l_1, \cdots, l_n]^\top$ *such that the polynomial* $s^n + l_1 s^{n-1} + \cdots + l_{n-1}s + l_n$ *is Hurwitz.*

Under these conditions and using observer (2.12), the estimation error

$\|\tilde{y}_e\| = \|\hat{y}_e - y_e\|$ *is uniformly ultimately bounded and its ultimate bound, which depends on* δ, *can be reached arbitrarily fast.*

Before proving the theorem, we clarify requirement (2.19). Requirement (2.19) is a Lipschitz inequality which has to be satisfied at every time with a fixed Lipschitz constant. Local Lipschitz continuity of α and β, and boundedness of $x(t)$ and $v(t)$ for all $t \geq 0$ are not sufficient for requirement (2.19) to be satisfied. Assuming that α and β are globally Lipschitz and $v(t)$ is bounded will fulfill the requirement, but we will show in a later section that using an appropriate dynamic projection for $\hat{x}$ onto a compact set, requirement (2.19) is always satisfied and we do not need to assume global Lipschitz continuity for α and β.

Proof. We use superposition to find the response of the observer error $\tilde{y}_e(t)$ for the two inputs (2.18) and $G(\bar{\delta})$. First, we consider the case when $G = 0$. Using the coordinate transformation

$$\tilde{\nu} = \mathcal{E}'\tilde{y}_e, \quad \mathcal{E}' \triangleq \text{diag}\left[\frac{1}{\rho^{n-1}}, \frac{1}{\rho^{n-2}}, \ldots, 1\right], \tag{2.20}$$

the error dynamics (2.17) is

$$\dot{\tilde{\nu}} = \frac{1}{\rho}(A_c - LC_c)\tilde{\nu} + B_c\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v - \frac{\partial\phi_{n-1}(x,z)}{\partial x}\delta(t)\right]. \tag{2.21}$$

By assumption, $A_c - LC_c$ is Hurwitz. Let P be the solution to the Lyapunov equation

$$P(A_c - LC_c) + (A_c - LC_c)^\top P = -I, \tag{2.22}$$

and consider the Lyapunov function candidate $V_o(\tilde{\nu}) = \tilde{\nu}^\top P\tilde{\nu}$. Calculating the time derivative of V_o along the $\tilde{\nu}$ trajectories we have

$$\dot{V}_o = -\frac{\tilde{\nu}^\top\tilde{\nu}}{\rho} + 2\tilde{\nu}^\top PB_c\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) + \beta(y_e, z)v - \frac{\partial\phi_{n-1}(x,z)}{\partial x}\delta(t)\right]. \tag{2.23}$$

Using requirement (2.19) we have the following inequality

$$\|\alpha(\hat{y}_e(t), z(t)) + \beta(\hat{y}_e(t), z(t))v(t) - \alpha(y_e(t), z(t)) - \beta(y_e(t), z(t))v(t)\| \leq \gamma_1\|\tilde{y}_e(t)\|. \tag{2.24}$$

Since $X \in \Omega$, a compact set, and H_1 is smooth, for the last element of $\dot{V}_o$ in (2.23) we can write

$$\left\|\frac{\partial\phi_{n-1}(x,z)}{\partial x}\right\| \|\delta(t)\| \leq \gamma_2\|\delta(t)\| \tag{2.25}$$

for some positive real number γ_2. Using inequalities (2.24) and (2.25) we have

$$\dot{V}_o \leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|\tilde{\nu}^\top\|\|P\|(\gamma_1\|\tilde{y}_e\| + \gamma_2\|\delta(t)\|). \tag{2.26}$$

$$\dot{V}_o \leq -\underbrace{\left(\frac{1}{\rho} - 2\|P\|\gamma_1\right)}_{\eta}\|\tilde{\nu}\|^2 + 2\|\tilde{\nu}\|\|P\|\gamma_2 d \tag{2.27}$$

where in the last inequality we used the fact that $\|\tilde{y}_e\| \leq \|\tilde{\nu}\|$ (see (2.20)).

Choosing $\frac{1}{\rho} > 2\|P\|\gamma_1$, for all $0 < \epsilon < 1$ we have

$$\dot{V}_o \leq -\eta(1-\epsilon)\|\tilde{\nu}\|^2 \qquad \forall\, \|\tilde{\nu}\|^2 \geq \frac{2\|P\|}{\epsilon\eta}\gamma_2 d. \tag{2.28}$$

The ultimate bound for $\|\tilde{\nu}\|$ (and hence for $\|\tilde{y}_e\|$) is

$$b(\rho) = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}} \frac{2\|P\|}{\epsilon\left(\frac{1}{\rho} - 2\|P\|\gamma_1\right)} \gamma_2 d. \tag{2.29}$$

It is clear that $b(\rho)$ can be made arbitrarily small by choosing a sufficiently small ρ.

As for the time needed for $\|\tilde{\nu}(t)\|$ (and hence $\|\tilde{y}_e(t)\|$) to reach this bound, notice that

$$\lambda_{\min}(P)\|\tilde{\nu}\| \leq V_o = \tilde{\nu}^\top P \tilde{\nu} \leq \lambda_{\max}(P)\|\tilde{\nu}\|. \tag{2.30}$$

Using (2.30) and (2.28), we have

$$\dot{V}_o(t) \leq \frac{-\eta(1-\epsilon)}{\lambda_{\max}(P)} V_o(t). \tag{2.31}$$

Therefore, by the comparison lemma (see [14] Lemma 2.5), $V_o(t)$ satisfies the following inequality,

$$V_o(t) \leq V_o(0) \exp\left\{\frac{-\eta(1-\epsilon)}{\lambda_{\max}(P)} t\right\}. \tag{2.32}$$

But we know that

$$V_o(0) = \tilde{\nu}(0)^\top P \tilde{\nu}(0) \leq \lambda_{\max}(P)\|\tilde{\nu}(0)\|^2.$$

So (2.32) can be written as

$$V_o(t) \leq \lambda_{\max}(P)\|\tilde{\nu}(0)\|^2 \exp\left\{\frac{-\eta(1-\epsilon)}{\lambda_{\max}(P)} t\right\} \tag{2.33}$$

An upper bound T on the time needed for the trajectory to reach the bound b in (2.29) can be found by the following equation,

$$\lambda_{\min}(P) b^2 \leq V_o(b) \leq \lambda_{\max}(P)\|\tilde{\nu}(0)\|^2 \exp\left\{\frac{-\eta(1-\epsilon)}{\lambda_{\max}(P)} t\right\} \tag{2.34}$$

$$t \geq T = \frac{\lambda_{\max}(P)}{\eta(1-\epsilon)} \ln \frac{\lambda_{\max}(P)\|\tilde{\nu}(0)\|^2}{\lambda_{\min}(P)b^2} \tag{2.35}$$

and, for all $t \geq T$ we have $\|\tilde{\nu}\| \leq b$.

Using the definitions of η in (2.27) and b in (2.29), for sufficiently small ρ we can write

$$T \simeq a_1\rho + a_2\rho\ln(a_3\rho) \tag{2.36}$$

for some constants a_1, a_2 and a_3.

Using Mercator series $\left(\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right)$, we can write

$$\lim_{\rho\to 0} \rho\ln(a_3\rho) = 0.$$

Therefore $\lim_{\rho\to 0} T = 0$. In other words T can be made arbitrarily small by choosing a sufficiently small ρ.

Now we return to (2.17) and continue the superposition argument by assuming that $G(\bar{\delta}) \neq 0$ and $B_c = 0$, so that $\dot{\tilde{y}}_e = (A_c - \mathcal{E}^{-1}LC_c)\tilde{y}_e + G(\bar{\delta})$. Therefore the error dynamics can be viewed as the states of a linear system $\dot{w} = Aw + Bu$, with A Hurwitz, $B = 1$ and $u = [u_1\ u_2\ \cdots\ u_{n-1}\ 0]^\top = G(\bar{\delta})$. To find the solution for this system notice that

$$A = A_c - \mathcal{E}^{-1}LC_c = \begin{bmatrix} -L_1 & 1 & 0 & \cdots & 0 \\ -L_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -L_{n-1} & 0 & \cdots & 0 & 1 \\ -L_n & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $[L_1 \quad L_2 \quad \cdots \quad L_n] = [l_1/\rho \quad l_2/\rho^2 \quad \cdots \quad l_n/\rho^n]$. In Laplace domain we have $(sI-$

$A)W(s) = U(s)$, where the components of $W(s)$ are given by

$$\begin{aligned} W_1(s) =& \frac{s^n U_1(s) + s^{n-1}U_2(s) + \cdots + sU_{n-1}}{s^n + L_1 s^{n-1} + \cdots + L_{n-1}s + L_n} \\ W_2(s) =& (s + L_1)W_1(s) - U_1(s) \qquad (2.37) \\ W_i(s) =& L_{i-1}W_1(s) + sW_{i-1}(s) - U_{i-1}(s) \quad \text{for } i = 3 \cdots n \qquad (2.38) \end{aligned}$$

Since the poles of $W_1(s)$ are in the left-half plane, the final value theorem gives

$$\lim_{t\to\infty} w_1(t) = \lim_{s\to 0} sW_1(s) = 0.$$

and, as a consequence, from (2.37), when $t \to \infty$ we have that

$$\|w_2(t)\| \leq \limsup \|u_1\|.$$

For $i = 3 \cdots n$ we have

$$\|w_i(t)\| \leq \limsup \|\dot{w}_{i-1}(t)\| + \|u_{i-1}(t)\|. \qquad (2.39)$$

To find an upper bound for $\|w_i(t)\|$ in (2.39), we need to find an upper bound for $\|\dot{w}_{i-1}(t)\|$. To do so, notice that from (2.38) we can write recursively the following equations

$$\begin{aligned} sW_{i-1}(s) =& sL_{i-2}W_1(s) + s^2 W_{i-2}(s) - sU_{i-2}(s) \\ s^2 W_{i-2}(s) =& s^2 L_{i-3}W_1(s) + s^3 W_{i-3}(s) - s^2 U_{i-3}(s) \\ &\vdots \\ s^{i-3}W_3(s) =& s^{i-3}L_2 W_1(s) + s^{i-2}W_2(s) - s^{i-3}U_2(s) \\ s^{i-2}W_2(s) =& s^{i-2}(s + L_1)W_1(s) - s^{i-2}U_1(s). \end{aligned}$$

Therefore when $t \to \infty$ we have

$$\begin{aligned}
&\|w_2^{(i-2)}\| \le \limsup \|u_1^{(i-2)}\| \triangleq \bar{u}_2 \\
&\|w_3^{(i-3)}\| \le \bar{u}_2 + \limsup \|u_2^{(i-3)}\| \triangleq \bar{u}_3 \\
&\quad \vdots \\
&\|\dot{w}_{(i-1)}\| \le \bar{u}_2 + \cdots + \bar{u}_{(i-2)} + \limsup \|\dot{u}_{(i-1)}\| \triangleq \bar{u}_{i-1} \\
&\|w_i\| \le \bar{u}_2 + \cdots + \bar{u}_{i-1} + \limsup \|u_{i-1}\| \triangleq \bar{u}_i
\end{aligned}$$

which shows that for some function g of $G(\bar{\delta}(t))$ (remember that $u = G(\bar{\delta}(t))$)

$$\lim_{t\to\infty} \|w(t)\| \le \limsup \|g(G(\bar{\delta}(t)))\| \triangleq \bar{G}. \tag{2.40}$$

Moreover the transient can be made arbitrarily fast by decreasing ρ.

In conclusion, using superposition, we have that $\limsup \|\tilde{y}_e\| \le \bar{G} + b(\rho)$ where $b(\rho)$ is the bound in (2.29). Since $b(\rho)$ can be made arbitrarily small and can be reached arbitrarily fast, we can conclude that $\|\tilde{y}_e\|$ can reach any neighborhood of $\bar{G}$ arbitrarily fast by decreasing ρ.

■

2.3 Projection

Expressing inequalities (2.30) as a function of $\tilde{y}_e$ ($\tilde{\nu} = \mathcal{E}'\tilde{y}_e$), we have

$$\lambda_{min}(P)\|\tilde{y}_e\|^2 \le V_o = \tilde{y}_e^\top \mathcal{E}' P \mathcal{E}' \tilde{y}_e \le \frac{1}{\rho^{2(n-1)}} \lambda_{max}(P)\|\tilde{y}_e\|^2. \tag{2.41}$$

Note that $\lambda_{min}(\mathcal{E}'P\mathcal{E}') \ge \lambda_{min}(\mathcal{E}')^2\lambda_{min}(P) = \lambda_{min}(P)$, since $\lambda_{min}(\mathcal{E}') = 1$, and $\lambda_{max}(\mathcal{E}'P\mathcal{E}') \le \lambda_{max}(\mathcal{E}')^2\lambda_{max}(P) = 1/(\rho^{2(n-1)})\lambda_{max}(P)$, since $\lambda_{max}(\mathcal{E}') = 1/\rho^{(n-1)}$.

Using inequalities (2.32) and (2.41) we will have the following result,

$$\|\tilde{y}_e\| \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \frac{1}{\rho^{n-1}} \|\tilde{y}_e(0)\| \exp\left\{\frac{-\eta(1-\epsilon)}{\lambda_{\max}(P)} t\right\}. \tag{2.42}$$

During the initial transient, $\tilde{y}_e(t)$ exhibits peaking, and the size of the peak grows larger as ρ decreases and convergence rate is made faster (peaking phenomenon). In order to isolate the peaking phenomenon from the system states, the approach generally adopted in several papers is to saturate the control input outside a compact set of interest to prevent it from growing above a given threshold. This technique, however, does not eliminate the peak in the observer estimate and, hence, cannot be used to control general systems like the ones satisfying assumption A1, since even when the system states lie in the observable region $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, the observer estimates may enter the unobservable domain where (2.12) is not well defined. In other words H_1^{-1} in assumption A1 is not defined outside the observable set $\mathcal{O}$, which implies that $\left[\frac{\partial H_1(\hat{x}, z)}{\partial \hat{x}}\right]^{-1}$ is not defined.

It appears that in order to deal with systems that are not completely observable, one has to eliminate peaking from the observer by forcing its estimates not to exit a pre-specified observable compact set. A common procedure in adaptive control literature for keeping estimates in a desired convex set is to use gradient projection (see [15]). This idea cannot be used here mainly because $\dot{\hat{x}}$ is not proportional to the gradient of the observer Lyapunov function. In other words, applying a standard gradient projection for for $\hat{x}$ over an arbitrarily convex set does not necessarily preserve the convergence properties of the estimate. For example there are situations where, if we use projection directly in $[x^\top, z^\top]^\top$ coordinates, one can introduce equilibrium points on the boundary of the convex set.

To solve this problem, we use projection in $Y = [{y_e}^\top, z^\top]^\top$ coordinates. In order to relate the projected estimates in Y coordinates to $[x^\top, z^\top]^\top$ coordinates, recall form (2.16) when

$\delta(t) = 0$ we have that $\hat{y}_e = H_1(\hat{x}, z)$, and hence

$$\begin{aligned}\dot{\hat{y}}_e &= \frac{\partial H_1}{\partial \hat{x}}\dot{\hat{x}} + \frac{\partial H_1}{\partial z}\dot{z} \\ \dot{\hat{x}} &= \left[\frac{\partial H_1}{\partial \hat{x}}\right]^{-1}\left(\dot{\hat{y}}_e - \frac{\partial H_1}{\partial z}\dot{z}\right).\end{aligned} \tag{2.43}$$

We will use equation (2.43) to estimate $\hat{x}$ when $\hat{y}_e$ is on the boundary of a specified set $\mathcal{C}$ in $Y = [{y_e}^\top, z^\top]^\top$ coordinates (see equation (2.49)). In other words we confine $\hat{y}_e$ and z in that specific set $\mathcal{C}$ in the mentioned coordinate. In the next sections we show the construction and properties of the set $\mathcal{C}$ and demonstrate how to modify the observer using such a projection set.

2.3.1 Projection Sets

Consider system (2.8) and assume there is no disturbance $\delta(t)$. Using assumption A2 and Remark 2 we conclude the existence of a smooth stabilizing control $v(X) = \phi(x, z)$ which makes the origin of (2.8) an asymptotically stable equilibrium point with domain of attraction $\mathcal{D}$. By the converse Lyapunov theorem found in [13], there exists a continuously differentiable function V defined on $\mathcal{D}$ satisfying, for all $X \in \mathcal{D}$,

$$\alpha_1(\|X\|) \leq V(X) \leq \alpha_2(\|X\|) \tag{2.44}$$

$$\lim_{X \to \partial\mathcal{D}} \alpha_1(\|X\|) = \infty \tag{2.45}$$

$$\frac{\partial V}{\partial X}\left(f_e(X) + g_e\,\phi\right) \leq -\alpha_3(\|X\|) \tag{2.46}$$

where $\alpha_i,\ i = 1, 2, 3$ are class $\mathcal{K}$ functions (see [14] for a definition), and $\partial\mathcal{D}$ is the boundary of the set $\mathcal{D}$. Given any scalar $c > 0$, define

$$\Omega_c \triangleq \{X \in \mathbb{R}^{n+n_u} \,|\, V \leq c\}$$

where $\Omega_c \subset \mathcal{D}$ for all $c > 0$ and, from (2.45), $\Omega_c \to \mathcal{D}$ as $c \to \infty$. Next, we define set C as follows.

Assumption A3(Properties of $\mathcal{C}$): Assume that there exists a constant $c_2 > 0$ and a set $\mathcal{C}$ such that

$$F_1(\Omega_{c_2}) \subset \mathcal{C} \subset F_1(\mathcal{O}), \tag{2.47}$$

where Ω_{c_2} is a level set of Lyapunov function V. Assume that $\mathcal{C}$ has the following properties

(i) There exists a C^1 function $g : \mathcal{C} \to \mathbb{R}$ such that $\partial\mathcal{C} = \{Y \in \mathcal{C} \,|\, g(Y) = 0\}$, and $(\partial g/\partial Y)^\top \neq 0$ on $\partial\mathcal{C}$, which $\partial\mathcal{C}$ is the boundary of $\mathcal{C}$.

(ii) $\mathcal{C}^{\bar{z}} = \{y_e \in \mathbb{R}^n \,|\, [{y_e}^\top, \bar{z}^\top]^\top \in \mathcal{C}\}$ is convex for all $\bar{z} \in \mathbb{R}^{n_u}$.

(iii) $\dfrac{\partial g(y_e, \bar{z})}{\partial y_e} \neq 0 \quad \forall \bar{z} \in \mathbb{R}^{n_u}$.

(iv) $\bigcup_{\bar{z} \in \mathbb{R}^{n_u}} \mathcal{C}^{\bar{z}}$ is compact.

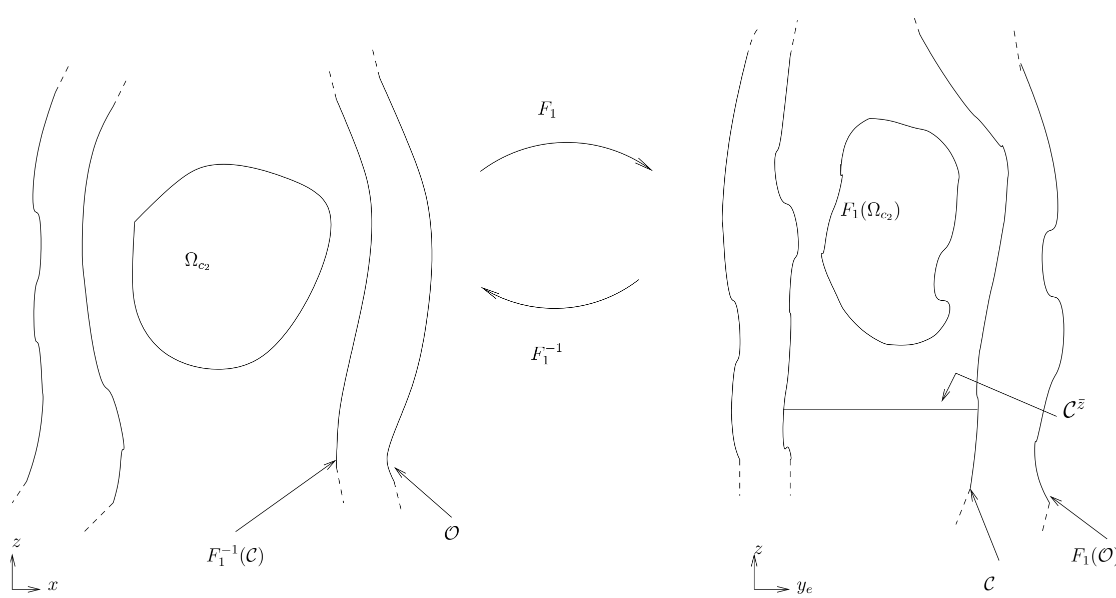

Figure 2.2: Projection mechanism.

