Robust Output Feedback Tracking With a Matching Condition

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Abstract

We study the tracking problem in the presence of smooth, bounded uncertainty and find sufficient conditions so that, if the uncertainty satisfies a suitable matching condition, one can design a partial information controller (i.e., an output feedback controller) achieving arbitrarily small steady-state tracking error.

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1 Introduction

In [1] and [2] the notion of a practical internal model was introduced as a paradigm to solve the output feedback (or partial information) tracking problem for nonlinear systems. The word *practical internal model* was chosen to indicate the fact that this paradigm allows to solve the tracking problem *practically* (i.e., to an arbitrary degree of accuracy), rather than asymptotically, and that its solution relies on the existence of a compensator (the practical internal model) which has a conceptually similar role to a nonlinear internal model in output regulation theory (see, e.g., [3] for an introduction to the output regulation problem and the definition of nonlinear internal model). In [2] it was also showed that, when the tracking problem is posed within an output regulation framework with appropriate restrictions, the practical internal model can be replaced by an internal model and the paradigm can still be employed. As pointed out in [1] and [2], this theory is still far from being self-contained and leaves several open questions. One of them is the extension of the results in [1, 2] to the case when the system is affected by disturbances. The present paper represents a first step in this direction.

Assuming that the disturbances satisfy a matching condition, we derive a set of sufficient conditions on the existence of a dynamic extension leading to the solution of the practical tracking problem using certainty equivalence. We show that, under suitable conditions, two classes of compensators represent feasible dynamic extensions, namely chains of integrators and linearizing compensators, and for these we provide a constructive procedure to design feedback control laws.

Throughout this paper we use $\mathrm{col}(a, b)$ to indicate the vector $[a^\top, b^\top]^\top$. If v is a n-dimensional vector, v_i, $i = 1, \ldots, n$, are its components. Given real numbers a, b, c, $diag[a, b, c]$ denotes the matrix with a, b, c on the diagonal and zeros elsewhere. Given matrices A, B, C, we denote by $block\text{-}diag[A, B, C]$ the matrix formed by placing A, B, C on the diagonal and zeros elsewhere.

To illustrate the main ideas of this paper, we will resort to a simple example.

Example 1 Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + u_1 + \Delta_1(t) \\ \dot{x}_3 &= x_4 - x_1^2 - u_1 - \Delta_1(t) \\ \dot{x}_4 &= u_2 + \Delta_2(t) \\ y &= \mathrm{col}(x_1, x_3), \end{aligned} \tag{1}$$

where $\Delta(t) = \mathrm{col}(\Delta_1(t), \Delta_2(t))$ is an unknown smooth function of time which is bounded with bounded time derivatives, u is the control input, and y is the measurable output (x is not available for feedback). Given a smooth reference trajectory $r(t) = \mathrm{col}(r_1(t), r_2(t))$, we seek to find a *partial information controller* (i.e., an output feedback controller) using only the information given by y and r to make $y(t)$ track $r(t)$. We begin by noticing that Δ satisfies a matching condition in that letting

$$\tilde{u} = u + \Delta,$$

the plant can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + \tilde{u}_1 \\ \dot{x}_3 &= x_4 - x_1^2 - \tilde{u}_1 \\ \dot{x}_4 &= \tilde{u}_2 \\ y &= \mathrm{col}(x_1, x_3), \end{aligned} \tag{2}$$

which is a system free of disturbance where, however, $\tilde{u}$ can not be freely assigned because Δ is not known. Given a smooth reference trajectory $r(t)$, the smooth functions of time

$$x^r(t) = \mathrm{col}\left(r_1(t), \dot{r}_1(t), r_2(t), \ddot{r}_1(t) + \dot{r}_2(t)\right), \quad u^r(t) = \mathrm{col}\left(\ddot{r}_1(t) - r_1^2(t), \ddot{r}_2(t) + \dddot{r}_1(t)\right)$$

are feasible state and input trajectories for (2) since

$$\begin{aligned} \dot{x}_1^r &= x_2^r \\ \dot{x}_2^r &= (x_1^r)^2 + u_1^r \\ \dot{x}_3^r &= x_4^r - (x_1^r)^2 - u_1^r \\ \dot{x}_4 &= u_2^r. \end{aligned} \tag{3}$$

Further, it is readily seen that the output of (3) is precisely r. Thus the problem of tracking can be converted to one of stabilization by setting $\tilde{x} = x - x^r$ and computing the associated error dynamics. By doing so, one finds that the controller

$$\tilde{u} = \bar{\tilde{u}}(x, x^r, u^r) = \mathrm{col}\left(u_1^r + (x_1^r)^2 - x_1^2 + K_1{}^\top(x - x^r), u_2^r + K_2{}^\top(x - x^r)\right) \tag{4}$$

globally uniformly asymptotically stabilizes the equilibrium $\tilde{x} = 0$ of the error dynamics and hence solves the tracking problem globally. This solution, however, presents some problems. Firstly, the feedback controller $\bar{\tilde{u}}$ is not implementable because Δ is unknown. Secondly, the pair (x^r, u^r), called the stable inverse of (2) (see [4]), may in general be difficult or even impossible to exactly compute and therefore it would be desirable to develop a solution that does not rely on its knowledge. Finally, $\bar{\tilde{u}}$ depends on x which is not available for feedback.

View (3) as a copy of the disturbance-free plant (2) with unknown state x^r, unknown input u^r, known output r, and augment it with the following compensator

$$\begin{aligned}\dot{\zeta}_1^r &= \zeta_2^r + \zeta_3^r \\ \dot{\zeta}_2^r &= v_1^r \\ \dot{\zeta}_3^r &= v_2^r \\ u^r &= \mathrm{col}(\zeta_1^r, \zeta_2^r).\end{aligned} \tag{5}$$

Define $X_1 \triangleq \mathrm{col}(x^r, \zeta^r)$ and form the observability mapping of the augmented system

$$\begin{aligned} y_{X_1} &\triangleq \mathrm{col}(r_1, \dot{r}_1, \ddot{r}_1, \dddot{r}_1, r_2, \dot{r}_2, \ddot{r}_2) \\ &= \left(x_1^r,\, x_2^r,\, (x_1^r)^2 + \zeta_1^r,\, 2x_1^r x_2^r + \zeta_2^r + \zeta_3^r,\, x_3^r,\, x_4^r - \zeta_1^r - (x_1^r)^2,\, -\zeta_3^r - 2x_1^r x_2^r\right) \triangleq \mathcal{H}_X(X_1). \end{aligned}$$