In assumption A3 it is required that $\mathcal{C}$ possesses some topological properties which we clarify here: part (i) means that the boundary of $\mathcal{C}$ is continuously differentiable, part (ii) means that every slice of $\mathcal{C}$, $\mathcal{C}^{\bar{z}}$, which is obtained by holding z constant at $\bar{z}$, is convex, part (iii) means that normal vectors to each slice $\mathcal{C}^{\bar{z}}$ do not vanish anywhere on the slice and, finally, part (iv) means that the set $\mathcal{C}$ is compact in the y_e direction.

Remark 4: From assumption A1 we know that $\hat{Y} = F_1(\hat{x}, z) + F_2(\bar{\delta}) \triangleq \hat{Y}^1 + F_2(\bar{\delta})$ in which we do not know $F_2(\bar{\delta})$. Therefore we use $\hat{Y}^1$ in the projection. To do so we need to find set $\mathcal{C}$ such that whenever $\hat{Y}^1 \in \mathcal{C}$ we have $F_1^{-1}(\hat{Y}^1) \in \mathcal{O}$. This requirement is represented in condition (2.47). See Figure 2.2 for a pictorial representation of this condition. This assumption requires that there exists a set $\mathcal{C}$ in Y coordinates which contains the image of Ω_{c_2} under F_1 and is contained in the image of the observable set $\mathcal{O}$ under F_1. This guarantees that if estimates in Y coordinates do not leave set $\mathcal{C}$ then any trajectory starting in Ω_{c_2} remains in the observable set $\mathcal{O}$.

After defining $\mathcal{C}$, $\Omega_c \triangleq \{X \in \mathbb{R}^{n+n_u} \,|\, V \leq c\}$ is chosen such that it is the greatest set which $\Omega_c \subset F_1^{-1}(\mathcal{C})$.

Remark 5: To further clarify the requirements of set $\mathcal{C}$, consider the sets $\mathcal{C}_1$ to $\mathcal{C}_4$ in Figure 2.3 and Figure 2.4 for the case $y_e \in \mathbb{R}^2$, $z \in \mathbb{R}$. While they all satisfy part (i), $\mathcal{C}_1$ does not satisfy part (ii) since its slices along z are not convex. $\mathcal{C}_2$ satisfies part (ii) but not part (iii) because the normal vector to one of the slices has no components in the y_e direction. $\mathcal{C}_3$ satisfies parts (i)-(iii) but not part (iv) since the area of its slices grows unbounded as $z \to \infty$. $\mathcal{C}_4$ satisfies all the properties for $\mathcal{C}$.

When $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^{n_u}$ and $F_1(\mathbb{R}^{n+n_u}) = \mathbb{R}^{n+n_u}$, A3 is always satisfied by a sufficiently

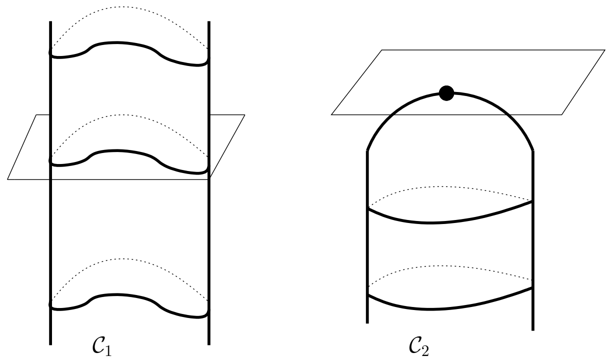

Figure 2.3: Properties of $\mathcal{C}$.

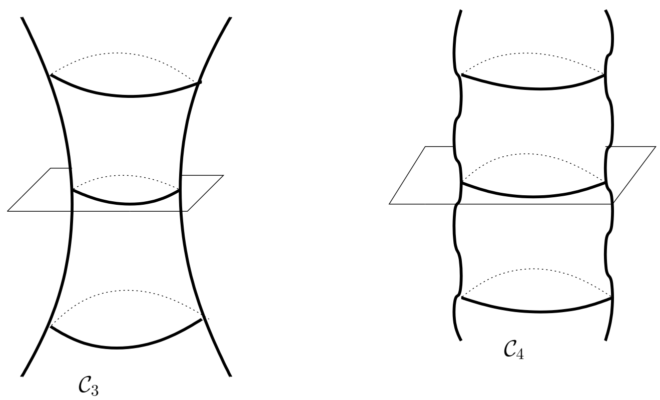

Figure 2.4: Properties of $\mathcal{C}$,cont.

large set $\mathcal{C}$ and any $c_2 > 0$. In order to see that, pick any c_2 and choose $\mathcal{C}$ to be any cylinder $\{Y \in \mathbb{R}^{n+n_u} \,|\, \|y_e\| \leq M\}$, where $M > 0$, containing $F_1(\Omega_{c_2})$. $\mathcal{C}$ always exists since the set $F_1(\Omega_{c_2})$ is bounded. Generally, the same holds when $F_1(\mathbb{R}^{n+n_u})$ is not all of $\mathbb{R}^{n+n_u}$ and $\mathcal{Y}^{\bar{z}} \triangleq \{y_e \in \mathbb{R}^n \,|\, F_1(x, \bar{z})\}$ is convex for all $\bar{z} \in \mathbb{R}^{n_u}$.

When $\mathcal{O} = \mathcal{X} \times \mathbb{R}^{n_u}$, where $\mathcal{X}$ is an open set which is not all of $\mathbb{R}^n$, and the the system is globally ISS (A2 holds globally), one can choose $\mathcal{C} = D \times \mathbb{R}^{n_u}$, where $D \subset \mathbb{R}^n$ is any convex compact set with smooth boundary contained in $H_1(\mathcal{X})$ and containing the origin of y_e coordinates, $H_1(0, 0)$. The scalar c_2 is then the largest scalar such that $F_1(\Omega_{c_2}) \subset \mathcal{C}$ (c_2 does not need to be known for design purposes). In the particular case when $n_u = 0$

(control input u does not affect the mapping H_1) and A2 holds globally, one can choose $\mathcal{C} = D$, where D is defined above.

2.3.2 Observer Estimates Projection

Recall the coordinate transformation defined in (1) and let

$$\hat{y}_e^P = H_1(\hat{x}^P, z), \quad \tilde{y}_e^P = \hat{y}_e^P - y_e, \quad \hat{Y}^P = [\hat{y}_e^{P\top}, z^\top]^\top, \tag{2.48}$$

where $\hat{x}^P$ is the state of the *projected* observer defined as

$$\begin{aligned}\dot{\hat{x}}^P &= \Upsilon(\hat{x}^P, z, y)\\ &= \begin{cases} \left[\dfrac{\partial H_1}{\partial \hat{x}^P}\right]^{-1} \left\{ \dot{\hat{y}}_e^1|_{\hat{x}^P} - \Gamma \dfrac{N_{y_e}(\hat{Y}^P)\left(N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e^1|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z}\right)}{N_{y_e}(\hat{Y}^P)^\top \Gamma N_{y_e}(\hat{Y}^P)} - \dfrac{\partial H_1}{\partial z}\dot{z} \right\} \\ \qquad\qquad \text{if } N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e^1|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z} \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ \hat{f}(\hat{x}^P, z, y) \qquad \text{otherwise} \end{cases}\end{aligned} \tag{2.49}$$

where $\hat{f}(\hat{x}^P, z, y)$ is defined in the observer equation (2.12), $\dot{\hat{y}}_e^1|_{\hat{x}^P}$ denotes the time derivative of $\hat{y}_e^P = H_1(\hat{x}^P, z)$ when $\hat{x}^P$ evolves according to the observer dynamics (2.12), i.e.,

$$\dot{\hat{y}}_e^1|_{\hat{x}^P} = \frac{\partial H_1}{\partial \hat{x}^P}\hat{f}(\hat{x}^P, z, y) + \frac{\partial H_1}{\partial z}\dot{z},$$

$\Gamma = (S\mathcal{E}')^{-1}(S\mathcal{E}')^{-\top}$, $S = S^\top$ denotes the matrix square root of P (used for the Lyapunov function in the proof of Theorem 1) and

$$N_{y_e}(\hat{Y}^P) = \left[\frac{\partial g}{\partial \hat{y}_e^P}(\hat{y}_e^P, z)\right]^\top, \; N_z(\hat{Y}^P) = \left[\frac{\partial g}{\partial z}(\hat{y}_e^P, z)\right]^\top$$

are the y_e and z components of the normal vector $N(\hat{Y}^P)$ to the boundary of $\mathcal{C}$ at $\hat{Y}^P$, i.e., $N(\hat{Y}^P) = [N_{y_e}(\hat{Y}^P)^\top, N_z(\hat{Y}^P)^\top]^\top$ (the function g is defined in Assumption A3). Notice that the dynamic projection (2.49) is well-defined since assumption A3, part (iii), guarantees that N_{y_e} does not vanish.

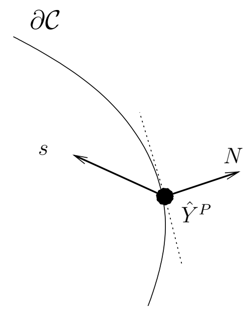

Figure 2.5: Geometric Interpretation of the projection condition.

See Figure 2.5 for a pictorial representation of the condition in the first case of equation (2.49). This condition is a constraint on the inner product of the two vectors $s = [\dot{\hat{y}}_e^1|_{\hat{x}^P}{}^\top \; \dot{z}^\top]^\top$ and $N = [N_{y_e}{}^\top \; N_z{}^\top]^\top$. Whenever $\hat{Y}^P$ is on the boundary of $\mathcal{C}$ and the vector field s is pointing outside of, or is tangent to, the boundary of $\mathcal{C}$, this product is greater than or equal to zero. As we mentioned earlier we want to prevent the observer trajectories from exiting the set $F_1^{-1}(\mathcal{C})$. A simple explanation of the modification in the observer equation is that when trajectories are on the boundary of $\mathcal{C}$, $\partial\mathcal{C}$, we eliminate those components of the vector field in the direction of the normal to $\partial\mathcal{C}$ which force the trajectories to exit $\mathcal{C}$. This in turn keeps $\hat{x}^P$ within the set $F_1^{-1}(\mathcal{C})$ which is a subset of the observable set $\mathcal{O}$. When $\hat{x}^P$ is not on the boundary of $\mathcal{C}$, we let the trajectories flow along the vector field.

The following theorem shows that (2.49) guarantees boundedness and preserves convergence for $\hat{x}^P$.

Theorem 2 *: If A3 holds and (2.49) is used:*

(i) Boundedness: if $(\hat{x}^P(0), z(0)) \in F_1^{-1}(\mathcal{C})$, then $(\hat{x}^P(t), z(t)) \in F_1^{-1}(\mathcal{C})$ for all t.

If, in addition, $[x(t)^\top, z(t)^\top]^\top \in \Omega_{c2}$ for all $t \geq 0$ and the assumptions of Theorem 1 are satisfied, then the following property holds for the flow of the projected observer dynamics

(2.49)

(ii) Requirement (2.19) in Theorem 1 is satisfied provided $\sup_{t\geq 0} v(t) < \infty$.

Notice that part (i) of the theorem shows that if $\hat{x}^P$ starts in an observable subset of $\mathcal{O}$, it remains in that set for all t. It does not give any information about the behavior of our system states $x(t)$ or estimation error. Remark 6 deals with the estimation error and in Section 2.4 we discuss the behavior of $x(t)$.

Proof. In order to prove part (i) of the theorem, it is sufficient to show that set $\mathcal{C}$ is a positively invariant set for $\hat{Y}^P$ trajectories under the projection (2.49). This will guarantee that $(\hat{x}^P(t), z(t)) = F_1^{-1}(\hat{Y}^P)$ is contained in the set $F_1^{-1}(\mathcal{C}) \subset \mathcal{O}$ for all $t \geq 0$.

We prove part (i) in a new coordinate $[\zeta^\top, z^\top]^\top$, with the following transformation,

$$\zeta = S\mathcal{E}' y_e \tag{2.50}$$

Similarly, $\hat{\zeta}^P = S\mathcal{E}'\hat{y}_e^P$, $\tilde{\zeta}^P = S\mathcal{E}'\tilde{y}_e^P$. Consider the mapping

$$\mathcal{G} = diag\left[S\mathcal{E}', I_{n_u \times n_u}\right]$$

and define $\mathcal{C}'$ as the image of the set $\mathcal{C}$ under the linear map $\mathcal{G}$, i.e.,

$$\mathcal{C}' = \mathcal{G}(\mathcal{C}) \triangleq \left\{[\zeta^\top, z^\top]^\top \in \mathbb{R}^{n+n_u} \,|\, \mathcal{G}^{-1}[\zeta^\top, z^\top]^\top \in \mathcal{C}\right\}.$$

Notice that $\mathcal{G}$ has no effect on z component. Figure 2.6 shows a pictorial representation of the sets under consideration.

Define $N'_\zeta(\zeta, z)$, $N'_z(\zeta, z)$ as the ζ and z components of the normal vectors to the boundary of $\mathcal{C}'$. In order to relate $N'_\zeta(\hat{\zeta}^P, z)$, $N'_z(\hat{\zeta}^P, z)$ to $N_{y_e}(\hat{y}_e^P, z)$, $N'_z(\hat{\zeta}^P, z)$, recall from A3 that the boundary of $\mathcal{C}$ is the set $\partial\mathcal{C} = \{Y \in \mathbb{R}^{n+n_u} \,|\, g(Y) = 0\}$ and hence the boundary of $\mathcal{C}'$ is the set

$$\partial\mathcal{C}' = \{\zeta \in \mathbb{R}^n \,|\, g((S\mathcal{E}')^{-1}\zeta, z) = 0\}.$$

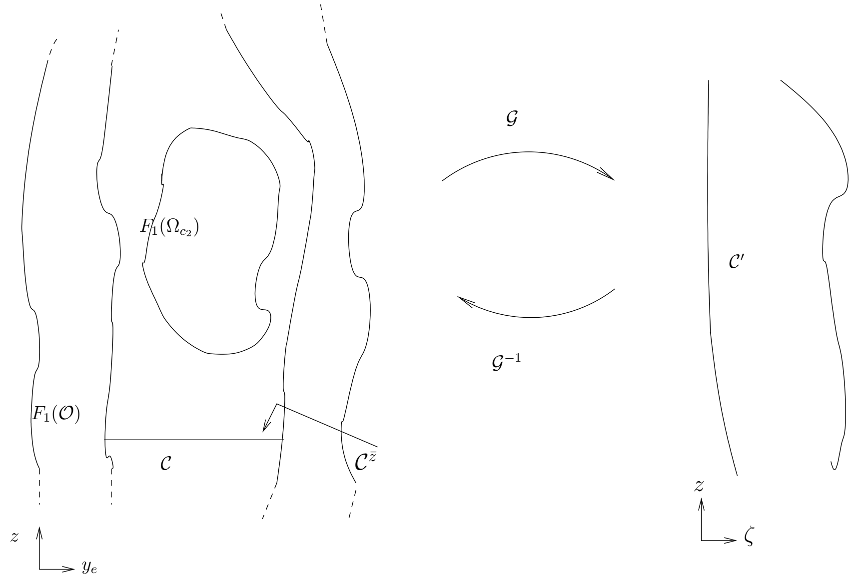

Figure 2.6: $\mathcal{G}$ Transformation.

From this definition we find the expression of N'_ζ and N'_z as

$$N'_\zeta(\hat{\zeta}^P, z) = (S\mathcal{E}')^{-\top}[\partial g(\hat{Y}^P)/\partial y_e^P]^\top = (S\mathcal{E}')^{-\top} N_{y_e}(\hat{Y}^P),$$

$$N'_z(\hat{\zeta}^P, z) = N_z(\hat{Y}^P).$$

Here again we express the projected observer in y_e coordinate using coordinate transformation (2.48) as follows

$$\dot{\hat{y}}_e^P = \frac{d}{dt}\left\{H_1(\hat{x}^P, z)\right\} = \left[\frac{\partial H_1}{\partial \hat{x}^P}\dot{\hat{x}}^P + \frac{\partial H_1}{\partial z}\dot{z}\right]$$
$$= \begin{cases} \dot{\hat{y}}_e^1|_{\hat{x}^P} - \Gamma \dfrac{N_{y_e}(\hat{Y}^P)\left(N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e^1|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z}\right)}{N_{y_e}(\hat{Y}^P)^\top \Gamma N_{y_e}(\hat{Y}^P)} \\ \qquad \text{if } N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e^1|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z} \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ \dot{\hat{y}}_e^1|_{\hat{x}^P} \qquad \text{otherwise} \end{cases} \tag{2.51}$$

Knowing that $y_e = (S\mathcal{E}')^{-1}\zeta$ from (2.50), the projection (2.49) in ζ coordinates is

$$\dot{\hat{\zeta}}^P = S\mathcal{E}'\dot{\hat{y}}_e^P = \begin{cases} S\mathcal{E}'\dot{\hat{y}}_e^1|_{\hat{x}^P} - (S\mathcal{E}')^{-\top}\dfrac{N_{y_e}\left(N_{y_e}^\top \dot{\hat{y}}_e^1|_{\hat{x}^P} + N_z^\top \dot{z}\right)}{N_{y_e}^\top \Gamma N_{y_e}} & \\ \text{if } N_{y_e}^\top \dot{\hat{y}}_e^1|_{\hat{x}^P} + N_z^\top \dot{z} \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} & \\ & \\ S\mathcal{E}'\dot{\hat{y}}_e^1|_{\hat{x}^P} & \text{otherwise} \end{cases} \tag{2.52}$$

and then substituting $N'_\zeta = (S\mathcal{E}')^{-\top}N_{y_e}$, $N'_z = N_z$, and $\dot{\hat{y}}_e^1|_{\hat{x}^P} = (S\mathcal{E}')^{-1}\dot{\hat{\zeta}}|_{\hat{x}^P}$ (since we only use $\dot{\hat{y}}_e^1|_{\hat{x}^P}$, for simplicity we do not use the notation $\dot{\hat{\zeta}}^1|_{\hat{x}^P}$), we have

$$\dot{\hat{\zeta}}^P = \begin{cases} \dot{\hat{\zeta}}|_{\hat{x}^P} - \dfrac{N'_\zeta\left(N'^\top_\zeta \dot{\hat{\zeta}}|_{\hat{x}^P} + N'^\top_z \dot{z}\right)}{N'^\top_\zeta N'_\zeta} & \text{if } N'^\top_\zeta \dot{\hat{\zeta}}|_{\hat{x}^P} + N'^\top_z \dot{z} \geq 0 \text{ and } [\hat{\zeta}^{P\top}, z^\top]^\top \in \partial\mathcal{C}' \\ \dot{\hat{\zeta}}|_{\hat{x}^P} & \text{otherwise} \end{cases} \tag{2.53}$$

Next, we show that the boundary of the set $\mathcal{C}'$ is positively invariant with respect to (2.53). In order to do that, consider the continuously differentiable function $V_{\mathcal{C}'} = g\left((S\mathcal{E}')^{-1}\zeta, z\right)$ and calculate its time derivative along the trajectory of (2.53) when $[\hat{\zeta}^{P\top}, z^\top]^\top \in \partial\mathcal{C}'$,

$$\dot{V}_{\mathcal{C}'} = N'_\zeta(\hat{\zeta}^P, z)^\top \dot{\hat{\zeta}}^P + N'_z(\hat{\zeta}^P, z)\dot{z} \tag{2.54}$$

$$= N'^\top_\zeta \dot{\hat{\zeta}}|_{\hat{x}^P} - \frac{N'^\top_\zeta N'_\zeta\left(N'^\top_\zeta \dot{\hat{\zeta}}|_{\hat{x}^P} + N'^\top_z \dot{z}\right)}{N'^\top_\zeta N'_\zeta} + N'_z \dot{z} \tag{2.55}$$

$$= 0 \tag{2.56}$$

which shows that the trajectory $[\hat{\zeta}^{P\top}(t)^\top, z^\top(t))]^\top$ cannot cross the boundary of $\mathcal{C}'$, which in turn implies that $[\hat{y}_e^{P\top}(t), z^\top(t)]^\top$ cannot cross $\partial\mathcal{C}$ and, therefore, $\hat{x}^P(t) \in H_1^{-1}(\mathcal{C})$ for all t.