Since $\mathcal{H}_X(X_1)$ is everywhere smooth and bijective (a diffeomorphism), from y_{X_1} one can calculate $X_1 = \mathrm{col}(x^r, \zeta^r) = \mathcal{H}_X^{-1}(y_{X_1})$ from which one gets the stable inverse $(x^r, u^r) = (x^r, (\zeta_1^r, \zeta_2^r))$. In conclusion, through the compensator (5), which we call a *practical internal model*, one can formulate the problem of calculating the stable inverse (x^r, u^r) as that of estimating some time derivatives of r (the vector y_{X_1}) and then inverting the mapping $\mathcal{H}_X(X_1)$. Notice that the practical internal model is not *directly* implemented, as it is only used for estimation purposes.

We now turn our attention to the disturbance-free plant (2) and augment it with a compensator with identical structure to (5)

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 + \zeta_3 \\ \dot{\zeta}_2 &= v_1 \\ \dot{\zeta}_3 &= v_2 \\ \tilde{u} &= \mathrm{col}(\zeta_1, \zeta_2).\end{aligned} \tag{6}$$

Define $X_2 \triangleq \mathrm{col}(x, \zeta)$ and note that since the augmented system (2), (6) has the same structure as (3), (5), its observability mapping is given by

$$y_{X_2} \triangleq \mathrm{col}(y_1, \dot{y}_1, \ddot{y}_1, \dddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2) = \mathcal{H}_X(X_2).$$

Since $\mathcal{H}_X(X_2)$ is a diffeomorphism, we conclude that from y and its time derivatives one gets $X_2 = \mathrm{col}(x, \zeta) = \mathcal{H}_X^{-1}(y_{X_2})$ and hence also $\tilde{u} = \mathrm{col}(\zeta_1, \zeta_2)$. Recalling that $\tilde{u} = u + \Delta$, estimating $\tilde{u}$ is equivalent to estimating Δ as

$$\Delta = \mathrm{col}(\zeta_1, \zeta_2) - u.$$

Summarizing our observations so far, using two practical internal models and estimating the time derivatives of r (the vector y_{X_1}) and y (the vector y_{X_2}), one can estimate the stable inverse of the system, the state of the plant, and the disturbance. Unfortunately, however, such estimates cannot be employed in the feedback controller (4) because the vector relative degree of the disturbance-free system (2) is $\{2, 1\}$ while the number of time derivatives of its output that need to be estimated is $\{3, 2\}$, and thus the resulting closed-loop system would not be proper. To address this problem we employ an input dynamic extension with the property that the relative degree of the extended system[1] is equal to the relative degree of the system augmented with a practical internal model (in this example, $\{3, 2\}$). Such an input dynamic extension cannot always be found, however we show that under suitable conditions it can take the form of chains of integrators or linearizing compensators.

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2 Problem Statement and Assumptions

Given the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u, \Delta(t)) \\ y &= h(x), \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurable output, and $\Delta(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is an unknown smooth function of its arguments which is bounded with bounded derivatives, we seek to find a tracking controller solving the following problem

Problem 1 (Output Feedback Practical Tracking): *Given the dynamical system (7) and a sufficiently smooth reference trajectory $r(t) = col(r_1(t), \ldots, r_m(t))$, design a dynamic output feedback controller*

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r) \\ u &= h_c(x_c, y) \end{aligned} \tag{8}$$

where f_c and h_c are sufficiently smooth, such that the closed-loop system (7)-(8) has the property that there exists a $T > 0$ such that $\|e(t)\| \le e_0$ for all $t \ge T$, and such that the internal states x and x_c are bounded for all $t \ge 0$, and for all initial conditions $(x(0), x_c(0)) \in \mathcal{A}$, for some closed set $\mathcal{A}$.

In [1], we have showed that, when no uncertainty affects the system, if there exists a practical internal model then Problem 1 has a solution. We start by assuming that the uncertainty $\Delta(t)$

[1]Note that in the sequel we do not need the relative degree to be well-defined.

satisfies a matching condition.

Assumption A1 (Matching Condition): There exists a smooth function $m(x, u, \Delta(t)) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that $m(x, u, 0) = u$ and, setting $\tilde{u} = m(x, u, \Delta(t))$, (7) can be rewritten as

$$\begin{aligned} \dot{x} &= f(x, \tilde{u}, 0) \\ y &= h(x), \end{aligned} \tag{9}$$

and the function $m(x, u, \Delta)$ is a diffeomorphism with respect to its second and third argument, i.e., there exist smooth functions $m_\Delta^{-1}(x, u, \tilde{u})$ and $m_u^{-1}(x, \tilde{u}, \Delta)$ such that

$$\Delta = m_\Delta^{-1}(x, u, \tilde{u}), \ u = m_u^{-1}(x, \tilde{u}, \Delta). \tag{10}$$

This assumption is rather restrictive and is not *strictly* necessary in our framework (this point is made clear in Example 4).

Example 2 In Example 1, $m(x, u, \Delta) = u+\Delta$, $m_u^{-1}(x, \tilde{u}, \Delta) = \tilde{u}-\Delta$, and $m_\Delta^{-1}(x, u, \tilde{u}) = \tilde{u}-u$. More in general, if the system vector field in (7) is given by

$$\dot{x} = f(x) + g(x)[u + \Phi(x)\Delta] \tag{11}$$

where $\Phi(x) : \mathbb{R}^n \to \mathbb{R}$ is bounded away from zero, then

$$\tilde{u} = m(x, u, \Delta) = u + \Phi(x)\Delta, \quad u = m_u^{-1}(x, \tilde{u}, \Delta) = \tilde{u} - \Phi(x)\Delta, \quad \Delta = m_\Delta^{-1}(x, u, \tilde{u}) = \frac{\tilde{u} - u}{\Phi(x)}.$$

$\triangle$

The new plant (9) obtained using A1 and letting $\tilde{u}$ be the new control input is free of disturbance (however, since $\Delta(t)$ is not known, $\tilde{u}$ cannot be freely assigned). In what follows we make additional assumptions allowing us to define a controller $\tilde{u}$ to solve Problem 1 for the disturbance-free plant (9). This, together with the estimation of $\Delta(t)$, will allow us to derive a controller for the original plant (7).

The following is a basic requirement for the solution of the tracking problem (see [4]).

Assumption A2 (Stable Inverse): Given $r(t)$, there exist sufficiently smooth and bounded functions $x^r(t)$ and $u^r(t)$ such that

$$\begin{aligned} \dot{x}^r(t) &= f(x^r(t), u^r(t), 0) \\ r(t) &= h(x^r(t)) \end{aligned} \tag{12}$$

for some initial condition $x^r(0), u^r(0)$, and all $t \geq 0$.

Consider the change of coordinates $\tilde{x} = x - x^r(t)$, rewrite (9) in new coordinates as

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, \tilde{u}), \tag{13}$$

and notice that the asymptotic stability of the origin of (13) is equivalent to the stability of the trajectory $x^r(t)$. We now introduce a condition to estimate the functions $x^r(t)$ and $u^r(t)$ on-line. It is useful to think of (12) as a copy of the plant with unknown state x^r, unknown input u^r, but a known output which is the reference trajectory $r(t)$. Consider a compensator of the type

$$\begin{aligned} \dot{\zeta}^r &= a(\zeta^r, x^r, v^r) \\ u^r &= b(\zeta^r, x^r), \end{aligned} \tag{14}$$

where $\zeta^r \in \mathbb{R}^q$ $(q \geq m)$, $v^r \in \mathbb{R}^m$, a and b are sufficiently smooth, and v^r is the new input of the composite system (12),(14). Let $X_1 = \mathrm{col}(x^r, \zeta^r)$ and rewrite (12), (14) as

$$\begin{aligned} \dot{X}_1 &= F(X_1, v^r) \\ r &= H(X_1) \end{aligned} \tag{15}$$

(with obvious definition of F and H). Define the observability mapping associated with X_1 in (15) as

$$\begin{aligned} y_{X_1} &\triangleq \mathrm{col}\left(r_1, \ldots, r_1^{(\bar{k}_1-1)}, \ldots, r_m, \ldots, r_m^{(\bar{k}_m-1)}\right) \\ &\triangleq \mathcal{H}_X\left(X_1, v^r, \ldots, (v^r)^{(\bar{n}_u-1)}\right), \end{aligned}$$

where $\sum_{i=1}^{p} \bar{k}_i = n + q$, $0 \leq \bar{n}_u \leq \max\{\bar{k}_1, \ldots, \bar{k}_m\} - 1$.

Assumption A3 (Practical Internal Model [1]): There exists a compensator of the form (14), which we call a *practical internal model*, which is regular (i.e., for each $x(0)$ and $u(t)$ there exist $\zeta(0)$ and $v(t)$ such that $b(\zeta(t), x(t)) = u(t)$, for all $t \geq 0$) and such that the following two properties hold for the composite system (12), (14).

(i) $\mathcal{H}_X$ does not depend on v^r and its derivatives, i.e., $\mathcal{H}_X = \mathcal{H}_X(X_1)$.

(ii) There exists a set of indices $\{\bar{k}_1, \ldots, \bar{k}_m\}$ and a set $\mathcal{X} \subset \mathbb{R}^{n+q}$ such that the mapping $\mathcal{H}_X : \mathcal{X} \to \mathcal{H}_X(\mathcal{X})$ defined by

$$y_{X_1} = \mathcal{H}_X(X_1)$$

is a diffeomorphism.

Notice that, by replacing x^r, ζ^r, u^r, and v^r in (12), (14) by x, ζ, $\tilde{u} = m(x, u, \Delta(t))$, and v, we get an observability assumption for (9) augmented with a practical internal model with state

ζ and input v. Thus, letting $X_2 = \mathrm{col}(x, \zeta)$, the dynamics of the two augmented systems can be written as

$$\begin{aligned} \dot{X}_i &= F(X_i, v^i) \\ y^i &= H(X_i),\ i = 1, 2, \end{aligned} \tag{16}$$

where $v^i = v^r$, $v^2 = v$, $y^1 = r = H(X_1)$, $y^2 = y = H(X_2)$. A3 guarantees that from y^i, $i = 1, 2$, and its time derivatives (i.e., the vectors $y_{X_1} = \mathrm{col}\left(r_1, \ldots, r_1^{(\bar{k}_1 - 1)}, \ldots, r_m, \ldots, r_m^{(\bar{k}_m - 1)}\right)$, $y_{X_2} = \mathrm{col}\left(y_1, \ldots, y_1^{(\bar{k}_1 - 1)}, \ldots, y_m, \ldots, y_m^{(\bar{k}_m - 1)}\right)$) one can get $X_1 = (x^r, \zeta^r)$ and $X_2 = (x, \zeta)$, respectively, and thus also $u^r = b(\zeta^r, x^r)$ and $\tilde{u} = b(\zeta, x)$. We will use this fact, together with A1, to estimate x and $\Delta(t)$. We stress that the two practical internal models with state ζ^r and ζ are not *directly* implemented. Rather, they are used to define estimators for x and x^r.

Assumption A4 (Input Dynamic Extension): There exists a compensator

$$\begin{aligned} \dot{\xi} &= c(\xi, x, w) \\ u &= d(\xi, x) \end{aligned}, \quad \xi \in \mathbb{R}^{q'}, q' \geq q. \tag{17}$$

where $w \in \mathbb{R}^m$ is the new control input, such that

(i) *(Compensator Relative Degree).* The augmented system

$$\begin{aligned} \dot{x} &= f(x, d(\xi, x), \Delta) \\ \dot{\xi} &= c(\xi, x, w) \\ y &= h(x) \end{aligned} \tag{18}$$

has the property that y_{X_2}, calculated along the vector field (18), does not depend on w.