Next, for part (ii), we want to show that if $X(t) = [x^\top(t), z^\top(t)]^\top$ is contained in a positively invariant, compact set Ω for all $t \geq 0$, then inequality (2.19) holds for all $t \geq 0$

using $\hat{y}_e^P(t)$ from part (i) instead of $\hat{y}_e(t)$, provided that $v(t)$ is uniformly bounded. Note that $z(t)$ is contained in the compact set $\Omega^z = \{z \in \mathbb{R}^{n_u} \mid X(t) \in \Omega\}$ for all $t \geq 0$. Using part (i) of this theorem and part (iv) in assumption A3 we have

$$[\hat{y}_e^{P\top}(t), z(t)^\top]^\top \in \bar{\mathcal{C}} = \left(\bigcup\nolimits_{z \in \Omega^z} \mathcal{C}^z\right) \times \Omega^z, \text{ for all } t \geq 0, \tag{2.57}$$

where $\bar{\mathcal{C}}$ is a compact set.

Now, part (ii) is proved by noticing that inequality (2.19) follows directly from compactness of Ω, Ω^z, $\bar{\mathcal{C}}$, and the boundedness of $v(t)$, and the local Lipschitz continuity of α and β. As already mentioned, the local Lipschitz continuity of α and β alone is not sufficient for (2.19). Part (i) of this theorem is the key feature: it proves that $\hat{y}_e^P(t)$ is contained in a compact set whose size is independent of ρ. This makes it possible to establish (2.19), where the Lipschitz constant γ_1 is independent of ρ (notice that bounded $\hat{y}_e^P(t)$ means bounded $\hat{x}^P(t)$).

■

Remark 6: Notice that in Theorem 2 we do not discuss the properties of the estimation error $\tilde{x}^P$ as we did for the estimation error in Theorem 1; in other words we have to show that the properties that we established in Theorem 1 remain valid for $\hat{x}^P$. We could not prove this part explicitly because its difficulty is beyond the scope of this thesis. Our major problem was that we worked on a Lyapunov-based proof. But so far this path has failed because it relies on finding an explicit Lyapunov function which can be used to show the boundedness of observer error dynamics in equation (2.17) when $G \neq 0$, and not using a superposition-based proof. The problem lies in the transformation (2.20). When the mentioned transformation is applied, it produces two problems; the first problem is that it renders a Lyapunov function dependent on $\mathcal{E}'$, while the second problem is the multiplication of $G(\bar{\delta}(t))$ by $\mathcal{E}'$. At the end these two problems provide us with an estimation error that depends on ρ, in a way that decreasing of ρ produces an increase in estimation error.

But based on our simulations, we conjecture that $\hat{x}^P$ has the same properties as $\hat{x}$ in Theorem 1. Therefore for now we assume that the following statement is true

Conjecture 1: Preservation of the original convergence characteristics
Properties established by Theorem 1 remain valid for $\hat{x}^P$.

For a proof of this statement when $\delta(t) \equiv 0$ one can refer to [16], Lemma 2.

2.4 Boundedness and Closed Loop Stability

So far we have proved that the projected state estimate, $\hat{x}^P$, is bounded. Also based on Conjecture 1 in Remark 6 we assume that the projected estimation error, $\tilde{x}^P = \hat{x}^P - x$, is bounded with an ultimate bound dependent on $\bar{G}$. Here, we study the behavior of the closed-loop system using the projected observer, i.e. the behavior of the system

$$\dot{X} = f_e(X) + g_e v + \delta_e(t) \tag{2.58}$$

when $v = \phi(\hat{x}^P, z)$.

Theorem 3 *Consider system (2.58) and the set* Ω_c *defined in Remark 4. Suppose Assumptions A1, A2, A3 and Conjecture 1 hold. For all* $\underline{c} \in (0, c)$, $b(\rho) > 0$, $d = \sup(\|\delta(t)\|)$ *and* $0 < \theta < 1$ *such that*

$$d_\epsilon = \alpha_2 \circ \alpha_3^{-1}(A(d + \bar{\gamma}\kappa(\bar{G}, b(\rho)))/\theta) < c \tag{2.59}$$

(where α_2, α_3, A *and* κ *will be defined later) there exists a number* $\bar{\rho} \in (0, 1)$ *such that for all* $\rho \in (0, \bar{\rho}]$ *and*

$$\forall (X(0), \hat{x}^P(0)) \in \left\{ (X, \hat{x}^P) \,|\, X \in \Omega_{\underline{c}}, (\hat{x}^P, z) \in F_1^{-1}(\mathcal{C}) \right\},$$

the phase curves $X(t)$ of the closed-loop system remain in Ω_c, and system (2.58) is uniformly ultimately bounded.

Proof. First we show that the needed time for every trajectory starting in a set $\Omega_{\underline{c}}$ to reach the boundary of Ω_c is finite. Since $(\hat{x}^P(0), z(0)) \in F_1^{-1}(\mathcal{C})$, by Theorem 2 we have that $(\hat{x}^P(t), z(t)) \in F_1^{-1}(\mathcal{C})$ for all $t \geq 0$. Let $\Omega_c^z = \{z \in \mathbb{R}^{n_u} \,|\, (x, z) \in \Omega_c\}$. When $X(t) \in \Omega_c$ from (2.57) we have that

$$[\hat{x}^{P\top}(t), z(t)^\top]^\top \in F_1^{-1}\left(\bigcup\nolimits_{z\in\Omega_c^z} \mathcal{C}^z \times \Omega_c^z\right), \text{ for all } t \geq 0,$$

is a compact set independent of ρ. Hence there exists a bounded positive real number D independent of ρ such that for all $X \in \Omega_c$ and all $(\hat{x}^P(t), z(t)) \in F_1^{-1}(\mathcal{C})$, we have that $\|f_e(X) + g_e\phi(\hat{x}^p, z)\| \leq D$. Therefore we have

$$\|f_e(X) + g_e\phi(\hat{x}^p, z) + \delta_e(t)\| \leq D + d.$$

In other words, the maximum rate change of $X(t)$ is $D+d$. Therefore $\|X(t) - X(0)\| \leq (D+d)t$ for all $X(t) \in \Omega_c$. If $l = dist(\Omega_c, \Omega_{\underline{c}}) = \inf_{X_1\in\Omega_c, X_2\in\Omega_{\underline{c}}} \|X_1 - X_2\|$, then $T = l/(D+d)$ is a uniform lower bound for the exit time from Ω_c which is independent of ρ.

Now choose $T_0 \in (0, T)$. From assumption A2 we know that there exists a Lyapunov function $V(X)$ for system (2.58) such that for the disturbance $\delta(\cdot)$ and some positive definite, class K_∞ functions $\alpha_i(\cdot)$ and some K function $\chi(\cdot)$ the following inequalities hold

$$\alpha_1(\|X\|) \leq V(X) \leq \alpha_2(\|X\|) \tag{2.60}$$

$$\dot{V}(X) \leq -\alpha_3(\|X\|) + \chi(\|\delta(t)\|) \tag{2.61}$$

Choose $A = \sup_{X\in\Omega_c} \|\partial V/\partial X\|$. From the Conjecture 1 we know that $\forall t \in [T_0, T)$ there exists $\bar{\rho} \in (0, 1)$ such that for all $\rho \in (0, \bar{\rho}]$, $\|y_e(t) - \hat{y}_e^P(t)\| \leq \bar{G} + b(\rho)$, which by Assumption A1 implies that $\|x(t) - \hat{x}^P(t)\| \leq \kappa(\bar{G}, b(\rho))$ for all $t \in [T_0, T)$ and some smooth function κ with $\kappa(0,0) = 0$. Choose $\bar{\gamma}$ to be the Lipschitz constant of ϕ over the set

$$F_1^{-1}\left(\bigcup_{z\in\Omega_c^z}\mathcal{C}^z \times \Omega_c^z\right).$$

Then we have

$$\|\phi(x(t), z(t)) - \phi(\hat{x}^P(t), z(t))\| \leq \bar{\gamma}\kappa(\bar{G}, b(\rho))$$

Using a Lypunov analysis, for all $t \in [T_0, T)$ we have

$$\begin{aligned}
\dot{V} =& \frac{\partial V}{\partial X}\left[\, f_e(X) + g_e\phi(\hat{x}^P, z) + \delta_e(t)\right] \\
\dot{V} =& \frac{\partial V}{\partial x}\left[\, f_e(X) + g_e\phi(\hat{x}^P, z) - g_e\phi(x, z) + g_e\phi(x, z) + \delta_e(t)\right] \\
\dot{V} =& \frac{\partial V}{\partial x}\left[\, f_e(X) + g_e\phi(x, z)\right] + \frac{\partial V}{\partial x}\left[\, g_e\phi(\hat{x}^P, z) - g_e\phi(x, z) + \delta_e(t)\right] \\
\dot{V} \leq& -\alpha_3(\|X\|) + A(\|\delta\| + \bar{\gamma}\kappa(\bar{G}, b)) \qquad (2.62) \\
\dot{V} \leq& -\alpha_3 \circ \alpha_2^{-1}(V) + A(d + \bar{\gamma}\kappa(\bar{G}, b)). \qquad (2.63)
\end{aligned}$$

For any $V \geq d_\epsilon$ we have that

$$\dot{V} \leq -\frac{1-\theta}{\theta}A(d + \bar{\gamma}\kappa(\bar{G}, b(\rho)))$$

which implies that Ω_c is invariant since $d_\epsilon < c$.

Since $X(t)$ stays in the set Ω_c for all time, using (2.63) and, e.g. Lemma 5.3 in [14], we can find an ultimate bound for $X(t)$ which is given by

$$M = \alpha_1^{-1}\left(\alpha_2\left(\alpha_3^{-1}\left(\frac{A(d + \bar{\gamma}\kappa(\bar{G}, b(\rho))}{\theta}\right)\right)\right) \qquad (2.64)$$

■

The main result of this theorem is to show that if we use the projected observer in the the feedback loop of the extended system (2.58), or in other words $v = \phi(\hat{x}^P, z)$, with a sufficiently small ρ, we can guarantee that the observer estimates will converge to a neighborhood of the system states in finite time and before the system states exit the set Ω_c. When ρ is made smaller, the projected observer does not allow large peaks in the observer estimates, therefore preventing large control inputs which may drive $X(t)$ outside the set Ω_c in shorter time.

When $X(t)$ stays in Ω_c, the Lyapunov analysis shows that $X(t)$ is ultimately bounded. With (2.59) we guarantee that this bound is inside the set Ω_c. The major difficulty here is to find an analytical solution for c and d ($d = \sup \|\delta(t)\|$) such that (2.59) holds. Notice that from (2.29) and (2.40), when $\delta(t) = 0$ ($d = 0$), we have that $M = 0$, which implies that $X(t)$ approaches the origin. For a complete discussion on asymptotic stability of the origin when $\delta(t) = 0$ and the projected observer is used, one can refer to [16].

Notice that in general, when the extended system $\dot{X} = f_e(X) + g_e \phi(\hat{x}^P, z)$ is asymptotically stable at the origin, adding small disturbance $\delta_e(t)$ to the system (see (2.58)) will render the extended system UUB. In this way, the domain of attraction is local and the bound for $\delta(t)$ is not known, while in our method we have a regional result and a bound for the disturbance defined in (2.59).

2.5 Concluding Remarks

In this chapter we developed an output feedback control theory for a class of nonlinear systems subjected to disturbances. We specified allowable disturbances and system models by imposing some restrictions on the observability mapping (2.2). Next, by introducing the projected observer (2.49) in the feedback loop and the concept of projection sets, we showed that we could achieve boundedness and stability in the presence of bounded disturbances.

One of the benefits of the approach introduced in this chapter is that the knowledge of the inverse of the observability mapping H_1 is not needed. Notice that the ISS assumption A2 does not affect our assumptions regarding observability mapping or disturbances, which implies that, having observability assumption A1 satisfied for a system, we can use standard state feedback control design techniques to achieve input to state stability.

Although, when the disturbance $\delta(t)$ is small, one can get local results by the method introduced in this chapter for a linearized model of the original system (2.1), notice that the observer designed in this way would not preserve the boundedness for the original nonlinear system.

In the next chapters we will describe the difficulties that may be encountered when using this technique in practice.

Chapter 3

Application to Axial Flow Compressors

In this chapter we apply the techniques we developed in the previous chapter to the surge and stall problem of axial air compressors. Surge and stall are instability phenomena limiting the range of operation for compressors.

The instability problem has been studied and industrial solutions based on *surge avoidance* are well established. These solutions are based on keeping the operating point to the right of the compressor surge line using a surge margin (this point will be explained shortly). However, there is potential for increasing the efficiency of compressors by allowing the operation closer to the surge line and increasing the range of mass flow over which the compressor can operate stably. This, however, raises the need for control techniques which stabilize the compressor also to the left of the surge line. This approach is known as *active surge control*.

In the following sections, we give a brief introduction to compressors, axial flow compressor model, and surge and stall problem. Next we use the technique in Chapter 2 to stabilize axial flow compressors in the presence of disturbances. The introduction to compressors, surge and stall, and Moore-Greizter model are based on the material in [31],

[38] and [39]. Parts of the state feedback stabilizer design and the projection set rely on the work in [23].

3.1 Compressors

3.1.1 Types of Compressors

The function of a compressor can be defined as *to raise pressure of a specified mass flow of gas by a prescribed amount using the minimum power input.*

Compressors are used in a variety of applications such as, turbojet engines used in aerospace propulsion, power generation using industrial gas turbines, pressurization of gas and fluids in the process industry, transport of fluids in pipelines and so on.

Compressors can be divided into four general types: reciprocating, rotatory, centrifugal and axial. Reciprocating, rotatory and centrifugal compressors are not studied in this work. Axial compressors, also known as turbo-compressors or continuous flow compressors, work by the principle of accelerating the fluid to a high velocity and then converting the kinetic energy and pressure rise by decelerating the gas in diverging channels. In axial compressors the deceleration takes place in the stator blade passages.

3.1.2 The Axial Compressor, Principles of Operation

A stage of an axial compressor consists of a row of rotor blades and a row of stator blades, as shown in Figure 3.1, where C_1, C_2 are the velocities of the flow through the rotor and stator respectively, C_3 is the velocity of the flow exiting the stator, and U is the rotor velocity.

A single stage has often a low pressure ratio 1.1:1 to 1.2:1. Therefore it is very common to use multi-stage axial compressors whereby the compressor has a series of stages, each having a row of rotor blades and a row of stator blades, where the number of stages depends of the desired pressure ratio.

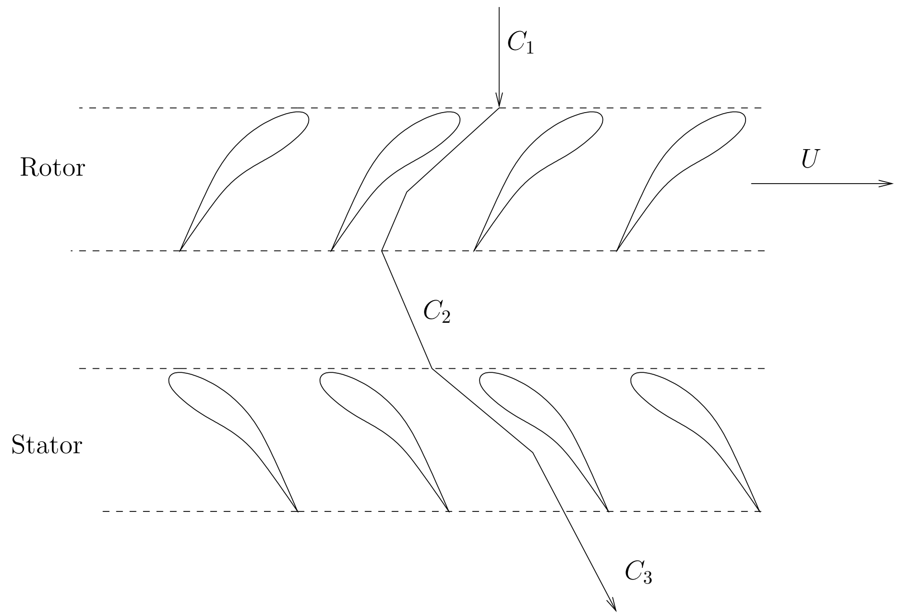

Figure 3.1: Blade rows.

In axial compressors, gas is accelerated in the rotor blades, and decelerated in the stator blades, converting kinetic energy gained in the rotor into pressure rise. All the power is transferred to the gas in the rotor, the stator merely transforms kinetic energy to an increase in static pressure with the temperature remaining constant.

3.1.3 Surge

Surge is an axisymmetrical oscillation of the flow through the compressor, and is characterized by a limit cycle in the compressor characteristic. Surge oscillations are in most applications unwanted, and in extreme cases can damage the compressor.

The mechanism behind surge can be explained as follows. First, consider a compressor operating between two constant pressure reservoirs. The compressor is equipped with a downstream throttle valve. First the compressor operates at point A on the compressor characteristic of Figure 3.2. Consider a disturbance, in the form of partially closing the

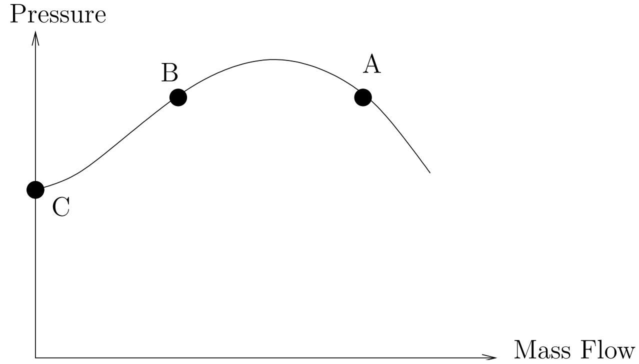

Figure 3.2: Compressor characteristic.

throttle valve and thereby temporarily reducing the flow (not beyond the maximum of the characteristic). This results in an increase in the delivery pressure from the compressor and a reduction in the compressor flow. The increased delivery pressure produces a larger mass flow trough the throttle, reducing compressor delivery pressure and increasing compressor flow. This is therefore self compensating, stable system.

Now, consider the compressor operating at point B in Figure 3.2. A reduction in the mass flow would now result in reduced compressor delivery pressure, reducing flow through the throttle, moving the operating point further and further to the left. Eventually, the mass flow reduction would be so great that the pressure upstream of the throttle falls below the compressor delivery pressure. Mass flow will then increase until the system is drawn back to the operating point B, and the whole cycle repeats. This instability phenomenon is referred to as surge.

As a rule, stall and surge occur at the local maximum of the compressor characteristic or at a point of the compressor characteristic with a certain positive slope. The surge point will be located at some small distance to the left of the peak. Figure 3.3 is a representation of the compressor characteristic for different values of rotor speed (n_i's). Surge points are located on the so-called *surge line*, which can be considered as a barrier that separates

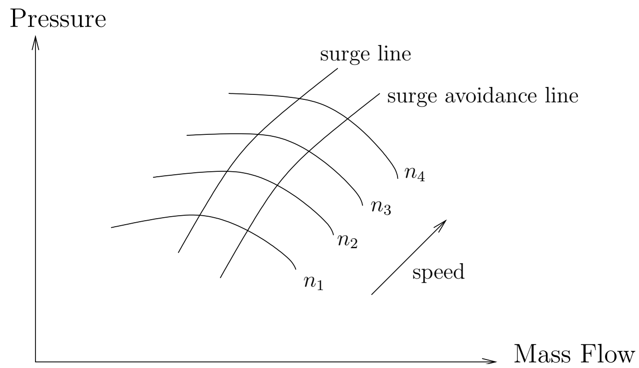

Figure 3.3: Compressor characteristic with surge and surge avoidance lines.

regions of stable and unstable operation in the compressor map. In surge avoidance techniques, a *surge avoidance line* in the compressor map is introduced which is in some distance (e.g., 10% of the flow rate) from the actual surge line, and the compressor state is not allowed to cross to the left of this line, see Figure 3.3.

3.1.4 Rotating Stall

Rotating stall is an instability which arises when the circumferential flow pattern is disturbed. Rotating stall occurs when one or more stall cells of reduced, or stalled flow propagate around the compressor annulus at the fraction of the rotor speed. According to Greitzer, the stall cell propagation may be 20-70% of the rotor speed. Rotating stall leads to a reduction of the pressure rise of the compressor, which in the compressor characteristic corresponds to the compressor operating on the in-stall characteristic[1], see Figure 3.4 (this point will be made clear in what follows).

A basic explanation of the rotating stall mechanism can be summarized as follows. Consider a row of axial compressor blades operating at a high angle of attack, as shown in

[1]In-stall or stalled flow characteristic is a set of operating points which define the pressure rise and mass flow relation when the compressor is in stall. These operating points may be stable and the compressor remain there, until measures are taken to bring it back to the un-stalled characteristic. In Figure 3.4 un-stalled characteristic is indicated by compressor char. A technique for extracting the in-stall characteristic for an axial flow compressor is presented in [22].