(ii) *(Information Vector).* For any $\vartheta \in (0, 1)$ there exist a smooth function $\bar{w}(x^r, \zeta^r, x, \zeta, \xi) = \bar{w}(X_1, X_2, \xi)$, a positive integer n_Δ, a smooth function $\gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)})$, a C^1 function $V(\tilde{x}, \tilde{\xi}) : \tilde{D} \to \mathbb{R}^+$, with $\tilde{\xi} = \xi - \gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)})$, and a real number $c^* \geq 1$ such that $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q'} \,|\, V(\tilde{x}, \tilde{\xi}) \leq c^*\}$ is a compact subset of $\tilde{D}$ and the time derivative of V along the trajectories of

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, m(x, d(\xi, x), \Delta)) \\ \dot{\tilde{\xi}} &= c(\xi, x, \bar{w}(X_1, X_2, \xi)) - \dot{\gamma}. \end{aligned} \tag{19}$$

satisfies

$$\dot{V} \leq -\Phi(\tilde{x}, \tilde{\xi}),$$

where $\Phi(\tilde{x}, \tilde{\xi})$ is continuous on $\tilde{D}$ and positive definite on the set $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q'} \,|\, \vartheta \leq V(\tilde{x}, \tilde{\xi}) \leq c^*\}$.

Part (ii) of this assumption, derived from Assumption ULP in [5], implies that the smooth feedback $\bar{w}(x^r, \zeta^r, x, \zeta, \xi)$ practically stabilizes the origin of (19) and the set $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q'} \,|\, V(\tilde{x}, \tilde{\xi}) \leq c^*\}$ is included in its domain of attraction. Moreover, it requires that the information needed to do so is contained in the vector $(x^r, \zeta^r, x, \zeta, \xi) = (X_1, X_2, \xi)$. This is useful because from A3 one can estimate X_1 and X_2 from r and y, respectively, while ξ being the state of the controller is available for feedback. Thus, A3 and A4 allow to use a separation principle to solve Problem 1. The existence of $\gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)})$ ensures that the boundedness of X_1 and $\tilde{\xi}$ implies the boundedness of ξ. In the next section we specify two classes of compensators satisfying A4.

Next, we need to guarantee that the reference trajectory is contained in within an observable region.

Assumption A5 (Reference Trajectory): The reference trajectory $r(t)$ is such that, for all $t \geq 0$,

$$y_{X_1} \in \mathcal{C}_1 \subset \mathcal{H}_X(\mathcal{X}),$$

for some convex compact set $\mathcal{C}_1$ with boundary $\partial\mathcal{C}_1 = \{X_1 \in \mathbb{R}^{n+q} \,|\, g^1(X_1) = 0\}$, where $g^1 : \mathbb{R}^{n+q} \to \mathbb{R}$ is a C^1 function for which 0 is a regular value, i.e., $\forall X_1 \in \partial\mathcal{C}_1$, $\partial g/\partial X_1 \neq 0$.

Notice that A3 and A5 imply A2 which, therefore, is redundant and is introduced solely for the sake of illustration.

We now use V to characterize a set which is positively invariant and is contained in within the observable set $\mathcal{X}$ of X_2. This puts a constraint on the topology of the set $\mathcal{X}$. First recall that, from A2 and A3, $x^r(t)$ and $\zeta^r(t)$, and thus $X_1(t)$, are bounded functions of time. For any positive real number $c \leq c^*$, let

$$\Omega_c = \{(x, \xi) \in \mathbb{R}^{n+q'} \,|\, V(\tilde{x}, \tilde{\xi}) \leq c\}.$$

Since $\Delta(t)$ and its time derivatives are uniformly bounded, $\gamma(X_1(t), \Delta(t), \ldots, \Delta^{(n_\Delta)}(t))$ is also uniformly bounded, and thus, by the definition of $\tilde{\xi}$ in A4, Ω_c is a compact set. From the definition of $n_1, \ldots, n_m$, we have that

$$(x, \xi) \text{ bounded} \quad \overset{\text{A4}}{\Longrightarrow} \quad y_{X_2} \text{ bounded} \quad \overset{\text{A3}}{\Longleftrightarrow} \quad (x, \zeta) = X_2 \text{ bounded}$$

that is, there exists a compact set $\Sigma_c \subset \mathbb{R}^{n+q}$ such that

$$(x, \xi) \in \Omega_c \quad \Rightarrow \quad X_2 \in \Sigma_c.$$

Assumption A6 (Topology of $\mathcal{X}$): There exists a positive scalar $\bar{c} \le c^*$ such that

$$\mathcal{H}_X(\Sigma_{\bar{c}}) \subset \mathcal{C}_2 \subset \mathcal{H}_X(\mathcal{X}),$$

for some convex compact $\mathcal{C}_2$ with boundary $\partial\mathcal{C}_2 = \{X_2 \in \mathbb{R}^{n+q} \,|\, g^2(X_2) = 0\}$, where $g^2 : \mathbb{R}^{n+q} \to \mathbb{R}$ is a C^1 function for which 0 is a regular value.

3 Compensator Choice

In this section we focus our attention on two classes of compensators, namely chains of integrators and linearizing compensators. In both cases we provide sufficient conditions for A4 to be satisfied and a constructive procedure to find the feedback controller $\bar{w}(X_1, X_2, \xi)$.

3.1 Chains of Integrators

The main idea in this section is illustrated in the following example.

Example 3 Go back to Example 1 and recall that, setting $\tilde{u} = \bar{\bar{u}}(x, x^r, u^r)$ (with $\bar{\bar{u}}(x, x^r, u^r)$ defined in (4)), we have that the origin $\tilde{x} = 0$ of the error dynamics is globally uniformly asymptotically stable. Recall further that if we express the stable inverse (x^r, u^r), the disturbance $\Delta(t)$, and the state x by means of y_{X_1} and y_{X_2}, such expressions cannot be used to control (1) because the vector relative degree of (1) is $\{2, 1\}$, while the number of time derivatives of its output that need to be estimated is $\{3, 2\}$. As we argued in Example 1, this would result in a non-proper closed-loop system. To address this problem the most obvious choice for dynamic extension is two chains of integrators of length $\{2, 1\}$, yielding the extended system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1^2 + \xi_1^1 + \Delta_1(t) \\
\dot{x}_3 &= x_4 - x_1^2 - \xi_1^1 - \Delta_1(t) \\
\dot{x}_4 &= \xi_1^2 + \Delta_2(t) \\
\dot{\xi}_1^1 &= \xi_2^1, \quad \dot{\xi}_2^1 = w_1 \\
\dot{\xi}_1^2 &= w_2 \\
y &= \mathrm{col}(x_1, x_3),
\end{aligned} \tag{20}$$

where $\xi = \mathrm{col}(\xi_1^1, \xi_2^1, \xi_1^2)$ is the state of the dynamic extension and w is the new control input. Indeed, notice that y_{X_2} calculated along the vector field (20) is independent of the control input

w,

$$y_{X_2} = \mathrm{col}(y_1, \dot{y}_1, \ddot{y}_1, \dddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2)$$
$$= \mathrm{col}\Big(x_1, x_2, x_1^2 + \xi_1^1 + \Delta_1, 2x_1x_2 + \xi_2^1 + \dot{\Delta}_1, x_3, x_4 - x_1^2 - \xi_1^1 - \Delta_1,$$
$$\xi_1^2 + \Delta_2 - 2x_1x_2 - \xi_2^1 - \dot{\Delta}_1\Big).$$