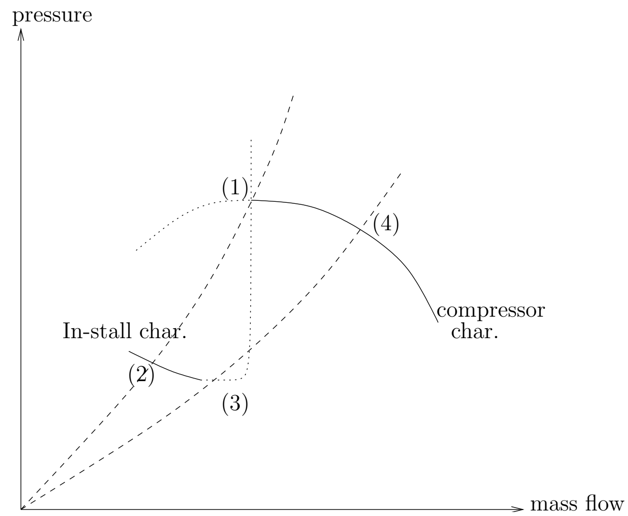

Figure 3.4: Shematic drawing of hysteresis caused by rotating stall. Solid lines represent stable equilibria and dotted lines represent unstable equilibria. The dashed lines are the throttle lines (throttle pressure-flow characteristic) for the onset and clearing of stall.

Figure 3.5. Suppose that there is a non-uniformity in the inlet flow such that a locally higher angle of attack is produced on blade B which is enough to stall it. The flow now separates from the suction surface of the blade, producing a flow blockage between B and C. This blockage causes a diversion of the inlet flow away from B towards A and C, resulting in an increased angle of attack on C, causing it to stall. Thus the stall cell propagates along the blade row.

Another consequence of rotating stall is hysteresis occurring when trying to clear the stall by using the throttle. This situation is shown in Figure 3.4 and can be described as follows. Initially the compressor is operating stably (1), then a disturbance drives the compressor to rotating stall, and an operating point on the low pressure in-stall characteristic (2). By opening the throttle to clear the stall, a mass flow corresponding to (4) which is higher than the initial mass flow corresponding to (1), is required, before the operating point is back on the un-stalled compressor characteristic.

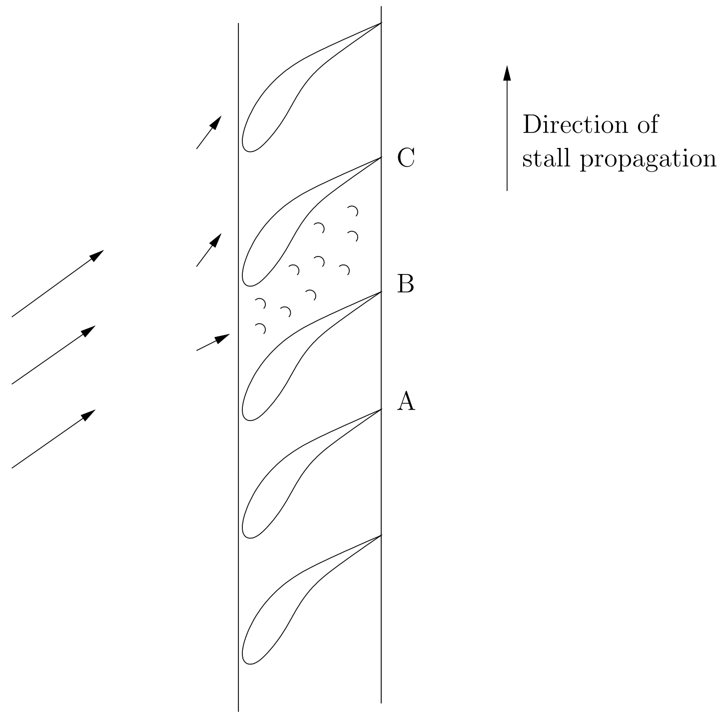

Figure 3.5: Physical mechanism for inception of rotating stall.

3.1.5 Modelling of Axial Compression Systems

Although dynamic models of basic compression systems have been available since 1955, a major step in this field was made in 1976 when a nonlinear dynamic model for a basic axial compression system was presented by Greitzer. A major drawback of earlier models was that they were linearized, and therefore unable to describe the large amplitude pulsations during the surge cycle due to their restriction to small perturbations from an equilibrium. The main contribution of Greitzer is the finding of B parameter (which will be defined shortly) and showing that $B > B_{crit}$ leads to surge and $B < B_{crit}$ leads to rotating stall.

While the model of Greitzer is capable of simulating surge oscillations, rotating stall is described as a pressure drop. Motivated by stall problems in gas turbines, Moore and Greitzer (1986) proposed a model for multistage axial compressors where rotating stall is included as a state. The low order model of Moore and Greitzer captures the post stall transients of a low speed axial compressor-plenum-throttle model (as shown in Figure 3.6).

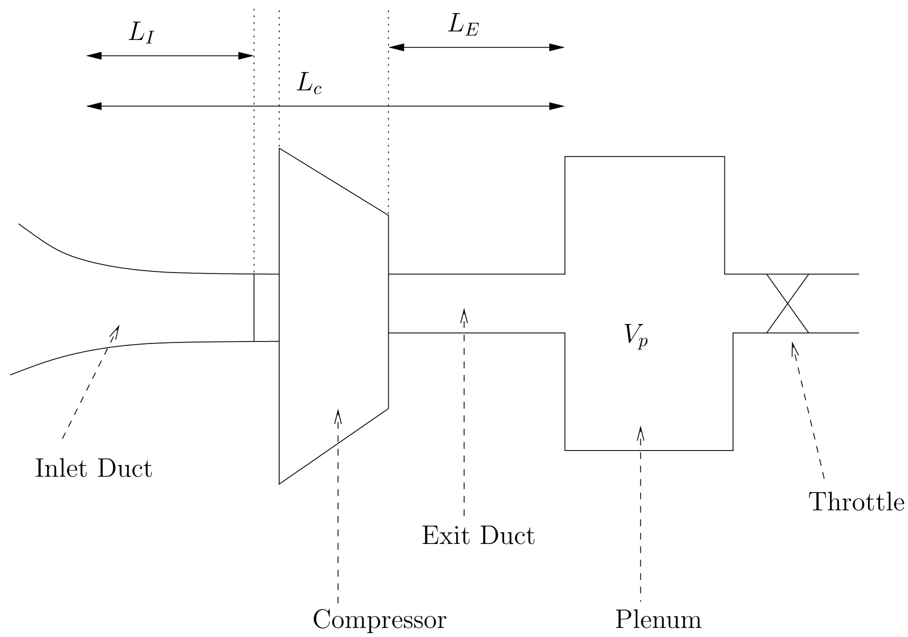

Figure 3.6: Compressor system.

The three differential equations of the model resulting from a Galerkin approximation of the original PDE model are

$$\dot{\Psi}_1 = \frac{W/H}{4B^2}\left(\frac{\Phi_1}{W} - \frac{1}{W}\Phi_{T1}(\Psi_1)\right)\frac{H}{L_c} \tag{3.1}$$

$$\dot{\Phi}_1 = \frac{H}{L_c}\left(-\frac{\Psi_1 - \psi_{1,c0}}{H} - \frac{1}{2}\left(\frac{\Phi_1}{W} - 1\right)^3 + 1 + \frac{3}{2}\left(\frac{\Phi_1}{W} - 1\right)\left(1 - \frac{J}{2}\right)\right) \tag{3.2}$$

$$\dot{J} = J\left(1 - \left(\frac{\Phi_1}{W}\right)^2 - \frac{J}{4}\right)\sigma \tag{3.3}$$

where

- H is the semi-height of the compressor characteristic (see Figure 3.7)
- W is the semi-length of the compressor characteristic (see Figure 3.7)

- Φ_1 is the compressor mass flow
- Ψ_1 is the plenum pressure rise
- J is the squared amplitude of the rotating stall
- $\Phi_{T1}(\Psi_1)$ is the mass flow through the throttle
- L_c is the effective flow-passage length of the compressor and the ducts defined as

$$L_c \triangleq L_I + \frac{1}{a} + l_E$$

  where the positive constant a is the reciprocal time-lag parameter of the blade passage.

- Constant $B > 0$ is the Greitzer B-parameter defined as

$$B \triangleq \frac{U}{2a_s}\sqrt{\frac{V_p}{A_c L_c}}$$

  where U is the constant compressor tangential speed, a_s is the speed of sound, V_p is the plenum volume, A_c is the flow area and L_c is the length of the duct and the compressor.

- Constant $\sigma > 0$ is defined as

$$\sigma = \frac{3aH}{(1 + ma)W}$$

  where m is the compressor-duct flow parameter.

Notice that all the time derivatives in equations (3.1)-(3.3) are calculated with respect to the non-dimensional time ξ_1, defined as

$$\xi_1 \triangleq \frac{Ut}{R}$$

where t is the actual time and R is the mean compressor radius.

The pressure rise of the compressor is a nonlinear function of the mass flow. This function, $\Psi_{1,c}(\Phi_1)$, is known as the compressor characteristic. One expression for this characteristic which has found widespread acceptance in the control literature is the characteristic of Moore and Greitzer

$$\Psi_{1,c}(\Phi) = \psi_{1,c0} + H\left(1 + \frac{3}{2}\left(\frac{\Phi_1}{W} - 1\right) - \frac{1}{2}\left(\frac{\Phi_1}{W} - 1\right)^3\right) \tag{3.4}$$

The characteristic is shown in Figure 3.7. Notice that in (3.2)

$$\dot{\Phi}_1 = \frac{H}{l_c}\left(\frac{-\Psi_1}{H} + \frac{\Psi_{1,c}(\Phi_1)}{H} - \frac{3}{4}J\left(\frac{\Phi_1}{W} - 1\right)\right). \tag{3.5}$$

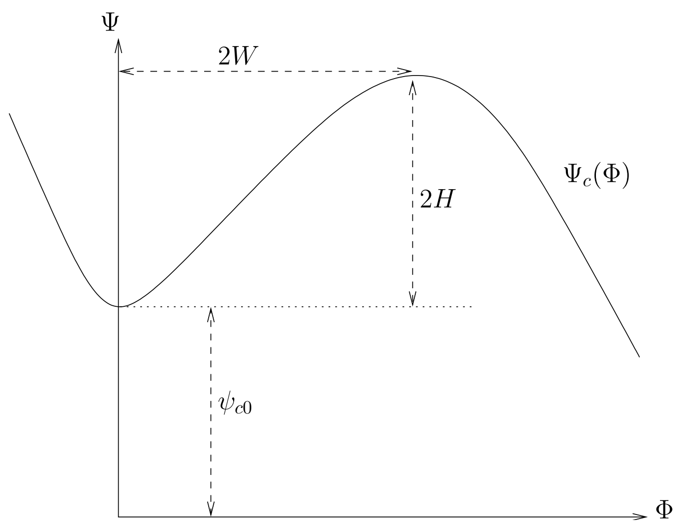

Figure 3.7: Cubic compressor characteristic of Moore and Greitzer. The constants W and H are known as the semi width and semi hight, respectively.

3.1.6 Disturbances

As in [19], the effect of pressure and flow disturbances are considered in this chapter. Pressure disturbances, which may arise from combustion-induced fluctuations, accelerate the flow. Flow disturbances may arise from processes upstream of the compressor, other compressors in series, or an air cleaner in the compressor in-let duct (see Figure 3.6). In case of an aircraft jet engine, large angle of attack or altitude variations may cause mass flow disturbances.

Here we assume that disturbances, $\delta(t)$, are bounded, and they are either slow-varying or constant. The constant disturbance case is of particular interest when, e.g., a constant negative mass flow disturbance pushes the equilibrium over the surge line, initiating surge or rotating stall. A constant disturbance in equation (3.1) can be considered as an uncertainty in the throttle characteristic. Likewise a constant disturbance in equation (3.2) can be considered as an uncertainty in pressure characteristic $\Psi_{1,c}$.

3.1.7 Change of Variables

Consider system (3.1)-(3.3). We use the following change of variables to obtain a different form for the Moore-Greitzer equations which is easier to use. The change of variables is,

$$\Phi = \frac{\Phi_1}{W} - 1 \quad \Psi = \frac{\Psi_1}{W} \quad R = \frac{J}{4} \quad \xi = \frac{H}{WL_c}\xi_1. \tag{3.6}$$

Applying the above changes, system (3.1)-(3.3) is given by

$$\begin{aligned} \dot{\Phi} &= -\Psi + \Psi_C(\Phi) - 3\Phi R \\ \dot{\Psi} &= \frac{1}{\beta^2}(\Phi - \Phi_T) \\ \dot{R} &= \sigma R(1 - \Phi^2 - R), \ \ R(0) \geq 0 \end{aligned} \tag{3.7}$$

where $\beta = \frac{2BH}{W}$, Φ represents the mass flow, Ψ is the plenum pressure rise, $R \geq 0$ is the normalized stall cell squared amplitude, and Φ_T is the mass flow through the throttle. Referring to (3.4) here we have $\Psi_C(\Phi) = \frac{\Psi_{1,c}}{H} = \Psi_{C0} + 1 + \frac{3}{2}\Phi - \frac{1}{2}\Phi^3$ where $\Psi_{C0} = \frac{\psi_{1,c0}}{H}$. $\Phi_T(\Psi)$ is the throttle characteristic, and is defined as $\Psi = \frac{1}{\gamma^2}(1 + \Phi_T(\Psi))^2$, where γ is the throttle opening, the control input. For our purpose we assign $\sigma = 7$, and $\beta = 1/\sqrt{2}$. We call (3.7) the **MG3** (Moore-Greitzer three state) model.

3.2 Output Feedback Control for MG3

Our control objective is to stabilize system (3.7) around the critical equilibrium $R^e = 0, \Phi^e = 1, \Psi^e = \Psi_C(\Phi^e) = \Psi_{C_0} + 2$, which achieves the peak operation on the compressor characteristic, in the presence of external disturbances. We shift the origin to the desired equilibrium with the change of variables $\phi = \Phi - 1, \psi = \Psi - \Psi_{C_0} - 2$. System (3.7) then becomes

$$\begin{aligned}\dot{R} &= -\sigma R^2 - \sigma R(2\phi + \phi^2) \triangleq f_1(R, \phi, \psi) \\ \dot{\phi} &= -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \triangleq f_2(R, \phi, \psi) \\ \dot{\psi} &= -\frac{1}{\beta^2}(\Phi_T - 1 - \phi).\end{aligned} \tag{3.8}$$

We assume the pressure rise (and hence ψ) to be the only measurable state variable. System (3.8) including disturbances can be represented as following

$$\begin{aligned}\dot{x}_1 &= f_1(x) + \delta_1(t) \qquad \delta_1(t) > 0 \\ \dot{x}_1 &= f_2(x) + \delta_1(t) \\ \dot{x}_3 &= kx_2 - ku + \delta_3(t) \\ y &= x_3\end{aligned} \tag{3.9}$$

where $[x_1, x_2, x_3] = [R, \phi, \psi]$, $u = \Phi_T - 1$, $k = -\frac{1}{\beta^2}$, and δ_i is disturbance. It is clear that the system is in the form of system (2.1).

3.2.1 Observability Mappings

First we have to check the observability mapping in Assumption A1 of Chapter 2 to see if F has the desired form. The observability mapping for system (3.9) including disturbances is

$$
\begin{aligned}
y_{e1} &= y = x_3 \\
y_{e2} &= \dot{y} = \dot{x}_3 = kx_2 - ku + \delta_3(t) \\
y_{e3} &= \ddot{y} = k\dot{x}_2 - k\dot{u} + \dot{\delta}(t) = kf_2(x) - k\dot{u} + k\delta_2(t) + \dot{\delta}_3(t),
\end{aligned} \tag{3.10}
$$

so letting $z = [u, \dot{u}]^\top$ and $\bar{\delta} = [\delta, \dot{\delta}]^\top$, H_1 and H_2 in Assumption A1 have the expression

$$
\begin{aligned}
H(x, z, \bar{\delta}) &= H_1(x, z) + H_2(\bar{\delta}) \\
&= \begin{bmatrix} x_3 \\ kx_2 - ku \\ kf_2(x) - k\dot{u} \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_3(t) \\ k\delta_2(t) + \dot{\delta}_3(t) \end{bmatrix} \\
&= \begin{bmatrix} h(x) \\ \phi_1(x, z) \\ \phi_2(x, z) \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_1(\bar{\delta}) \\ \theta_2(\bar{\delta}). \end{bmatrix}
\end{aligned} \tag{3.11}
$$

Notice that $\dfrac{\partial H}{\partial x} = \dfrac{\partial H_1}{\partial x}$. Therefore F in A1 can be written as

$$
\begin{aligned}
F(R, \phi, \psi, z_1, z_2) &= \begin{bmatrix} \psi \\ 1/\beta^2\,(\phi - z_1)) \\ 1/\beta^2\,(-\psi - 3/2\phi^2 - 1/2\phi^3 - 3R\phi - 3R - z_2) \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_3(t) \\ k\delta_2(t) + \dot{\delta}_3(t) \\ 0 \\ 0 \end{bmatrix} \\
&= F_1(x, z) + F_2(\bar{\delta}).
\end{aligned} \tag{3.12}
$$

It is clear that F has the desired form. Notice that when $\phi = -1$, which corresponds to $\Phi = 0$, F_1 does not depend on R and is not invertible. Therefore when there is no mass flow through the compressor ($\Phi = 0$), a condition that we want to avoid during normal operation, R cannot be determined. In all other cases F_1 is a deffeomorphism. We thus conclude that assumption A1 is satisfied on the set

$$\mathcal{O} = \left\{ [R, \phi, \psi]^\top \in \mathbb{R}^3, z \in \mathbb{R}^2 \,|\, \phi > -1 \right\}.$$

3.2.2 State Feedback Control

Here we design a state feedback control law for system (3.8) and later we show that this control law input-to-state stabilizes system (3.8). By viewing system (3.8) as an interconnection of two subsystems, namely the R-subsystem and the (ϕ, ψ)-subsystem, we design a full-state feedback controller which makes the origin of (3.8) an asymptotically stable equilibrium point with domain of attraction $\{(R, \phi, \psi) \in \mathbb{R}^3 | R \geq 0\}$. We begin by assuming that $\delta(t) = 0$.

The following theorem and its proof are taken from [23] and will be used in the following to prove the ISS property in assumption A2 from Chapter 2.

Theorem 4 *For system (3.8), with the choice of the control law*

$$\bar{u} = (1 - \beta^2 k_1 k_2)\phi + \beta^2 k_2 \psi + 3\beta^2 k_1 R\phi \tag{3.13}$$

where k_1 and k_2 are positive scalars satisfying the inequalities,

$$k_1 > \frac{17}{8} + \frac{(2C\sigma + 3)^2}{2} \tag{3.14}$$

$$\left(C\sigma - \frac{105}{64}\right) k_1^2 + \frac{3}{4}\left(-\frac{1}{2}C\sigma + \frac{21}{4}\right) k_1 - (C\sigma + 3)^2 > 0 \tag{3.15}$$

$$k_2 > k_1 + \frac{9}{4}k_1^2 + \frac{9k_1}{4k_1 - 9/2} + \frac{(k_1^2 - 1)^2}{4} \tag{3.16}$$

$$C > \frac{3}{2\sigma} \tag{3.17}$$

the origin is an asymptotically stable equilibrium point with domain of attraction $\mathcal{A} = \{(R, \phi, \psi) \in \mathbb{R}^3 | R \geq 0\}$.

Proof. Without loss of generality and for simplicity let

$$u' = \frac{1}{\beta^2}(u - \phi),$$

be the control input. Therefore the last equation in (3.8) becomes $\dot{\psi} = -u'$. In the next step, we consider system (3.8) the interconnection of two subsystems $[S_1]$ and $[S_2]$ as follows

$$[S_1]\ \dot{R} = -\sigma R^2, \quad [S_2] \begin{cases} \dot{\phi} = -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ \dot{\psi} = -u' \end{cases}$$

A Lyapunov function for $[S_1]$, defined on the domain $\{R \in \mathbb{R} \,|\, R \geq 0\}$, is $V_1 = R$, and its time derivative is $\dot{V}_1 = -\sigma R^2$ thus showing that the origin of $[S_1]$ is an asymptotically stable equilibrium point of $[S_1]$, and its domain of attraction is $\{R \in \mathbb{R} \,|\, R \geq 0\}$. As for subsystem $[S_2]$ we use $V_2 = \frac{1}{2}\phi^2 + \frac{k_1}{8}\phi^4 + \frac{1}{2}(\phi - k_1\psi)^2$, where k_1 is a positive design constant. Furthermore, in [12], a stabilizing control law for $[S_2]$ is found to be $u = -c_1\phi + c_2\psi$, where c_1 and c_2 are two appropriate positive constants. In the following we will show that, in order to stabilize the interconnection of systems $[S_1]$ and $[S_2]$, one needs to add to $u = -c_1\phi + c_2\psi$ a term which is proportional to the product $R\phi$. Based on these considerations, consider the following candidate Lyapunov function for system (3.8),

$$V = CV_1 + V_2 = CR + \frac{1}{2}\phi^2 + \frac{k_1}{8}\phi^4 + \frac{1}{2}(\psi - k_1\phi)^2 \tag{3.18}$$

where $C > 0$ is a scalar. Notice that V is positive definite on the domain $\mathcal{A}$. Letting $\tilde{\psi} = \psi - k_1\phi$, we calculate the time derivative of V as follows,

$$\begin{aligned} \dot{V} =& - C\sigma R^2 - C\sigma R(2\phi + \phi^2) + \left(\phi + \frac{k_1}{2}\phi^3\right)\left(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R\right) + \\ &+ \tilde{\psi}\left(-u' + k_1\psi + \frac{3}{2}k_1\phi^2 + \frac{1}{2}k_1\phi^3 + 3k_1R\phi + 3k_1R\right) \end{aligned} \tag{3.19}$$

We use the identity $-\frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 = -\frac{1}{2}\left(\phi + \frac{3}{2}\right)^2\phi + \frac{9}{8}\phi$ to eliminate the potentially destabilizing term $-(\phi + k_1/2\phi^3)\,3/2\phi^2$. Next, substituting (3.13) into (3.19) (after taking into account the definition of u'), letting $\bar{k}_1 = k_1 - 9/8$, we get

$$\begin{aligned}\dot{V} =& -C\sigma R^2 - C\sigma R(2\phi + \phi^2) + \left(\phi + \frac{k_1}{2}\phi^3\right)\left(-\tilde{\psi} - \bar{k}_1\phi - \frac{1}{2}\left(\phi + \frac{3}{2}\right)^2\phi - 3R\phi - 3R\right) + \\ &+ \tilde{\psi}\left(-(k_2 - k_1)\tilde{\psi} + k_1^2\phi + \frac{3}{2}k_1\phi^2 + \frac{1}{2}k_1\phi^3 + 3k_1R\right)\end{aligned} \tag{3.20}$$

Notice that the expression $-\frac{\phi}{2}\left(\phi + \frac{k_1}{2}\phi^3\right)\left(\phi + \frac{3}{2}\right)^2$ can be discarded since it is negative definite, and that the term $\frac{k_1}{2}\phi^3\tilde{\psi}$ cancels out. After rearranging the remaining terms, we get

$$\begin{aligned}\dot{V} \leq& -C\sigma R^2 - (2C\sigma + 3)R\phi - (C\sigma + 3)R\phi^2 - \bar{k}_1\phi^2 - \left(\frac{k_1\bar{k}_1}{2} + \frac{3k_1}{2}R\right)\phi^4 - \frac{3k_1}{2}R\phi^3 + \\ &+ \tilde{\psi}\left(-(k_2 - k_1)\tilde{\psi} + (k_1^2 - 1)\phi + \frac{3}{2}k_1\phi^2 + 3k_1R\right)\end{aligned} \tag{3.21}$$

By using Young's inequality[2] we obtain the following inequalities

$$\begin{aligned}&-(2C\sigma + 3)R\phi \leq \frac{1}{2}R^2 + \frac{(2C\sigma + 3)^2}{2}\phi^2, \quad -\frac{3k_1}{2}R\phi^3 \leq \frac{3k_1}{2}\left(\frac{R\phi^2}{4} + R\phi^4\right), \\ &(k_1^2 - 1)\phi\tilde{\psi} \leq \phi^2 + \frac{(k_1^2 - 1)^2}{4}\tilde{\psi}^2, \quad 3k_1R\tilde{\psi} \leq R^2 + \frac{9}{4}k_1^2\tilde{\psi}^2, \quad \frac{3}{2}k_1\phi^2\tilde{\psi} \leq \frac{k_1\bar{k}_1}{4}\phi^4 + \frac{9k_1}{4\bar{k}_1}\tilde{\psi}^2.\end{aligned} \tag{3.22}$$

Applying the above inequalities to (3.21) we get

$$\begin{aligned}\dot{V} \leq& -\left(C\sigma - \frac{3}{2}\right)R^2 - \left(\bar{k}_1 - \frac{(2C\sigma + 3)^2}{2} - 1\right)\phi^2 - \left(k_2 - k_1 - \frac{9}{4}k_1^2 - \frac{9k_1}{4\bar{k}_1} - \frac{(k_1^2 - 1)^2}{4}\right)\tilde{\psi}^2 + \\ &- \left(C\sigma + 3 - \frac{3}{8}k_1\right)R\phi^2 - \frac{k_1\bar{k}_1}{4}\phi^4, \\ \leq& -\begin{bmatrix} R \\ \phi^2 \end{bmatrix}^\top \begin{bmatrix} C\sigma - \frac{3}{2} & \frac{1}{2}\left(C\sigma + 3 - \frac{3}{8}k_1\right) \\ \frac{1}{2}\left(C\sigma + 3 - \frac{3}{8}k_1\right) & \frac{1}{4}k_1\bar{k}_1 \end{bmatrix} \begin{bmatrix} R \\ \phi^2 \end{bmatrix} - \left(\bar{k}_1 - \frac{(2C\sigma + 3)^2}{2} - 1\right)\phi^2 + \\ &- \left(k_2 - k_1 - \frac{9}{4}k_1^2 - \frac{9k_1}{4\bar{k}_1} - \frac{(k_1^2 - 1)^2}{4}\right)\tilde{\psi}^2\end{aligned} \tag{3.23}$$

If the above quadratic form is positive definite, and the coefficients multiplying ϕ^2 and $\tilde{\psi}^2$ are positive, then $\dot{V}$ is negative definite on the domain $\mathcal{A}$. For the quadratic form to

[2]For any real numbers a and b, and any positive real k, one has that $ab \leq \frac{a^2}{4k} + kb^2$.

be positive definite we should have

$$C\sigma - \frac{3}{2} > 0\,, \quad \left(C\sigma - \frac{3}{2}\right)\frac{1}{4}k_1\bar{k}_1 - \frac{1}{4}\left(C\sigma + 3 - \frac{3}{8}k_1\right)^2 > 0.$$

For positivity of the coefficients of ϕ^2 and $\tilde{\psi}^2$ we need that

$$\bar{k}_1 > \frac{(2C\sigma + 3)^2}{2} + 1\,, \quad k_2 > k_1 + \frac{9}{4}k_1^2 + \frac{9k_1}{4\bar{k}_1} + \frac{(k_1^2 - 1)^2}{4}.$$

By using the definition of $\bar{k}_1$, inequalities (3.25), (3.26), (3.27), and (3.28) follow. In conclusion, if k_1, k_2, and C are chosen so that (3.25)-(3.28) hold, $\dot{V}$ will be negative definite on $\mathcal{A}$ which contains the origin. Moreover, the boundary of $\mathcal{A}$, $\partial\mathcal{A} = \{(R, \phi, \psi) \,|\, R = 0\}$, is an invariant manifold (when $R = 0$, $\dot{R} = 0$). Therefore the origin of the closed-loop system in an asymptotically stable equilibrium point and the set $\{(R, \phi, \psi) \,|\, V \leq K\} \cap \mathcal{A}$ is its region of attraction for any positive real number K. This in turn shows that $\mathcal{A}$ is the domain of attraction of the origin of the closed-loop system.

■

3.2.3 Input-to-State Stability

Now we have to check assumption A2 for system the (3.9) using the feedback controller $\bar{u}$ in (3.13).

Consider

$$\begin{aligned} \dot{R} &= f_1(x) + \delta_1(t) \\ \dot{\phi} &= f_2(x) + \delta_2(t) \\ \dot{\psi} &= -\frac{1}{\beta^2}(u - \phi) + \delta_3(t) \end{aligned} \tag{3.24}$$

where f_1 and f_2 are defined in (3.8), δ_i's are disturbances, and $x = [R, \phi, \psi]^\top$.

Theorem 5 *The closed-loop system given by (3.24) and (3.13) where k_1 and k_2 satisfy inequalities*

$$k_1 > \frac{17}{8} + \frac{(2C\sigma + 3)^2}{2} \tag{3.25}$$

$$\left(C\sigma - \frac{105}{64}\right) k_1^2 + \frac{3}{4}\left(-\frac{1}{2}C\sigma + \frac{21}{4}\right) k_1 - (C\sigma + 3)^2 > 0 \tag{3.26}$$

$$k_2 > k_1 + \frac{9}{4}k_1^2 + \frac{9k_1}{4k_1 - 9/2} + \frac{(k_1^2 - 1)^2}{4} \tag{3.27}$$

$$C > \frac{3}{2\sigma} \tag{3.28}$$

is input-to-state stable with respect to the disturbance input $[\delta_1(t), \delta_2(t), \delta_3(t)]^\top$.

Proof. Substituting $u = \bar{u}$ from (3.13) in the last equation of (3.24) we have the following

$$u = (1 - \beta^2 k_1 k_2)\phi + \beta^2 k_2 \psi + 3\beta^2 k_1 R\phi \stackrel{\triangle}{=} (1 - a)\phi + b\psi + dR\phi$$

$$\dot{\psi} = \underbrace{-\frac{1}{\beta^2}(-a\phi + b\psi + dR\phi)}_{f_3} + \delta_3(t)$$

$$\dot{\psi} = f_3(x) + \delta_3(t).$$

We have the following closed loop system

$$\begin{aligned} \dot{R} &= f_1 + \delta_1 \\ \dot{\phi} &= f_2 + \delta_2 \\ \dot{\psi} &= f_3 + \delta_3 \end{aligned} \tag{3.29}$$

where we assume that δ_i's are bounded.

Using the same Lyapunov function $V = CV_1 + V_2$ $(C > 0)$, as in the state feedback controller design section, we have the following

$$V_1 = R, \qquad V_2 = \frac{1}{2}\phi^2 + \frac{k_1}{8}\phi^4 + \frac{1}{2}\underbrace{(\psi - k_1\phi)^2}_{\tilde{\psi}}$$

$$\begin{aligned}
\dot{V} &= C\dot{R} + \frac{\partial V_2}{\partial \phi}\dot{\phi} + \frac{\partial V_2}{\partial \tilde{\psi}}\dot{\tilde{\psi}} \\
&= Cf_1 + C\delta_1 + \frac{\partial V_2}{\partial \phi}f_2 + \frac{\partial V_2}{\partial \phi}\delta_2 + \tilde{\psi}(\dot{\psi} - k_1\dot{\phi}) \\
&= \underbrace{Cf_1 + \frac{\partial V_2}{\partial \phi}f_2 + \tilde{\psi}(f_3 - k_1 f_2)}_{M} + \underbrace{C\delta_1 + \frac{\partial V_2}{\partial \phi}\delta_2 + \tilde{\psi}\delta_3 - k_1\tilde{\psi}\delta_2}_{N}.
\end{aligned}$$

Using Young's inequality for N we will have

$$\begin{aligned}
N &= C\delta_1 + (\phi + \frac{k_1}{2}\phi^3)\delta_2 + \tilde{\psi}\delta_3 - k_1\tilde{\psi}\delta_2 \\
&\leq C\delta_1 + \frac{\phi^2}{2} + \frac{\delta_2^2}{2} + \frac{k_1}{8}\phi^6 + \frac{k_1}{2}\delta_2^2 + \frac{\tilde{\psi}^2}{2} + \frac{\delta_3^2}{2} + \frac{k_1}{2}\tilde{\psi}^2 + \frac{k_1}{2}\delta_2^2 \\
&\leq \frac{k_1}{8}\phi^6 + \underbrace{C\delta_1 + \frac{\phi^2}{2} + \frac{\delta_2^2}{2} + \frac{k_1}{2}\delta_2^2 + \frac{\tilde{\psi}^2}{2} + \frac{\delta_3^2}{2} + \frac{k_1}{2}\tilde{\psi}^2 + \frac{k_1}{2}\delta_2^2}_{M'}
\end{aligned}$$

In the controller design section we discarded the negative definite term $-\frac{\phi}{2}\left(\phi + \frac{k_1}{2}\phi^3\right)\left(\phi + \frac{3}{2}\right)^2$, which is used here as follows

$$\begin{aligned}
-\frac{\phi}{2}(\phi + \frac{k_1}{2}\phi^3)(\phi + \frac{3}{2})^2 &= \underbrace{-\frac{\phi^2}{2}(\phi + \frac{3}{2})^2}_{\text{negative definite}} - \frac{k_1}{4}\phi^4(\phi + \frac{3}{2})^2 \\
&\leq -\frac{k_1}{4}\phi^6 - \frac{3}{4}k_1\phi^5 - \frac{9}{16}k_1\phi^4
\end{aligned}$$

then for $\dot{V}$ we have

$$\begin{aligned}
\dot{V} &\leq M - \frac{k_1}{4}\phi^6 - \frac{3}{4}k_1\phi^5 - \frac{9}{16}k_1\phi^4 + \frac{k_1}{8}\phi^6 + M' \\
&\leq M - \frac{k_1}{4}\phi^6 - \frac{3}{4}k_1\phi^5 - \frac{9}{16}k_1\phi^4 + M' \\
&\leq M - \frac{k_1}{2}\phi^4(\frac{\phi^2}{4} + \frac{3}{2}k_1\phi + \frac{9}{8}) + M' \\
&\leq M - \frac{k_1}{2}\phi^4\left[(\frac{\phi}{2} + \frac{3}{2})^2 - \frac{9}{8}\right] + M' \\
&\leq M \underbrace{- \frac{k_1}{2}\phi^4(\frac{\phi}{2} + \frac{3}{2})^2}_{\text{negative definite}} + \frac{9}{16}k_1\phi^4 + M'
\end{aligned}$$

Hence $\dot{V}$ satisfies the following inequality,

$$
\begin{aligned}
\dot{V} \leq & -\left(C\sigma - \frac{3}{2}\right) R^2 - \left(\overline{k}_1 - \frac{(2C\sigma+3)^2}{2} - 1\right)\phi^2 \\
& - \left(k_2 - \frac{3}{2}k_1 - \frac{9}{4}k_1^2 - 9\frac{k_1}{\overline{k}_1} - \frac{(k_1-1)^2}{4} - \frac{1}{2}\right)\tilde{\psi}^2 \\
& - (C\sigma + 3 - \frac{3}{8}k_1)R\phi^2 - \left(\frac{k_1\overline{k}_1}{4} - \frac{9k_1}{16}\right)\phi^4 \\
& + C\delta_1 + \left(\frac{1}{2} + k_1\right)\delta_2^2 + \frac{\delta_3^2}{2} \\
\leq & -\begin{bmatrix} R \\ \phi^2 \end{bmatrix}^T \begin{bmatrix} C\sigma - \frac{3}{2} & \frac{1}{2}(C\sigma + 3 - \frac{3}{8}k_1) \\ \frac{1}{2}(C\sigma + 3 - \frac{3}{8}k_1) & \frac{k_1\overline{k}_1}{4} - \frac{9k_1}{16} \end{bmatrix} \begin{bmatrix} R \\ \phi^2 \end{bmatrix} \\
& - \left(\overline{k}_1 - \frac{(2C\sigma+3)^2}{2} - 1\right)\phi^2 - \left(k_2 - \frac{3}{2}k_1 - \frac{9}{4}k_1^2 - 9\frac{k_1}{\overline{k}_1} - \frac{(k_1-1)^2}{4} - \frac{1}{2}\right)\tilde{\psi}^2 \\
& + C\delta_1 + \left(\frac{1}{2} + k_1\right)\delta_2^2 + \frac{\delta_3^2}{2}
\end{aligned}
\tag{3.30}
$$

Note that

$$
\begin{aligned}
C\delta_1 + \left(\frac{1}{2} + k_1\right)\delta_2^2 + \frac{\delta_3^2}{2} &\leq C|\delta_1| + \left(\frac{1}{2} + k_1\right)(\delta_2^2 + \delta_3^2) \\
&\leq C\|\delta\| + \left(\frac{1}{2} + k_1\right)\|\delta\|^2 \triangleq \chi(||\delta||)
\end{aligned}
$$

If the parameters of the inequality (3.30) satisfy the following inequalities,

$$
k_1 \geq \frac{17}{8} + \frac{(2C\sigma+3)^2}{2} \tag{3.31}
$$

$$
\left(C\sigma - \frac{105}{64}\right)k_1^2 + \frac{3}{4}\left(-\frac{5}{4}C\sigma + \frac{51}{8}\right)k_1 - (C\sigma+3)^2 \geq 0 \tag{3.32}
$$

$$
k_2 \geq \frac{3}{2}k_1 - \frac{9}{4}k_1^2 - 9\frac{k_1}{4k_1 - 9\backslash 2} - \frac{(k_1-1)^2}{4} - \frac{1}{2} \tag{3.33}
$$

$$
C\sigma \geq \frac{105}{64} \tag{3.34}
$$

then the quadratic form in equation (3.30) is positive definite and the coefficients of ϕ and $\tilde{\psi}$ are positive and we have

$$\begin{aligned}\dot{V} \leq & -\begin{bmatrix} R \\ \phi^2 \end{bmatrix}^T \begin{bmatrix} C\sigma - \frac{3}{2} & \frac{1}{2}(C\sigma + 3 - \frac{3}{8}k_1) \\ \frac{1}{2}(C\sigma + 3 - \frac{3}{8}k_1) & \frac{k_1\bar{k}_1}{4} - \frac{9k_1}{16} \end{bmatrix} \begin{bmatrix} R \\ \phi^2 \end{bmatrix} \\ & - \left(\bar{k}_1 - \frac{(2C\sigma+3)^2}{2} - 1\right)\phi^2 - \left(k_2 - \frac{3}{2}k_1 - \frac{9}{4}k_1^2 - 9\frac{k_1}{\bar{k}_1} - \frac{(k_1-1)^2}{4} - \frac{1}{2}\right)\tilde{\psi}^2 \\ & + \chi(||\delta||) \\ & \triangleq -\alpha(\|x\|) + \chi(||\delta||)\end{aligned}$$

which implies that the system (3.29) is input-to-state stable. Also notice that the set of triples (k_1, k_2, σ) satisfying (3.31)-(3.34) is non-empty and they also satisfy inequalities (3.25)-(3.28)

■

3.2.4 Stabilizing Control Law for Augmented System

Referring to equations (3.10), since H_1 depends on u and $\dot{u}$ (i.e. $n_u = 2$), following the procedure outlined in Section 2.1 we augment the system dynamics with two integrators at the input side, i.e.,

$$\begin{aligned} [P_1] & \begin{cases} \dot{R} = -\sigma R^2 - \sigma R(2\phi + \phi^2) + \delta_1(t) \\ \dot{\phi} = -\psi - 3/2\phi^2 - 1/2\phi^3 - 3R\phi - 3R + \delta_2(t) \\ \dot{\psi} = \frac{1}{\beta^2}(\phi - z_1) + \delta_3(t) \end{cases} \\ [P_2] & \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v \end{cases} \end{aligned} \tag{3.35}$$

Defining $x = [R, \phi, \psi]^\top$, we can write system $[P_1]$ as $\dot{x} = f(x) + g(x)z_1 + \delta(t)$. The following theorem shows that having the stabilizing controller (3.13), we can find a v such that the extended system is input-to-state-stable.

Theorem 6 *Consider system*

$$\dot{x} = f(x) + g(x)u + \delta(t) \tag{3.36}$$

Suppose there exists a smooth function $u = u_0(x)$ such that system (3.36) is ISS with $\delta(t)$ as an input. Then there exists a smooth function v such that system

$$\begin{aligned} \dot{x} &= f(x) + g(x)s_1 + \delta(t) \\ \dot{s}_1 &= s_2 \\ \dot{s}_2 &= v \end{aligned} \tag{3.37}$$

is ISS with $\delta(t)$ as the input.