We can now seek a controller for the extended system that employs y_{X_1}, y_{X_2}, and ξ as feedback variables, resulting in a proper closed-loop system. This is desirable in our framework because y_{X_1} and y_{X_2} can be easily estimated from r and y, respectively, while ξ being the state of the dynamic extension is available for feedback. Since $X_1 = \mathcal{H}_X^{-1}(y_{X_1})$ and $X_2 = \mathcal{H}_X^{-1}(y_{X_2})$ ($X_1 = \mathrm{col}(x^r, \zeta^r)$ and $X_2 = \mathrm{col}(x, \zeta)$), where $\mathcal{H}_X$ is a diffeomorphism, we equivalently seek a controller that is a function of (X_1, X_2, ξ). Later, in Theorem 1 we show how to estimate X_1 and X_2 from r and y *without using the inverse* $\mathcal{H}_X^{-1}$. Let

$$\bar{u}(x, x^r, u^r, \Delta) = \bar{\bar{u}}(x, x^r, u^r) - \Delta(t)$$
$$= \mathrm{col}\left(u_1^r + (x_1^r)^2 - x_1^2 + K_1{}^\top(x - x^r), u_2^r + K_2{}^\top(x - x^r)\right) - \Delta$$

so that, if $u = \bar{u}$ in (1), $\tilde{x} = 0$ is globally uniformly asymptotically stable. Use the fact that

$$u^r = \mathrm{col}(\zeta_1^r, \zeta_2^r), \qquad \Delta = \mathrm{col}(\zeta_1, \zeta_2) - u = \mathrm{col}(\zeta_1, \zeta_2) - \mathrm{col}(\xi_1^1, \xi_1^2),$$

to get

$$\bar{u} = \mathrm{col}\left(\zeta_1^r + (x_1^r)^2 - x_1^2 + K_1{}^\top(x - x^r) - \zeta_1 + \xi_1^1,\ \zeta_2^r + K_2{}^\top(x - x^r) - \zeta_2 + \xi_1^2\right). \tag{21}$$

Clearly, if $\dot{\bar{u}}_1$, $\ddot{\bar{u}}_1$, $\dot{\bar{u}}_2$, calculated along the vector fields (3), (5), (2), (6), and (20), could also be expressed as functions of (X_1, X_2, ξ), then by using integrator backstepping one could derive a feedback controller $\bar{w}(X_1, X_2, \xi)$ that globally uniformly asymptotically stabilizes the equilibrium $(x, \xi) = (x^r, \bar{u}_1, \dot{\bar{u}}_1, \bar{u}_2)$ of the extended dynamics (20). However, this is not the case, as it is easily seen that $\ddot{\bar{u}}_1$, $\dot{\bar{u}}_2$ depend on the inputs v^r, v, and w. Integrator backstepping can thus be applied only to the first integrator of the first chain, ξ_1^1, since $\dot{\bar{u}}_1$ *can* be expressed as a function of (X_1, X_2, ξ). For the remaining two integrators at the end of each chain, ξ_2^1 and ξ_1^2, one can resort to high-gain feedback to get a controller $\bar{w}(X_1, X_2, \xi)$ at the expense of loosing asymptotic stability of the equilibrium $(x, \xi) = (x^r, \bar{u}_1, \dot{\bar{u}}_1, \bar{u}_2)$ and achieving instead *practical stability*, i.e., regulation to an arbitrarily small residual set around $(x, \xi) = (x^r, \bar{u}_1, \dot{\bar{u}}_1, \bar{u}_2)$. This idea, which is the basis of the design developed in this section, is formalized in the proof of Lemma 1.

$\triangle$

In formalizing the idea described in Example 3, we begin by assuming that, at least in the ideal case when x, x^r, and u^r are available for feedback, there exists a smooth controller that uniformly asymptotically stabilizes the origin of (13).

Assumption A7 (Stabilizability of the Trajectory $x^r(t)$): There exist a smooth function $\bar{\bar{u}}(x, x^r, u^r)$, a C^1 function $V'(\tilde{x})$, $V' : \tilde{D}' \to \mathbb{R}^+$, and a real number $c' \geq 1$ such that $\bar{\bar{u}}(x^r, x^r, u^r) = u^r$, $\{\tilde{x} \in \mathbb{R}^n \,|\, V'(\tilde{x}) \leq c'\}$ is a compact subset of $\tilde{D}'$, and the time derivative of V' along the trajectories of

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, \bar{\bar{u}}(x, x^r, u^r))$$

satisfies

$$\dot{V}' \leq -\Phi'(\tilde{x}),$$

where $\Phi'(\tilde{x})$ is continuous on $\tilde{D}'$ and positive definite on the set $\{\tilde{x} \in \mathbb{R}^n \,|\, V'(\tilde{x}) \leq c'\}$.

Next, consider (7) and let $n_1, \ldots, n_m$ be the number of time derivatives of $u_1, \ldots, u_m$, respectively, appearing in $y_{X_2} = y_1^{(\bar{k}_1-1)}, \ldots, y_m^{(\bar{k}_m-1)}$ (if u_j does not appear in $y_1^{(\bar{k}_1-1)}, \ldots, y_m^{(\bar{k}_m-1)}$ we set $n_j = 0$). Consider the following choice for the compensator (17)

$$\begin{aligned} \dot{\xi} &= A_c\xi + B_c w, \quad \xi \in \mathbb{R}^{q'}, q' = n_1 + \ldots, n_m \\ u &= C_c \xi \end{aligned} \tag{22}$$

where the triple (A_c, B_c, C_c) is in controllable/observable canonical form with eigenvalues at zero. The compensator (22), depicted in Figure 1, is given by m chains of integrators - one chain for every input channel u_i - of order $n_1, \ldots, n_m$, respectively.

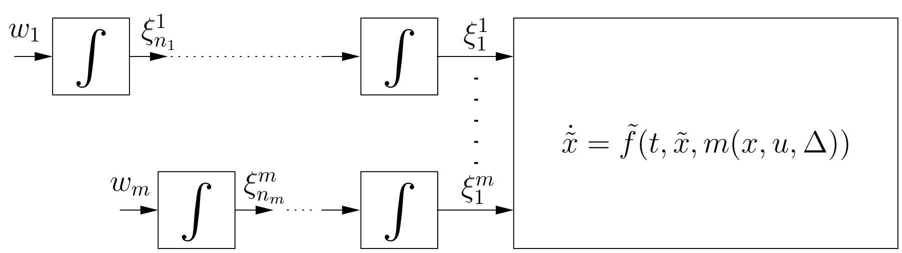

Figure 1: The system augmented with compensator (22).

Lemma 1 *Assume that A7 holds and that, for the system with outputs* $z^1, z^2 \in \mathbb{R}^m$

$$\begin{aligned}
\dot{X}_1 &= F(X_1, v) \\
\dot{X}_2 &= F(X_2, v^r) \\
\dot{\xi} &= A_c\xi + B_c\bar{w} \\
z^1 &= \alpha_1(X_1, X_2, \xi), \quad z^2 = \alpha_2(X_1, \Delta),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\alpha_1(X_1, X_2, \xi) &= m_u^{-1}\left(x, \bar{\bar{u}}(x, x^r, b(x^r, \zeta^r)), m_\Delta^{-1}(x, C_c\xi, b(\zeta, x))\right) \\
\alpha_2(X_1, \Delta) &= m_u^{-1}(x^r, b(\zeta^r, x^r), \Delta),
\end{aligned}$$

the output derivatives $[z_1^j, \ldots, (z_1^j)^{(n_1-1)}, \ldots, z_m^j, \ldots, (z_m^j)^{(n_m-1)}]$, $j = 1, 2$ *calculated along the vector field of (23), do not depend on* v, v^r, *and* w. *Then (22) satisfies A4.*

Proof. We begin by noting that by the definition of $n_1, \ldots, n_m$ the compensator (17) satisfies part (i) of A4.

From A1, letting $\bar{u}(x, x^r, u^r, \Delta) = m_u^{-1}(x, \bar{\bar{u}}(x, x^r, u^r), \Delta)$ we have $m(x, \bar{u}, \Delta) = \bar{\bar{u}}$, and thus by A7 the origin of

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, m(x, \bar{u}, \Delta)) \tag{24}$$

is uniformly asymptotically stable. Consider (24) augmented with (22)

$$\begin{aligned}
\dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, m(x, C_c\xi, \Delta(t))) \\
\dot{\xi} &= A_c\xi + B_c w,
\end{aligned} \tag{25}$$

and noticing that

$$\bar{u}(x^r, x^r, u^r, \Delta) = m_u^{-1}(x^r, \bar{\bar{u}}(x^r, x^r, u^r), \Delta) = m_u^{-1}(x^r, u^r, \Delta),$$

let

$$\bar{u}^r = \text{col}\,(\bar{u}_1^r, \ldots, \bar{u}_m^r) \triangleq m_u^{-1}(x^r, u^r, \Delta).$$

Next, for

$$\begin{aligned}
\dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, m(x, \text{col}(\xi_1^1, \bar{u}_2, \ldots, \bar{u}_m), \Delta)) \\
\dot{\xi}_i^1 &= \xi_{i+1}^1, \; i = 1, \ldots, n_1 - 2 \\
\dot{\xi}_{n_1-1}^1 &= \bar{\xi}_{n_1}^1,
\end{aligned} \tag{26}$$

which is (24) extended with $n_1 - 1$ integrators at the input channel u_1 (see Figure 2), using

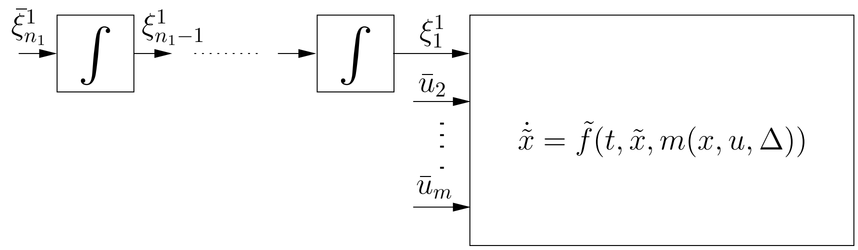

Figure 2: The augmented system (26).

integrator backstepping (see, e.g., Theorem 9.2.3 in [3]), one finds a smooth function

$$\bar{\xi}^1_{n_1}(x, \xi^1_1, \ldots, \xi^1_{n_1}, x^r(t), u^r(t), \bar{u}_1(t), \ldots, \bar{u}_1^{(n_1-1)}(t))$$

that uniformly asymptotically stabilizes the equilibrium

$$\big(\tilde{x}, \xi^1_1, \ldots, \xi^1_{n_1-1}\big) = \big(0, \bar{u}^r_1(t), \ldots, (\bar{u}^r_1)^{(n_1-2)}(t)\big)$$

of (26). Further, for

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, m(x, \mathrm{col}(\xi^1_1, \xi^2_1, \ldots, \xi^m_1), \Delta)) \\ \dot{\xi}^j_i &= \xi^j_{i+1},\ i = 1, \ldots, n_1 - 1,\ j = 1, \ldots, m \\ \dot{\xi}^j_{n_j} &= \bar{\xi}^j_{n_j}, \end{aligned} \tag{27}$$

which is (24) extended with $n_j - 1$ integrators at the input channel u_j, $j = 1, \ldots, m$ (see

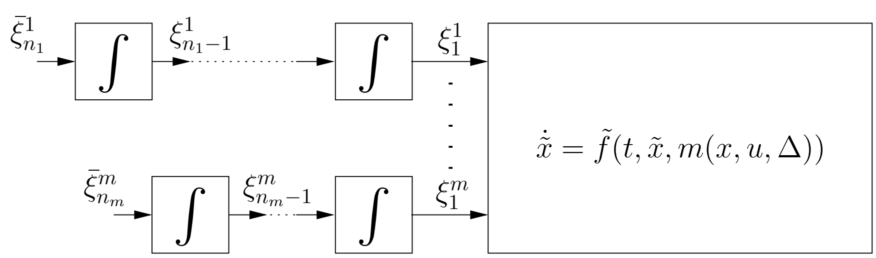

Figure 3: The augmented system (27).