Proof. Consider the following change of variables

$$z_1 = s_1 - u_0(x) \tag{3.38}$$

$$z_2 = s_2 - u_1(x, s_1) \tag{3.39}$$

Then we have

$$\begin{aligned} \dot{x} &= f(x) + g(x)s_1 + \delta(t) \\ \dot{z}_1 &= z_2 + u_1 - \underbrace{\frac{\partial u_0}{\partial x}\left[f + g(z_1 + u_0)\right]}_{A} - \frac{\partial u_0}{\partial x}\delta \\ \dot{z}_2 &= v - \underbrace{\frac{\partial u_1}{\partial x}\left[f + g(z_1 + u_0)\right]}_{B} - \frac{\partial u_1}{\partial x}\delta - \frac{\partial u_1}{\partial s_1}(z_2 + u_1) \end{aligned} \tag{3.40}$$

From ISS property for the system (3.36) we know that there exists a Lyapunov function $V(x)$ for that system such that for the disturbance $\delta(\cdot)$ and some positive definite, class K_∞ functions $\alpha_i(\cdot)$ and some K function $\chi(\cdot)$ the following inequalities hold

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{3.41}$$

$$\dot{V}(x) \leq -\alpha_3(\|x\|) + \chi(\|\delta(t)\|) \tag{3.42}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|) \tag{3.43}$$

Let $X = [x^\top, z_1, z_2]^\top$ and consider $W(X) = V(x) + \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ as a Lyapunov candidate for system (3.40). Then we have

$$\begin{aligned}
\dot{W} =& \frac{\partial V}{\partial x}\dot{x} + z_1\dot{z}_1 + z_2\dot{z}_2 \\
=& \frac{\partial V}{\partial x}\left[f + g(z_1 + u_0) + \delta\right] + z_1\left(z_2 + u_1 - A - \frac{\partial u_0}{\partial x}\delta\right) \\
&+ z_2\left[v - B - \frac{\partial u_1}{\partial x}\delta - \frac{\partial u_1}{\partial s_1}(z_2 + u_1)\right]
\end{aligned}$$

Choose u_1 and v as follows

$$u_1 = -c_1 z_1 - \frac{\partial V}{\partial x} g + A + K_1$$

$$v = -c_2 z_2 - z_1 + B + \frac{\partial u_1}{\partial s_1}(z_2 + u_1) + K_2 \triangleq \pi(x, z) \tag{3.44}$$

where $c_1, c_2 > 0$, and K_1 and K_2 will be defined later. Then we have

$$\begin{aligned}
\dot{W} \leq& -\alpha_3(\|x\|) + \chi(\|\delta(t)\|) \\
&- c_1 z_1^2 + z_1 K_1 - z_1 \frac{\partial u_0}{\partial x}\delta \\
&- c_2 z_2^2 + z_2 K_2 - z_2 \frac{\partial u_1}{\partial x}\delta \\
\leq& -\alpha_3(\|x\|) + \chi(\|\delta(t)\|) \\
&- c_1 z_1^2 + z_1 K_1 + z_1^2 \left\| \frac{\partial u_0}{\partial x} \right\|^2 + \frac{1}{4}\|\delta\|^2 \\
&- c_2 z_2^2 + z_2 K_2 + z_2^2 \left\| \frac{\partial u_1}{\partial x} \right\|^2 + \frac{1}{4}\|\delta\|^2.
\end{aligned}$$

Choosing $K_1 = -z_1 \left\| \frac{\partial u_0}{\partial x} \right\|^2$ and $K_2 = -z_2 \left\| \frac{\partial u_1}{\partial x} \right\|^2$, we have

$$\dot{W} \leq -\alpha_3(\|x\|) - c_1 z_1^2 - c_2 z_2^2 + \chi(\|\delta(t)\|) + \frac{1}{2}\|\delta\|^2$$

which implies that $\dot{W} \leq -\pi\left(\left\| \begin{bmatrix} x \\ z \end{bmatrix} \right\|\right) + \chi_1(\|\delta\|)$ for some K_∞ function π and K function χ_1.

■

3.2.5 Observability Set

We have that $\mathcal{Y} = \mathcal{F}_1(\mathcal{O}) = \{y_e^1 \in \mathbb{R}^3, z \in \mathbb{R}^2 \,|\, y_{e,2}^1 > \frac{1}{\beta^2}(-1 - z_1)\}$. Define set $\mathcal{C}$ as

$$\mathcal{C} = \left\{ [y_e^{1\top}, z^\top]^\top \in \mathbb{R}^5 \,|\, y_{e,1}^1 \in [a_1, b_1], y_{e,2}^1 \in \left[\frac{a_2 - z_1}{\beta^2}, \frac{b_2 - z_1}{\beta^2}\right], \right.$$
$$\left. y_{e,3}^1 \in \left[\frac{1}{\beta^2}(-z_2 + a_3), \frac{1}{\beta^2}(-z_2 + b_3)\right], z_1 \in [a_4, b_4], z_2 \in [a_5, b_5] \right\},$$

where a_i's and b_i's are scalars such that $a_i, b_i \in \mathbb{R}$, $a_i < b_i, i = 1, \dots, 5$. It can be verified that the for all $a_2 > -1$, $\mathcal{C} \subset \mathcal{Y}$. For requirement (ii) in assumption A3 to be satisfied, each slice $\mathcal{C}^{\bar{z}}$ should be convex. Here the slices are parallelpipeds in $\mathbb{R}^3$. The union of all slices is the set

$$\bigcup_{\bar{z} \in \mathbb{R}^2} \mathcal{C}^{\bar{z}} = \left\{ y_e \in \mathbb{R}^3 \,|\, y_{e,1} \in [a_1, b_1], y_{e,2} \in \left[\frac{a_2 - b_4}{\beta^2}, \frac{b_2 - a_4}{\beta^2}\right], \right.$$
$$\left. y_{e,3} \in \left[\frac{1}{\beta^2}(-b_5 + a_3), \frac{1}{\beta^2}(-a_5 + b_3)\right] \right\}, \tag{3.45}$$

which is compact and satisfying requirement (iv).

The vectors N_{y_e} and N_z are given by

$$N_{y_e}(\hat{y}_e, z) = \begin{cases} [1, 0, 0]^\top & \hat{y}_{e,1} = b_1 \\ [-1, 0, 0]^\top & \hat{y}_{e,1} = a_1 \\ [0, 1, 0]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(b_2 - z_1) \\ [0, -1, 0]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(a_2 - z_1) \\ [0, 0, 1]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(-z_2 + b_3) \\ [0, 0, -1]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(-z_2 + a_3), \end{cases} \tag{3.46}$$

$$N_z(\hat{y}_e, z) = \begin{cases} [0,0]^\top & \hat{y}_{e,1} = b_1 \text{ or } \hat{y}_{e,1} = a_1 \\ [1/\beta^2, 0]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(b_2 - z_1) \\ [-1/\beta^2, 0]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(a_2 - z_1) \\ [0, 1/\beta^2]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(b_3 - z_2) \\ [0, -1/\beta^2]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(a_3 - z_2) \\ [1,0]^\top & z_1 = b_4 \\ [-1,0]^\top & z_1 = a_4 \\ [0,1]^\top & z_2 = b_5 \\ [0,-1]^\top & z_2 = a_5. \end{cases} \tag{3.47}$$

Notice that N_{y_e} never vanishes on any slice $\mathcal{C}^{\bar{z}}$, which is the condition of the requirement (iii). Now we have to use the Lyapunov function V to find the largest value of c_2 such that $\Omega_{c_2} \subset \mathcal{O}$ and then pick some values for a_i and b_i, $i = 1, \ldots, 5$ such that $a_2 > -1$ and $F_1(\Omega_{c_2}) \subset \mathcal{C}$. A more practical way to design $\mathcal{C}$ is running a number of simulations for closed loop system under state feedback for some initial conditions $(x(0), z(0))$ and finding upper and lower bounds of $\psi(t)$, $\phi(t)$, $-\psi(t) - 3/2\phi^2(t) - 1/2\phi^3(t) - 3R(t)\phi(t) - 3R(t)$, $z_1(t)$, $z_2(t)$, which in turn will give us the values of a_i, b_i, $i = 1, \ldots, 5$, respectively. By doing that, we found that whenever

$$\begin{aligned} [x(0)^\top, z(0)^\top]^\top \in \Omega_0 \triangleq \big\{ & [x(0)^\top, z(0)^\top]^\top \in \mathbb{R}^5 \,|\, R \in [0, 0.1], \phi \in [-0.1, 0.1], \\ & \psi \in [-0.5, 0.5], z_1 \in [-0.1, 0.1], z_2 \in [-0.1, 0.1] \big\}, \end{aligned} \tag{3.48}$$

we have that $a_1 = -1.15$, $b_1 = 0.5$, $a_2 = -0.3$, $b_2 = -0.1$, $a_3 = -0.75$, $b_3 = 0.4$, $a_4 = -2$, $b_4 = 7$, $a_5 = -70$, $b_5 = 250$.

3.2.6 Observer Design

Now we are ready to design observer (2.49) for MG3. Letting $\hat{x}^P = [\hat{R}^P, \hat{\phi}^P, \hat{\psi}^P]^\top$, $\hat{f} = f(\hat{x}^P, z, y) = [\hat{f}_1, \hat{f}_2, \hat{f}_3]^\top$ is given by

$$\begin{aligned}
\hat{f}_1 =& -\sigma(\hat{R}^P)^2 - \sigma\hat{R}^P(2\hat{\phi}^P + (\hat{\phi}^P)^2) \\
& - \frac{l_1/\rho + \beta^2 l_2/\rho^2(3\hat{\phi}^P + 3\hat{R}^P + 3/2(\hat{\phi}^P)^2) + \beta^2 l_3/\rho^3}{3(1+\hat{\phi}^P)}(\psi - \hat{\psi}^P) \\
\hat{f}_2 =& -\hat{\psi}^P - 3/2\,(\hat{\phi}^P)^2 - 1/2\,(\hat{\phi}^P)^3 - 3\hat{R}^P\hat{\phi}^P - 3\hat{R}^P + \beta^2 l_2/\rho^2(\psi - \hat{\psi}^P) \\
\hat{f}_3 =& -\frac{z_1 - \hat{\phi}^P}{\beta^2} + l_1/\rho(\psi - \hat{\psi}^P)
\end{aligned}$$

$\hat{f}$ together with N_{y_e} and N_z in (3.46) and (3.47) respectively, and H_1 in (3.11), concludes the observer and the dynamic projection. The output feedback controller is given by $\hat{v} = \pi(\hat{x}^P, v)$ in (3.44).

3.2.7 Simulation Results

Here we represent the simulation results for the designed projected observer in the previous sections when applied to MG3 in the presence of disturbances. We choose $k_1 = 20.43$ and $k_2 = 4.43 \times 10^4$ to fulfill ISS inequalities, and $L = [6, 12, 18]$ so that the associated polynomial $s^3 + l_1 s^2 + l_2 s + l_3 = 0$ is Hurwitz. Notice that in all simulations $\hat{x}(0) = 0$.

As for the disturbances $\delta(t) = [\delta_1(t), \delta_2(t), \delta_3(t)]^\top$ we choose several cases. Figure 3.8 is the case when $\delta(t) = [0.001, 0.01sin(t), 0.002cos(t)]^\top$ affects the system. The simulation shows that the output feedback response follows the state feedback one. In Figure 3.9 the system is subjected to constant disturbances $\delta(t) = [0.003, 0.001, 0.002]^\top$ and the results are shown for two different values of ρ. It can be seen that by decreasing ρ, the convergence rate of the output feedback increases and the error between the output feedback and state feedback responses decreases. Figure 3.10 shows the system responses when $\delta(t) = [0.03e^{-t}, 0.05e^{-0.5t}, 0.02e^{-t}]^\top$, which is the case where $\|\delta(t)\| \to 0$ as $t \to \infty$.

It shows that both output feedback and state feedback responses approach to the origin as $t \to \infty$. In all three cases output feedback adds mild degradation of performance.

In Figure 3.11 phase curves of the system states and their estimates are represented when $\delta(t) = [0.001, 0.01, 0.002]^\top$ and $\rho = 0.1$. In this figure one can observe the effect of projection on the peaking phenomenon in the observer states, and it can be seen that the trajectories approach a neighborhood of the origin.

In the simulation for this example we encountered large peaks in $z_2 = \dot{u}$. This problem along with the difficulty of finding an appropriate projection set are the reasons that our initial condition set (3.48) is very conservative.

3.3 Concluding Remarks

In this chapter we applied the techniques we developed in Chapter 2 to the MG3 model and our simulation results confirmed our theoretical predictions in the previous chapter. The major difficulty of this technique is finding an appropriate projection set $\mathcal{C}$ in (y_e, z) coordinates when its dimension n is greater than 3, as we can see in the MG3 example. It appears that this particular problem needs further investigation which may lead to a systematic numerical technique in finding projection sets.

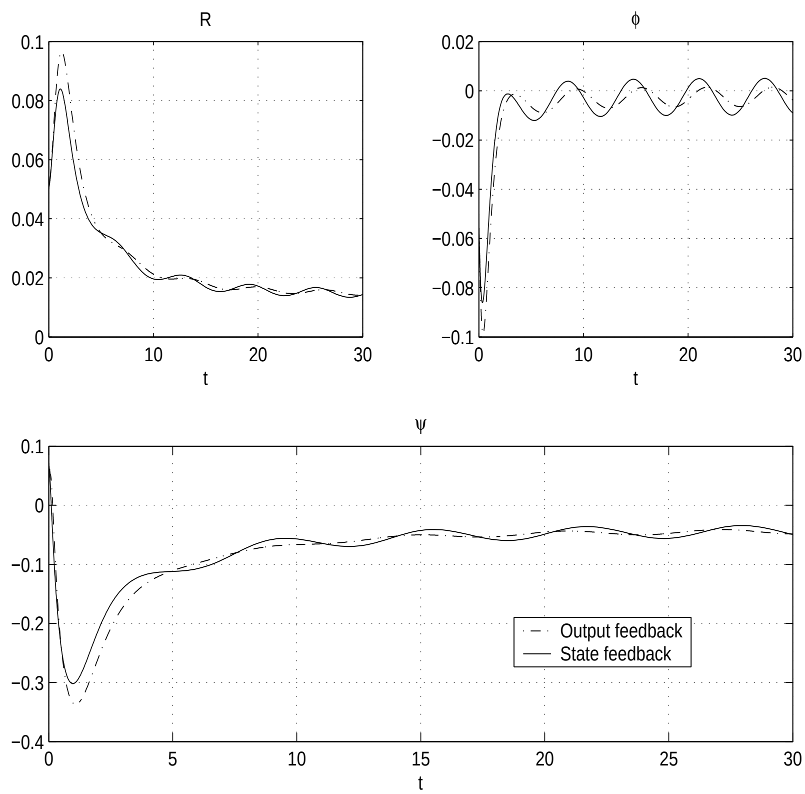

Figure 3.8: Sinusoidal disturbances ($\rho = 1/50$ in output feedback).

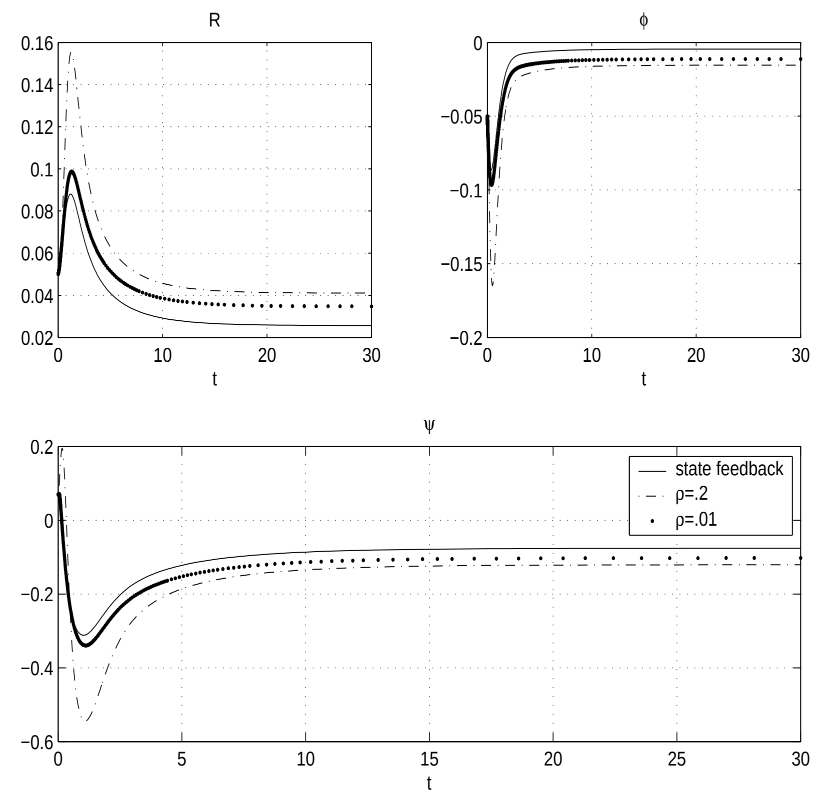

Figure 3.9: Constant disturbances with different values of ρ.

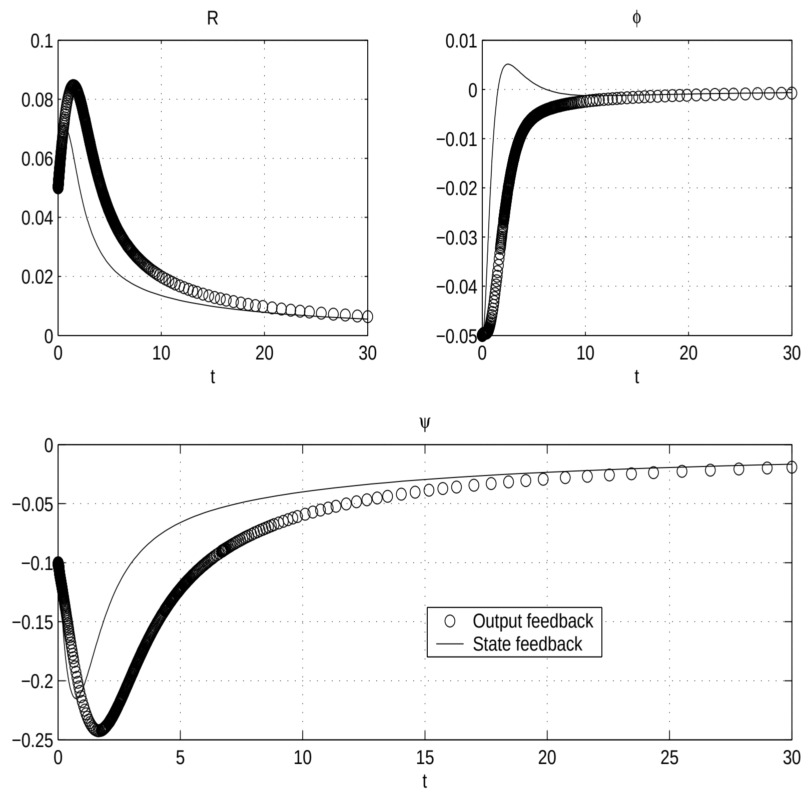

Figure 3.10: Decaying exponential disturbances ($\rho = \frac{1}{80}$ in output feedback).

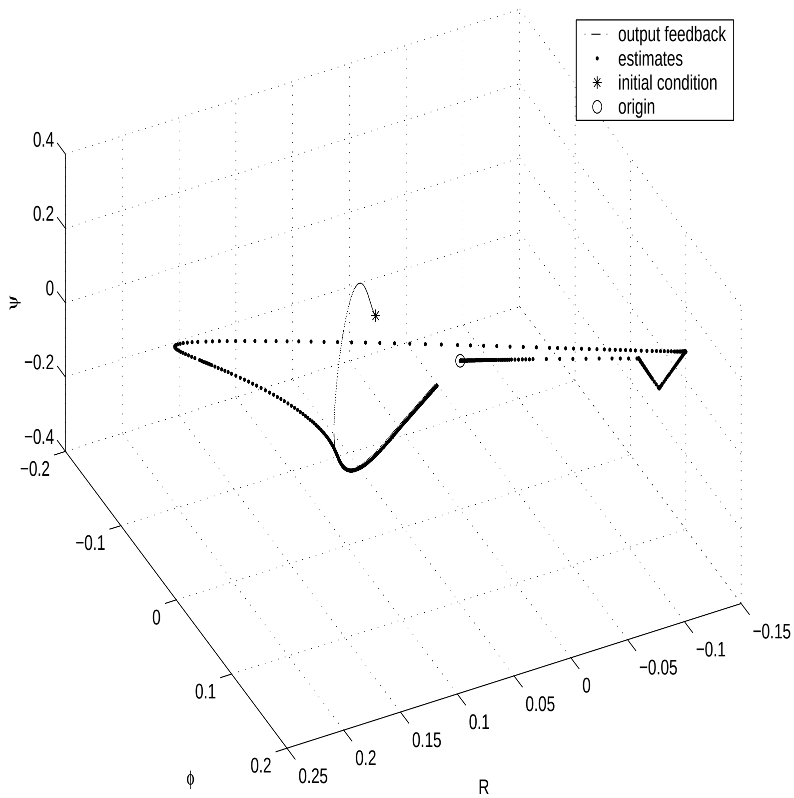

Figure 3.11: Peaking phenomenon.