Figure 3), applying the argument above sequentially to each of the m chains of integrators (see [6]) one finds m smooth functions

$$\bar{\xi}^j_{n_j}(x, \xi^1_1, \ldots, \xi^1_{n_1}, \ldots, \xi^j_1, \ldots, \xi^j_{n_j}, x^r(t), u^r(t), \bar{u}_j(t), \ldots, \bar{u}_j^{(n_j-1)}(t)),\ j = 1, \ldots, m$$

that uniformly asymptotically stabilize the equilibrium

$$\left(\tilde{x}, \xi_1^1, \ldots, \xi_{n_1-1}^1, \ldots, \xi_1^m, \ldots, \xi_{n_m-1}^m\right) = \left(0, \bar{u}_1^r(t), \ldots, (\bar{u}_1^r)^{(n_1-2)}(t), \ldots, \bar{u}_m^r(t), \ldots, (\bar{u}_m^r)^{(n_m-2)}(t)\right)$$

of (27). Augmenting (27) with one additional integrator for each channel we obtain (25) (i.e., the system in Figure 1). For $j = 1, \ldots, m$, let $e_{n_j} = \xi_{n_j}^j - \bar{\xi}_{n_j}^j$ and rewrite (25) as

$$\begin{aligned}
\dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, m(x, \mathrm{col}(\xi_1^1, \xi_1^2, \ldots, \xi_1^m), \Delta)) \\
\dot{\xi}_i^j &= \xi_{i+1}^j,\ i = 1, \ldots, n_j - 2 \\
\dot{\xi}_{n_j-1}^j &= \bar{\xi}_{n_j}^j + e_{n_j} \\
\dot{e}_{n_j} &= w_j - \dot{\bar{\xi}}_{n_j}^j,\ j = 1, \ldots, m.
\end{aligned} \tag{28}$$

If the signals $\bar{u}_j^{(n_j)}(t)$ (and hence $\dot{\bar{\xi}}_{n_j}^j(t)$) are not available, one cannot find a feedback controller uniformly stabilizing the equilibrium of (28). However, by applying Lemma 2.2 in [5] m times, we have that the controller

$$w_j = -K_j e_{n_j},\ j = 1, \ldots, m$$

achieves practical stabilization of the equilibrium of (28) for sufficiently large positive scalars K_j. In conclusion if

$$U \triangleq \left[\bar{u}_1, \ldots, \bar{u}_1^{(n_1-1)}, \ldots, \bar{u}_m, \ldots, \bar{u}_m^{(n_m-1)}\right]$$

is available then A7 implies that there exists a smooth function

$$\bar{w}(x^r, u^r, x, \xi, U) \tag{29}$$

achieving practical stabilization of the origin of

$$\begin{aligned}
\dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, m(x, C_c s, \Delta(t))), \\
\dot{\tilde{\xi}} &= A_c \tilde{\xi} + B_c(w - \mathrm{col}\left((\bar{u}_1^r)^{(n_1)}(t), \ldots, (\bar{u}_m^r)^{(n_m)}(t)\right)),
\end{aligned} \tag{30}$$

where $\tilde{\xi} = \xi - \mathrm{col}(\bar{u}_1^r, \ldots, (\bar{u}_1^r)^{(n_1-1)}, \ldots, \bar{u}_m^r, \ldots, (\bar{u}_m^r)^{(n_m-1)})$. Correspondingly, from the application of integrator backstepping and Lemma 2.2 in [5] we obtain a C^1 function $V(\tilde{x}, \tilde{\xi}) : \tilde{D} \to \mathbb{R}^+$ and a real number $c^* \geq 1$ such that $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^{n+q'} \mid V(\tilde{x}, \tilde{\xi}) \leq c^*\}$ is a compact subset of $\tilde{D}$ and for any $0 < \vartheta < 1$ there exist sufficiently large K_j, $j = 1, \ldots, m$ such that the time derivative of V along the trajectories of (30) satisfies

$$\dot{V} \leq -\Phi(\tilde{x}, \tilde{\xi}),$$

where $\Phi(\tilde{x}, \tilde{\xi})$ is positive definite on the set $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^{n+q'} \,|\, \vartheta \leq V(\tilde{x}, \tilde{\xi}) \leq c^*\}$. We now show that, under the assumptions of the lemma, U can be expressed as a function of X_1, X_2, and ξ. To this end, using the matching condition A1 we have that

$$\bar{u} = m_u^{-1}(x, \bar{\bar{u}}(x, x^r, u^r), \Delta) = m_u^{-1}(x, \bar{\bar{u}}(x, x^r, b(\zeta^r, x^r), m_\Delta^{-1}(x, C_c\xi, b(\zeta, x)))$$

is exactly the output z^1 in (23). Since, by assumption,

$$\left(z_1^1, \ldots, (z_1^1)^{(n_1-1)}, \ldots, z_m^1, \ldots, (z_m^1)^{(n_m-1)}\right)$$

does not depend on v, v^r, and w, we have that U is a function of X_1, X_2, and ξ. We thus conclude that $\bar{w}$ in (29) is a function of X_1, X_2, and ξ. In order to complete the proof we need to show that $\tilde{\xi}$ can be expressed as $\tilde{\xi} = \xi - \gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)})$, where γ is smooth. Noticing that

$$\bar{u}^r = m_u^{-1}(x^r, u^r, \Delta) = m_u^{-1}\left(x^r, b(\zeta^r, x^r), \Delta\right)$$

is exactly the output z^2 in (23), by assumption the derivatives of $\bar{u}_r$ calculated along the vector field of (23) do not depend on $v, v^r, \bar{w}$, i.e.,

$$\text{col}\left(\bar{u}_1^r, \ldots, (\bar{u}_1^r)^{(n_1-1)}, \ldots, \bar{u}_m^r, \ldots, (\bar{u}_m^r)^{(n_m-1)}\right) = \gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)}),$$

where $n_\Delta < \max\{n_1, \ldots, n_m\}$.