Chapter 4

Observability of Systems with Uncertainty

In this chapter we apply the observer we developed in Chapter 2 to the MG3 model with one uncertain parameter and explain the theoretical problem that is encountered when uncertainties appear in systems in such a way that the affect the observability map. Then we produce a simple example to show that our observer generates desirable results when uncertainties do not affect the observability map.

4.1 MG3 with Uncertainty

In Chapter 3 we used an MG3 model without any uncertain parameters. Here we use the following model,

$$\begin{aligned}\dot{\Phi} &= -\Psi + \Psi_C(\Phi,\theta) - 3\Phi R \\ \dot{\Psi} &= \frac{1}{\beta^2}(\Phi - \Phi_T) \\ \dot{R} &= \sigma R(1 - \Phi^2 - R),\ \ R(0) \geq 0\end{aligned} \tag{4.1}$$

where the difference from the model (3.7) is the presence of the uncertain parameter θ, which can be considered as an uncertainty in the compressor model $\Psi_C(\Phi) = \Phi_{C0} + 1 +$

$\theta\Phi^2 - \frac{1}{2}\Phi^3$. Notice that the technique used in Chapter 3 copes with small uncertainties in the constant term $\Phi_{C0} + 1$. A more general case of uncertainties will be discussed in Remark 7. Note that here we do not consider time-varying disturbances.

4.2 Controller Design

Following the procedure we outlined in Chapter 3 we first find a controller for the extended system (system with two integrators at the input side). Shifting the origin to the desired equilibrium $R^e = 0, \Phi^e = 1, \Psi^e = \Psi_C(\Phi^e) = \Psi_{C_0} + 2$ with the change of variables $\phi = \Phi - 1, \psi = \Psi - \Psi_{C_0} - 2$, we have

$$\begin{aligned}
\dot{R} &= -\sigma R^2 - \sigma R(2\phi + \phi^2) \triangleq f_1(x) \\
\dot{\phi} &= -\psi - \theta\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R = f_2(x, \theta) \\
\dot{\psi} &= -\frac{1}{\beta^2}(u - 1 - \phi).
\end{aligned} \tag{4.2}$$

where $x = (R, \phi, \psi)$, and Φ_T is considered as the control input u. Augmenting the system with two integrators we have

$$\dot{R} = f_1 \tag{4.3}$$

$$\dot{\phi} = -\psi + w_1\theta + B \tag{4.4}$$

$$\dot{\psi} = \frac{1}{\beta^2}(1 + \phi - \xi_1) \tag{4.5}$$

$$\dot{\xi}_1 = \xi_2 \tag{4.6}$$

$$\dot{\xi}_2 = v \tag{4.7}$$

where $B = \frac{1}{2}\phi^3 - 3R\phi - 3R$, $w_1 = \phi^2$. We design the controller in the following steps using the *tuning function approach* outlined in [12].

Step 1. First we consider 4.3-4.4. Using the change of variables

$$z_1 = \phi \tag{4.8}$$

$$z_2 = \psi - \alpha_1 \tag{4.9}$$

where $\alpha_1 = c_1\phi + w_1\hat{\theta}$, $c_1 > 0$, and $\hat{\theta}$ is an estimate of θ, we have

$$\dot{z}_1 = -z_2 - c_1 z_1 + w_1(\tilde{\theta}) + B$$

where $\tilde{\theta} = \theta - \hat{\theta}$. Using

$$V_1 = CR + \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \quad C > 0,\ \gamma > 0$$

as a Lyapunov candidate, letting $\tau_1 = z_1 w_1$, and noticing that $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$, we have that

$$\begin{aligned}\dot{V}_1 =& - C(\sigma R^2 - \sigma R(2\phi + \phi^2)) - c_1\phi^2 - \phi^4/4 - 3R\phi^2 - 3R\phi \\ &- z_1 z_2 + \tilde{\theta}(\tau_1 - \dot{\hat{\theta}}/\gamma).\end{aligned}$$

Using Young's inequality we have

$$\begin{aligned}\dot{V}_1 \leq& - C\sigma R^2 - c_1\phi^2 + 2C\sigma(R^2/4 + \phi^2) + 3R^2/4 + 3\phi^2 \\ &- z_1 z_2 + \tilde{\theta}(\tau_1 - \frac{1}{\gamma}\dot{\hat{\theta}}/\gamma) \\ \leq& - \left(\frac{C\sigma}{2} - \frac{3}{4}\right) R^2 - (c_1 - 2C\sigma - 3)\phi^2 \\ &- z_1 z_2 + \tilde{\theta}(\tau_1 - \dot{\hat{\theta}}/\gamma) \\ \leq& - \chi(R, \phi) - z_1 z_2 + \tilde{\theta}(\tau_1 - \dot{\hat{\theta}}/\gamma)\end{aligned}$$

where $\chi(R, \phi) = \left(\frac{C\sigma}{2} - \frac{3}{4}\right) R^2 + (c_1 - 2C\sigma - 3)\phi^2$. Choosing $c_1 > 2C\sigma + 3$ and $C\sigma > \frac{3}{2}$ we have that $\chi > 0$. We eliminate $z_1 z_2$ in the next step. Since ψ is not the control input, we postpone the choosing of the update law for $\hat{\theta}$ to the last step.

Step 2. Next we consider 4.3-4.5 and the change of variable

$$z_3 = \xi_1 - \alpha_2. \tag{4.10}$$

Then we have that

$$\dot{z}_2 = \frac{1}{\beta^2}(1 + \phi - z_3 - \alpha_2) - \dot{\alpha}_1.$$

Choosing $\alpha_2 = 1 + \phi - \alpha_2'/\beta^2$ we have

$$\dot{z}_2 = -z_3/\beta^2 + \alpha_2' - \dot{\alpha}_1.$$

After calculating $\dot{\alpha}_1$ we have

$$\begin{aligned}\dot{z}_2 =& - z_3/\beta^2 + \alpha_2' - \frac{\partial \alpha_1}{\partial \phi}(-\psi + B) + w_2\theta - w_1\dot{\hat{\theta}} \\ =& - z_3/\beta^2 + \alpha_2' - \frac{\partial \alpha_1}{\partial \phi}(-\psi + B) + w_2\tilde{\theta} + w_2\hat{\theta} - w_1\dot{\hat{\theta}}.\end{aligned}$$

where $w_2 = -\frac{\partial \alpha_1}{\partial \phi} w_1$. Using $V_2 = V_1 + \frac{1}{2}z_2^2$ as a Lyapunov candidate and letting $\tau_2 = z_2 w_2 + \tau_2$ we have that

$$\begin{aligned}\dot{V}_2 \leq& - \chi - z_1 z_2 + z_2\left(-z_3/\beta^2 + \alpha_2' - \frac{\partial \alpha_1}{\partial \phi}(-\psi + B) + w_2\hat{\theta} - w_1\dot{\hat{\theta}}\right) \\ &+ (\tau_2 - \dot{\hat{\theta}}/\gamma)\tilde{\theta}.\end{aligned}$$

Choosing α_2' as follows

$$\alpha_2' = z_1 + \frac{\partial \alpha_1}{\partial \phi}(-\psi + B) - w_2\hat{\theta} + \frac{\partial \alpha_1}{\partial \hat{\theta}}\gamma\tau_2 - c_2 z_2$$

where $c_2 > 0$, we have that

$$\dot{V}_2 \leq -\chi - c_2 z_2^2 - \frac{1}{\beta^2} z_2 z_3 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}(\gamma\tau_2 - \dot{\hat{\theta}}) + \tilde{\theta}(\tau_2 - \dot{\hat{\theta}})$$

and

$$\dot{z}_2 = -z_3/\beta^2 + z_1 + w_2\tilde{\theta} + \frac{\partial \alpha}{\partial \hat{\theta}}(\gamma\tau_2 - \dot{\hat{\theta}}).$$

Step 3. In this step we choose

$$z_4 = \xi_2 - \alpha_2. \tag{4.11}$$

Therefore we have

$$\begin{aligned}\dot{z}_3 =& z_4 + \alpha_3 - \dot{\alpha}_2 \\ =& z_4 + \alpha_3 + F + w_3\tilde{\theta} + w_3\hat{\theta} - \frac{\partial \alpha_2}{\partial \theta}\dot{\hat{\theta}}\end{aligned}$$

where $F = -\frac{\partial \alpha_2}{\partial R} f_1 - \frac{\partial \alpha_2}{\partial \phi}(-\psi + B) - \frac{\partial \alpha_2}{\partial \psi}(1 + \phi - \xi_1)/\beta^2$ and $w_3 = -\frac{\partial \alpha_2}{\partial \psi} w_1$. Choosing $V_3 = V_2 + \frac{1}{2} z_3^2$ as Lyapunov candidate and $\alpha_3 = -c_3 z_3 - F - w_3\hat{\theta} + \frac{\partial \alpha_2}{\partial \hat{\theta}}\gamma\tau_3 + z_2/\beta^2 + \nu_3$ we have

$$\dot{V}_3 \leq -\chi - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + \tilde{\theta}(\tau_3 - \dot{\hat{\theta}}) \left(z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}} \right) (\gamma\tau_3 - \dot{\hat{\theta}})$$

where $\tau_3 = \tau_2 + z_3 w_3,$ and we set $\nu_3 = \gamma z_2 w_3 \frac{\partial \alpha_1}{\partial \hat{\theta}}$. Here we used that fact that $\dot{\hat{\theta}} - \gamma\tau_2 = \dot{\hat{\theta}} - \gamma\tau_3 + \gamma z_3 w_3$. The time derivative of z_3 has the following form

$$\dot{z}_3 = z_4 - c_3 z_3 + w_3\tilde{\theta} + \frac{\partial \alpha_2}{\partial \hat{\theta}}(\gamma\tau_3 - \dot{\hat{\theta}}) + z_2/\beta^2 + \nu_3.$$

Step 4. In the last step of this recursive procedure we determine a control law for v. From (4.11) we have that

$$\begin{aligned}\dot{z}_4 =& v - \dot{\alpha}_3 \\ =& v - G + w_4\tilde{\theta} + w_4\hat{\theta} - \frac{\partial \alpha_3}{\partial \hat{\theta}}\dot{\hat{\theta}}\end{aligned}$$

where $G = \frac{\partial \alpha_3}{\partial R} f_1 - \frac{\partial \alpha_3}{\partial \phi}(-\psi + B) - \frac{\partial \alpha_3}{\partial \psi}(1 + \phi - \xi_1)/\beta^2 - \frac{\partial \alpha_3}{\partial \xi_1}\xi_2$ and $w_4 = \frac{\partial \alpha_3}{\partial \phi} w_1$. Using $V_4 = V_3 + \frac{1}{2} z_4^2$ as the Lyapunov candidate and choosing

$$v = -G - w_4\hat{\theta} + \frac{\partial \alpha_3}{\partial \hat{\theta}}\gamma\tau_4 - z_3 - c_4 z_4 + \nu_4 \tag{4.12}$$

where $\tau_4 = \tau_3 + z_4 w_4$, we have

$$\dot{V}_4 \leq -\chi_4 - c_2 z_2^2 - c_3 z_3^2 - c_4 z_4^2$$

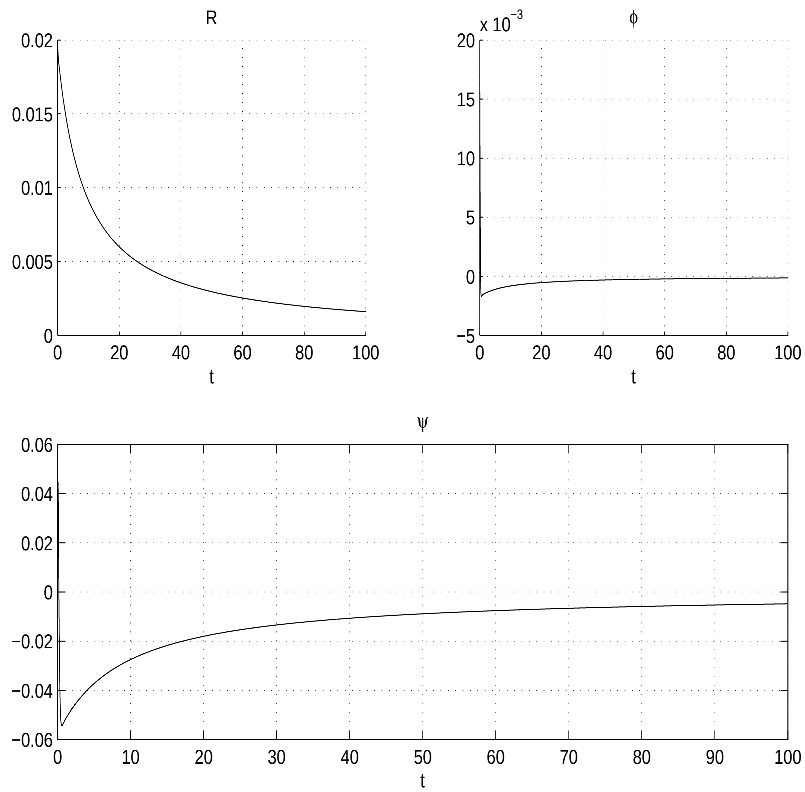

Figure 4.1: MG3 with adaptive controller.

where we use the fact that $\gamma(\tau_3 - \tau_4) = -\gamma z_4 w_4$ and the following

$$\begin{aligned}\nu_4 &= \Pi\gamma w_4 \\ \Pi &= z_2\frac{\partial\alpha_1}{\partial\hat{\theta}} + z_3\frac{\partial\alpha_2}{\partial\hat{\theta}} \\ \dot{\hat{\theta}} &= \gamma\tau_4.\end{aligned} \tag{4.13}$$

We use (4.13) as the update law for the uncertain parameter θ.

Figure 4.1 represents (R, ϕ, ψ) when controller (4.12) is applied which shows that the system state tend to zero as $t \to \infty$.

One drawback of tuning function method is that as the number of integrators increases, the complexity of the controller grows. Figure 4.1 shows R and ψ vary more slowly comparing with ϕ. This is due to the function α_1 in the first step of the controller design.

We can improve this problem by choosing a more complicated function α_1 at the first step, but at the expense of a more complicated final controller v. Due to this complexity, as it can be seen in the simulation results, we were forced to use small initial conditions.

Remark 7: When there is more than one uncertain parameter in equation (4.4), a more general controller can be designed using the same step-by-step procedure. Suppose (4.4) is written as

$$\dot{\phi} = f_2 + {w_1}^{\top}\theta \tag{4.14}$$

where θ is a vector of unknown parameters. To solve this problem we can start with $V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^{\top}\Gamma^{-1}\tilde{\theta}$, where Γ is the adaptation gain matrix and is positive definite. At the end the update law would be $\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta})$.

As another case which is more general, the authors in [24] introduce structured parametric uncertainty to the system (3.7) of the form Δ_i as follows

$$\begin{aligned}
\dot{R} &= f_1 + \Delta_1(R, \phi)\\
\dot{\phi} &= f_2 + \Delta_2(R, \phi)\\
\dot{\psi} &= \frac{1}{\beta^2}(\Phi - \Phi_T),
\end{aligned}$$

but their controller is full state feedback. In other words they assume that all the states can be measured.

4.3 Observability Map

Consider system (4.2) as follows

$$\begin{aligned}
\dot{R} &= f_1(x)\\
\dot{\phi} &= f_2(x, \theta)\\
\dot{\psi} &= k(u - 1 - \phi).
\end{aligned} \tag{4.15}$$

where $x = (R, \phi, \psi)$, and $k = -\frac{1}{\beta^2}$. Having $y = \psi = x_3$ as the output, the observability map for this system is

$$y_e = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} x_3 \\ k(1 + x_2) - ku \\ kf_2(x, \theta) - k\dot{u} \end{bmatrix} \triangleq H(x, z, \theta). \tag{4.16}$$

Here we have a major theoretical problem which is the dependence of H on θ, an unknown parameter. This problem makes the inverse H^{-1} unknown, which in our observer (2.12) in Chapter 2 translates to an undefined $\left[\frac{\partial H}{\partial x}\right]^{-1}$. Our attempt to apply the projected observer to this example has failed so far, and it seems that this problem needs a substantial investigation which is beyond the scope of this thesis.

But the projected observer in Chapter 2 can be applied to a class of systems with uncertainty which will be discussed in the next section.

4.4 Uncertainty and Observability Map

Consider the following system

$$\begin{aligned} \dot{x} &= f(x, u, \theta) \\ y &= h(x, u) \end{aligned} \tag{4.17}$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^p$ is a vector of unknown parameters, $u, y \in \mathbb{R}$, f and h are known smooth functions and $f(0, 0) = 0$. Applying the map (2.2) and using the notation in Chapter 2 we have

$$y_e = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(x, u) \\ \phi_1(x, z, \bar{\theta}) \\ \vdots \\ \phi_{n-1}(x, z, \bar{\theta}) \end{bmatrix} \tag{4.18}$$

where $z = (u, \dot{u}, \ddot{u}, \ldots, u^{(n_u-1)})$, $\bar{\theta} = (\theta, \dot{\theta}, \ldots, \theta^{(n_\theta)})$, $0 \leq n_u \leq n$ and $0 \leq n_\theta \leq n-1$ are integers. The appearance of the unknown term θ in H prevents us from applying the observer that we developed in Chapter 2. Therefore we need some assumption to overcome this problem.

Assumption A4(Observability and Uncertainty): Assume that the plant has the following property

$$\frac{\partial \phi_i}{\partial \theta} = 0, \text{ for } i = 0, \ldots, n-1. \tag{4.19}$$

Therefore (4.18) can be written as $y_e = H(x, z)$.

One class of systems that has this property is

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ &\vdots \\ \dot{x}_i &= f_i(x_1, \ldots, x_{i+1}) \\ &\vdots \\ \dot{x}_n &= f_n(x, u, \theta) \\ y &= h(x_1). \end{aligned} \tag{4.20}$$

For a further discussion on this problem one can refer to [25].

The next two examples are systems that have the structure of the form (4.20). The first example is a system that is observable everywhere while the second one does not have this property.

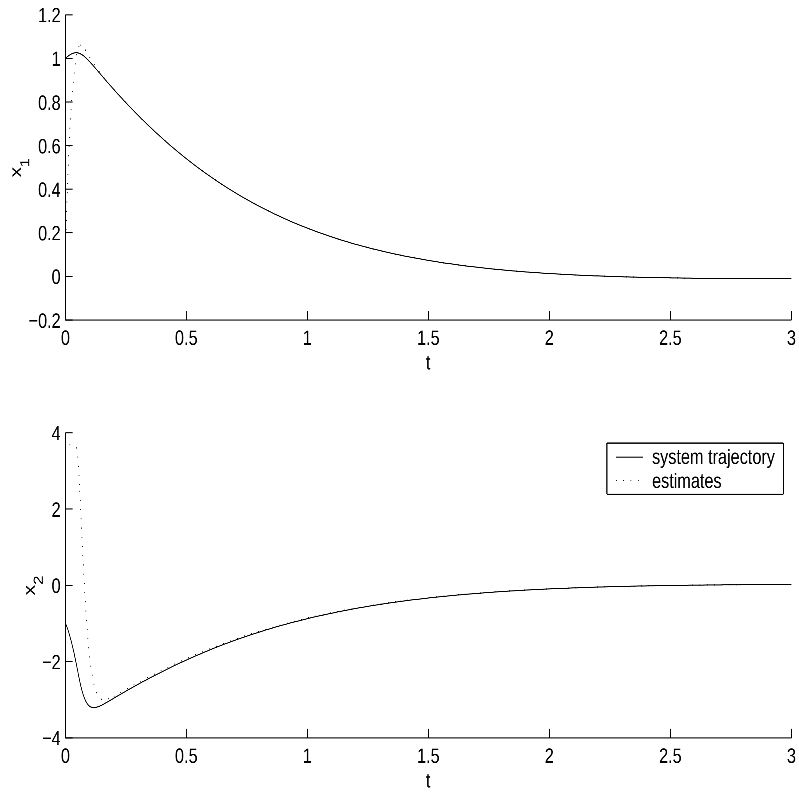

Figure 4.2: x and its estimate, Example 4.4.1.

Example 4.4.1 Consider the system $\dot{x} = f(x, \theta)$ as follows

$$\begin{aligned} \dot{x}_1 &= 2x_1 + x_2 \\ \dot{x}_2 &= u + w\theta \\ y &= x_1 \end{aligned} \tag{4.21}$$

where $w = x_1^2 + .5x_2$ and θ is an unknown parameter. This system has two interesting features. First, θ dose not appear in the observability map as we will show shortly. Second, the control input u does not appear in the observability map and we do not need to use integrators at the input side.

Using adaptive backstepping as in Section 4.2 we find the following state feedback con-

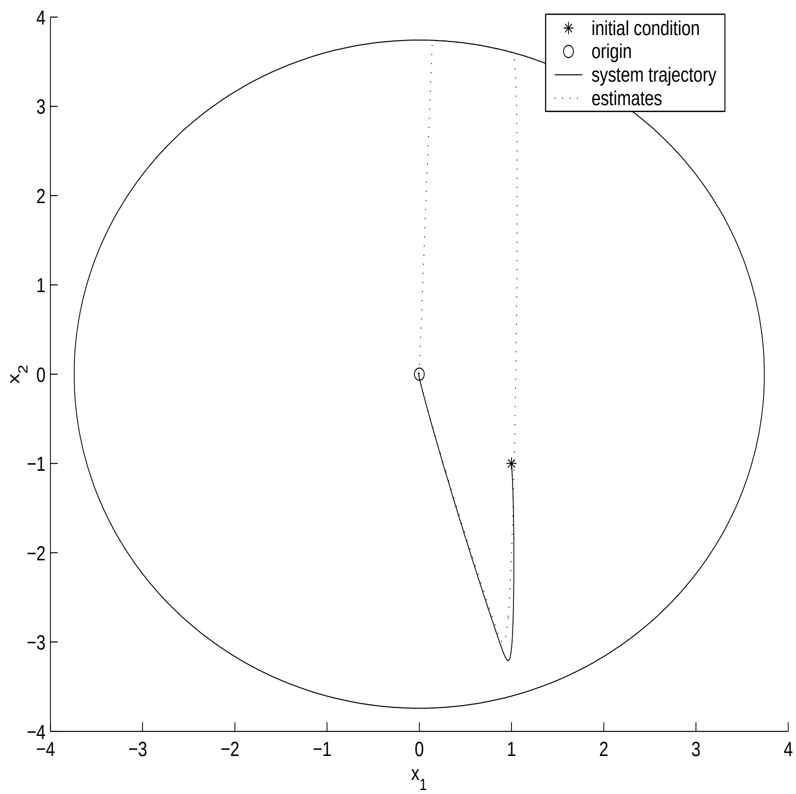

Figure 4.3: Projection effect, Example 4.4.1.

troller

$$u = -z_1 - c_2 z_2 + \dot{\alpha}_1 - w\hat{\theta} \tag{4.22}$$
$$z_2 = x_2 - \alpha_1$$
$$z_1 = x_1$$
$$\alpha_1 = -c_1 z_1$$

where $c_1 > 2$, $c_2 > 0$ and

$$\dot{\hat{\theta}} = \gamma z_2 w \tag{4.23}$$

where $\gamma > 0$ is the update law for the unknown parameter. Following the procedure to design an observer we have (notice that, due to the absence of disturbance, $H = H_1$)

$$y_e = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = H(x) \tag{4.24}$$
$$\frac{\partial H}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$L = [l_1 \; l_2]^\top = [2 \; 1]^\top$$
$$\mathcal{E} = \begin{bmatrix} \rho & 0 \\ 0 & \rho^2 \end{bmatrix}$$
$$\hat{f}(\hat{x}, u, y) = \begin{bmatrix} \hat{x}_1 \\ u(\hat{x}) + w(\hat{x}) + \theta_0 \end{bmatrix} + \mathcal{E}^{-1} L (y - \hat{x}_1)$$

where θ_0 is the nominal value of θ. As for the projection surface $\mathcal{C}$ we use the surface of a circle of radius $\sqrt{14}$ which is $x_1^2 + x_2^2 - 14 = 0$. Next we have to define the projection elements as follows

$$N_{y_e} = [\hat{x}_1^P \; \hat{x}_2^P]^\top \;, \; N_z = 0$$
$$\frac{\partial H}{\partial z} = 0. \tag{4.25}$$

The simulation results are depicted in Figures 4.2-4.3. In this simulation we have chosen $\rho = 0.02$, $\theta_0 = 1$ and $\theta = 3$. In Figure 4.3 one can see the effect of projection on circle surface for $\hat{x}$. Notice that the $\hat{x}(0) = 0$.

△△

Example 4.4.2 Consider the system $\dot{x} = f(x, \theta)$ as follows

$$\begin{aligned} \dot{x}_1 &= x_1^2 + 2x_2 \\ \dot{x}_2 &= u + w\theta \qquad (4.26) \\ y &= (x_1 - 1)^2 \end{aligned}$$

where $w = x_2^2 e^{x_1}$ and θ is an unknown parameter.

First we design a state feedback controller. Using adaptive backstepping and considering $V = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}$ as a Lyapunov candidate, we find the following state feedback controller

$$\begin{aligned} u &= -2z_1 - c_2 z_2 + \dot{\alpha}_1 - w\hat{\theta} \qquad (4.27) \\ z_2 &= x_2 - \alpha_1 \\ z_1 &= x_1 \\ \alpha_1 &= -c_1 z_1 - x_1^2 \end{aligned}$$

where $c_1 > 0$, $c_2 > 0$ and

$$\dot{\hat{\theta}} = \gamma z_2 w \qquad (4.28)$$

with $\gamma > 0$ is the update law for the unknown parameter. Using this controller we have

$$\dot{V} \leq -c_1 z_1^2 - c_2 z_2^2.$$

Following the procedure to design an observer we have (notice that, due to the absence of disturbance, $H = H_1$)

$$y_e = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} (x_1 - 1)^2 \\ 2(x_1^2 + 2x_2)(x_1 - 1) \end{bmatrix} = H(x) \tag{4.29}$$

$$\frac{\partial H}{\partial x} = \begin{bmatrix} 2(x_1 - 1) & 0 \\ 4(x_1^2 - x_1) + 2(x_1^2 + 2x_2) & 4(x_1 - 1) \end{bmatrix}. \tag{4.30}$$

Notice that $n_u = 0$ and H is invertible for $x_1 < 1$. Therefore we have to choose the set $\mathcal{C}$ in y_e coordinates such that it contains $H(0,0)$ and $y_{e,1} > 1$. For that purpose we choose $\mathcal{C}$ to be the following set

$$\mathcal{C} = \left\{(y_e \in \mathbb{R}^2 | (y_{e,1} - 1)^2/.9025 + y_{e,2}^2/4 \leq 1\right\}$$

which is an ellipse. Next we have to define the observer and the projection elements as follows

$$L = [l_1 \; l_2]^\top = [2 \; 1]^\top$$

$$\mathcal{E} = \begin{bmatrix} \rho & 0 \\ 0 & \rho^2 \end{bmatrix}$$

$$\hat{f}(\hat{x}, u, y) = \begin{bmatrix} \hat{x}_1 \\ u(\hat{x}) + w(\hat{x}) + \theta_0 \end{bmatrix} + \left[\frac{\partial H}{\partial x}\right]^{-1} \mathcal{E}^{-1} L(y - (\hat{x}_1 - 1)^2)$$

$$N_{y_e} = y_e + [-1/.9025 \; 1/4]^\top \;, \; N_z = 0$$

$$\frac{\partial H}{\partial z} = 0.$$

The simulation results are depicted in Figures 4.4, 4.5 and 4.6. In this simulation we have chosen $\rho = 0.02$, $\theta_0 = 1$ and $\theta = 3$. Notice that the $\hat{x}(0) = 0$. Figure 4.5 shows the effect of the projection on $\hat{x}$ in x coordinates, while Figure 4.6 shows the effect of the projection on the estimates when the hit the ellipse surface $(y_{e,1} - 1)^2/.9025 + y_{e,2}^2/4 = 1$ in y_e coordinates.

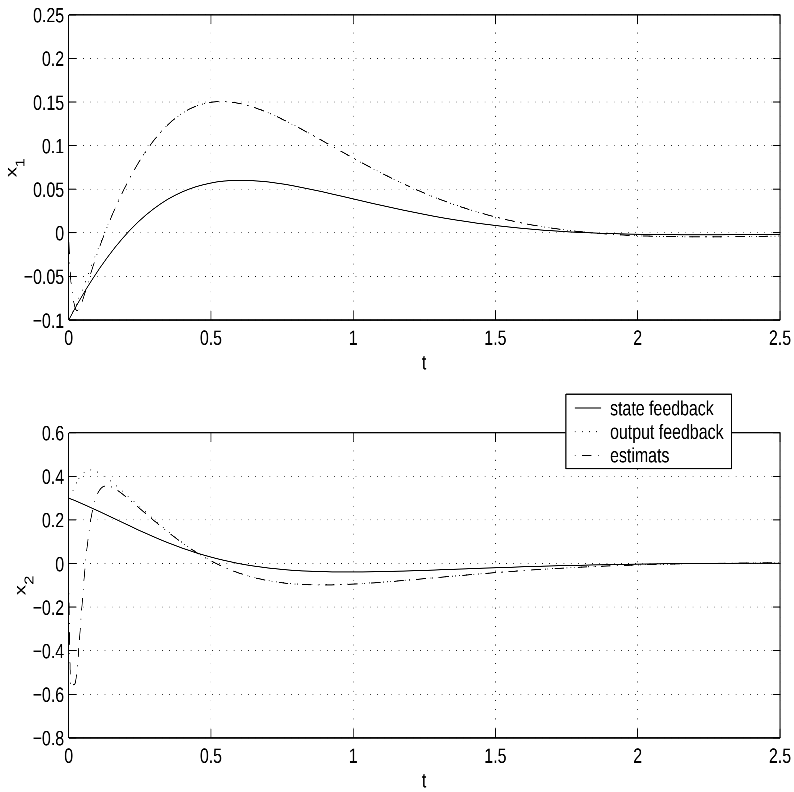

Figure 4.4: x and its estimate, Example 4.4.2.

Figures 4.7 and 4.8 represent the simulation results when disturbance $\delta(t) = [0.4sin(t) \;\; 0.3cos(t)]^{\top}$ is added to the system in the form of $\dot{x} = f(x, \theta) + \delta(t)$. This is an example when there are disturbances and uncertainty in system, in other words a combination of the results of Chapter 2 and this chapter. Figure 4.8 shows that, because of the existence of disturbance, the trajectories enter a neighborhood of the origin, which is an example of the ultimate boundedness result in Chapter 2.

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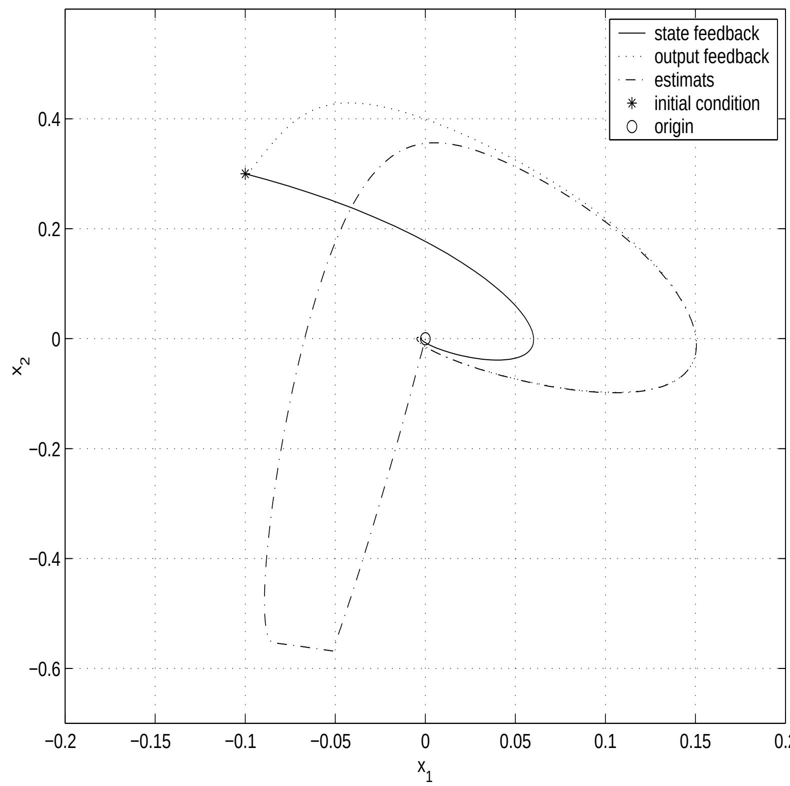

Figure 4.5: Projection effect in x coordinates, Example 4.4.2.

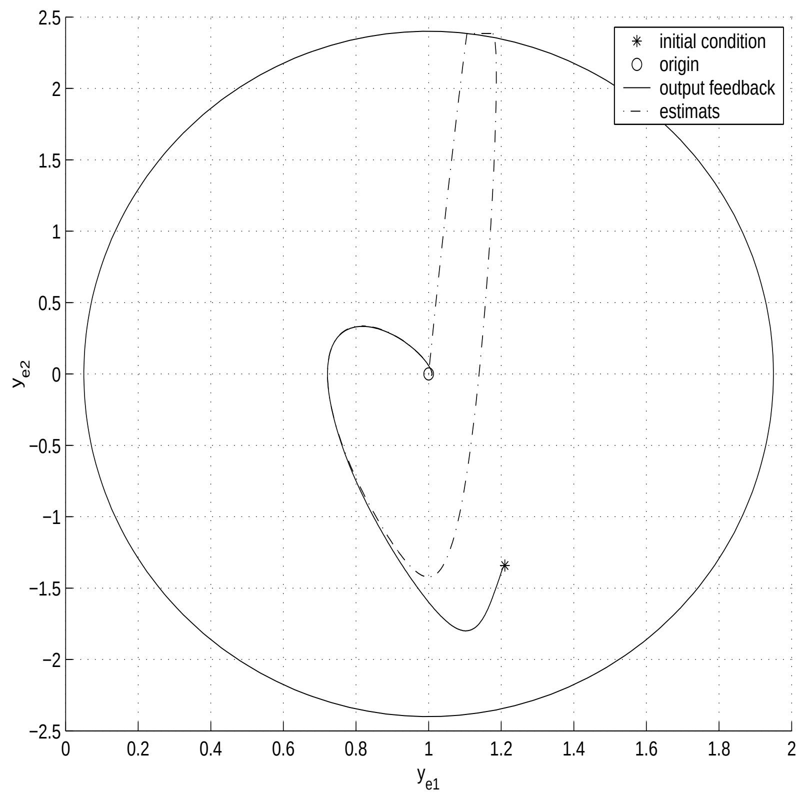

Figure 4.6: Projection effect in y_e coordinates, Example 4.4.2.

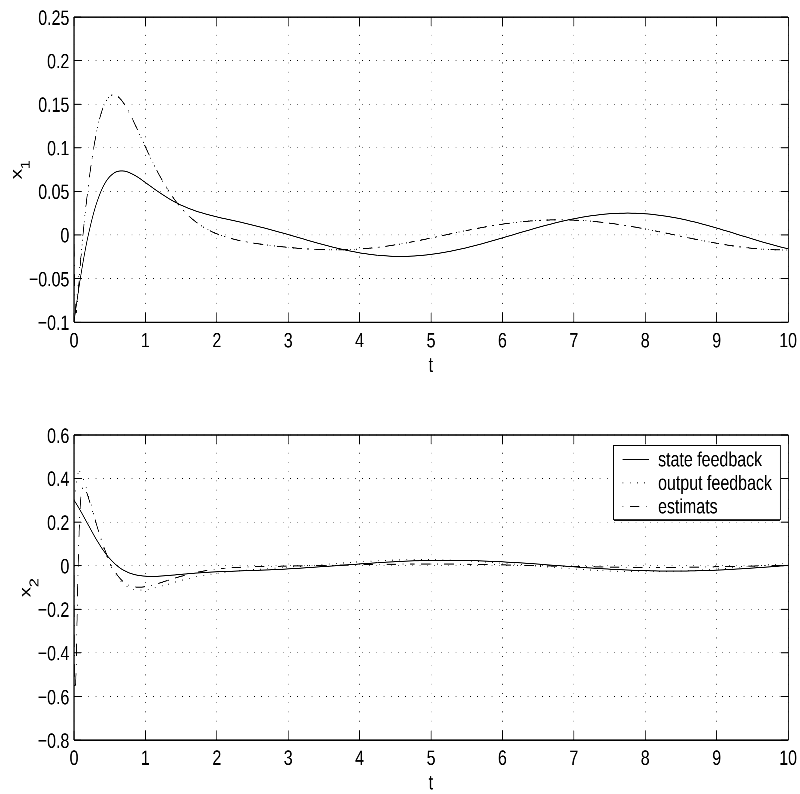

Figure 4.7: Disturbance and uncertainty, Example 4.4.2.

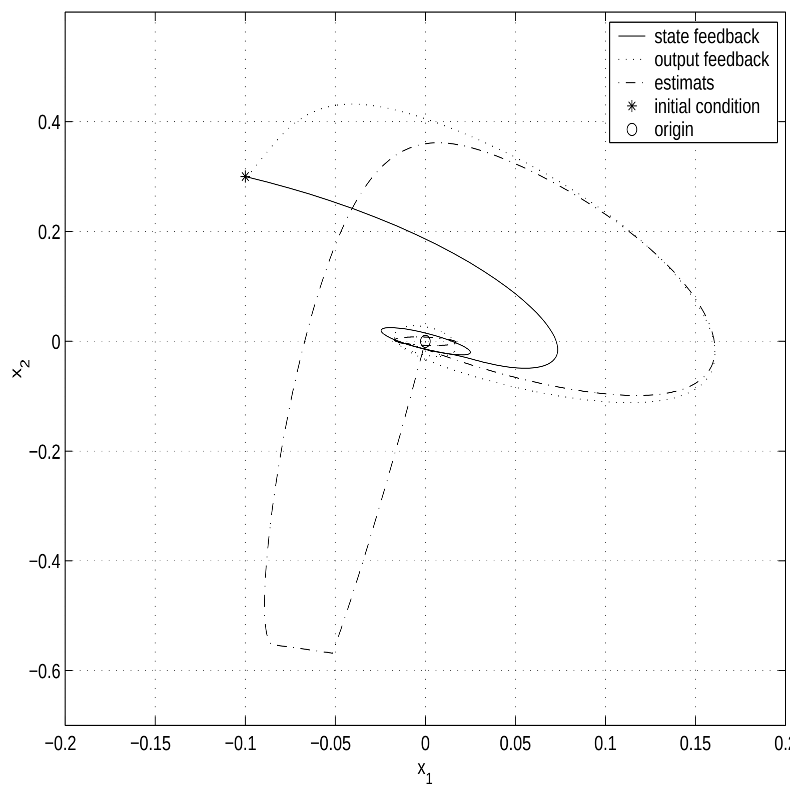

Figure 4.8: Disturbance and uncertainty, projection effect, Example 4.4.2.

Chapter 5

Conclusion

Our approach to the output feedback control of systems affected by disturbances in Chapter 2 allows us to use available state feedback control design techniques. However, in practice, during the design step one may encounter some limitations and difficulties as we did in Chapters 3 and 4.

First, our method requires adding integrators at the input side. This complicates the state feedback design and generally leads to a complex expression for the state feedback function in the last integrator. Second, it appears that finding an appropriate set $\mathcal{C}$ for the projection may be a difficult task and sometimes it may yields a very conservative set which limits our control effort and/or our ability to use the current simulation tools such as MATLAB. For example, due to our choice of set $\mathcal{C}$ in Chapter 3 for MG3, our choice of initial conditions for simulation is limited to small values. Third, in Chapter 3, our technique relies on the perfect knowledge of the compressor characteristic which is not a realistic assumption. Our attempts to add some limited kind of uncertainties in the compressor characteristic hasve failed as we discussed in Chapter 4. Additionally, we just solved the problem for a limited kind of disturbances with a conservative assumption.

Some points that we mentioned throughout this thesis and the limitations that we just mentioned set some possible directions for future research such as, finding a Lyapunov or non-Lyapunov based proof for the Conjecture in Chapter 2, a possible proof for the combination of the results in Chapters 2 and 4, modifying the observer to work with systems with a broader range of uncertainties and disturbances, developing numerical methods to automate the design of $\mathcal{C}$, and developing some estimation methods other than backstepping for the control input. Additionally, it would be interesting to find a possibly modified version of this method for systems that are unobservable at some individual points or even at the origin.

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