■

Remark 1: The feedback controller $\bar{w}(X_1, X_2, \xi)$ described above utilizes a high-gain feedback applied only to the last integrator of each of the m chains and, as such, does not yield peaking in the state of the extended system (x, ξ). See [7] for an investigation of the peaking phenomenon and its effects on nonlinear stabilization.

Remark 2: The requirement, in Lemma 1, that the output derivatives $[z_1^j, \ldots, (z_1^j)^{(n_1-1)}, \ldots, z_m^j, \ldots, (z_m^j)^{(n_m-1)}]$, $j = 1, 2$, calculated along the vector field of (23), do not depend on v, v^r, and w can be removed by stabilizing the cascade system (25) with high-gain feedback applied to the entire chain of integrators. In this case, however, the state ξ of the chains of integrators would be subject to peaking.

3.2 Linearizing Compensators

Assume that (9) is affine in the input, i.e., it reads as

$$\begin{aligned} \dot{x} &= f_1(x) + f_2(x)\tilde{u} = f_1(x) + f_2(x)m(x,u,\Delta) \\ y &= h(x). \end{aligned} \tag{31}$$

Assume further that (31), viewed as a system with input $\tilde{u}$, is dynamic feedback linearizable (differentially flat), i.e., there exists a *linearizing* compensator

$$\begin{aligned} \dot{\xi} &= c'(\xi,x,w), \qquad \xi \in \mathbb{R}^r \\ \tilde{u} &= d'(\xi,x,w) \end{aligned} \tag{32}$$

such that the plant augmented with such compensator yields the trivial system in output coordinates:

$$y_i^{(k_i)} = w_i, \quad k_1 + \ldots + k_m = n + r.$$

As shown in [2], a practical internal model satisfying A3 is given by (32) augmented with m integrators at the input side[2] (one for each input channel). Viewing (31) as a system with input u, consider the following dynamic extension

$$\begin{aligned} \dot{x} &= f_1(x) + f_2(x)m(x,d(\xi,x),\Delta) \\ \dot{\xi} &= c(\xi,x,w), \quad \xi \in \mathbb{R}^q \\ y &= h(x), \end{aligned} \tag{33}$$

where $q = r + m$ and $(c(\cdot,\cdot,\cdot), d(\cdot,\cdot))$ denote the vector field and output function of the augmented compensator above (clearly here, referring to A3, $(a(\cdot,\cdot,\cdot), b(\cdot,\cdot)) = (c(\cdot,\cdot,\cdot), d(\cdot,\cdot))$). From the dynamic feedback linearizability property and the fact that $m(x,u,0) = u$, when $\Delta = 0$ we have a well-defined vector relative degree $\{k_1 + 1, \ldots, k_m + 1\}$. Additionally, when $\Delta = 0$, A3 ensures that $(x,\xi) = \mathcal{H}_X^{-1}(y_{X_2})$, and thus in particular ξ can be expressed as a function of y_{X_2}. In Lemma 2 we show that if the two properties above are preserved when Δ and its derivatives are not zero, then besides providing a practical internal model satisfying A3, the pair $(c(\cdot\cdot,\cdot), d(\cdot,\cdot))$ is a valid input dynamic extension fulfilling the requirements in A4. Before stating the lemma, we illustrate the main idea in the following example.

Example 4 We return to Example 1 and this time employ a linearizing compensator as dynamic extension. Notice that the disturbance-free system (2) is dynamic feedback linearizable

[2]The integrators are not needed when d' is independent of w, i.e., when $d' = d'(\xi,x)$.

since the system augmented with a practical internal model has a well-defined full vector relative degree $\{4, 3\}$,

$$y_1^{(4)} = 2x_2^2 + 2x_1(x_1^2 + \zeta_1) + v_1 + v_2, \qquad y_2^{(3)} = -2x_2^2 - 2x_1(x_1^2 + \zeta_1) - v_2.$$

Choose, as dynamic extension, a copy of the practical internal model (6) with a feedback transformation,

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \xi_3 \\ \dot{\xi}_2 &= w_1 + w_2 \\ \dot{\xi}_3 &= -2x_2^2 - 2x_1(x_1^2 + \xi_1) - w_2 \\ u &= \mathrm{col}(\xi_1, \xi_2), \end{aligned} \tag{34}$$

so that, when $\Delta = 0$, $y_1^{(4)}$ and $y_2^{(3)}$, calculated along the vector fields (1), (34) are given by

$$y_1^{(4)} = w_1, \qquad y_2^{(3)} = w_2.$$

On the other hand, when $\Delta \neq 0$, one has

$$y_1^{(4)} = w_1 + 2x_1\Delta_1, \qquad y_2^{(3)} = w_2 - 2x_1\Delta_1 + \dot{\Delta}_2,$$

and thus the vector relative degree is unchanged. Recalling that

$$y_{X_1} = \mathrm{col}(r_1, \dot{r}_1, \ddot{r}_1, \dddot{r}_1, r_2, \dot{r}_2, \ddot{r}_2), \qquad y_{X_2} = \mathrm{col}(y_1, \dot{y}_1, \ddot{y}_1, \dddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2),$$

and letting $e = \mathrm{col}(e_1^1, \ldots, e_4^1, e_1^2, \ldots, e_3^2) = y_{X_1} - y_{X_2}$, we have

$$\begin{aligned} \dot{e}_i^1 &= e_{i+1}^1, i = 1, \ldots, 3 \\ \dot{e}_4^1 &= r_1^{(4)} - w_1 - 2x_1\Delta_1 \\ \dot{e}_j^2 &= e_{j+1}^2, \;\; j = 1, \ldots, 2 \\ \dot{e}_3^2 &= r_2^{(3)} - w_2 + 2x_1\Delta_1 - \dot{\Delta}_2. \end{aligned} \tag{35}$$

It is immediately clear that a standard high-gain controller can be employed to practically stabilize the origin $e = 0$ of (35), thus yielding practical tracking since $r - y = \mathrm{col}(e_1^1, e_1^2)$. Furthermore, if the reference trajectory $r(t)$ and its time derivatives (the vector y_{X_1}) are bounded functions of time, then the boundedness of e implies the boundedness of y_{X_2} and thus the boundedness of $X_2 = \mathcal{H}_X^{-1}(y_{X_2})$. Since $X_2 = \mathrm{col}(x, \zeta)$, we have that x is bounded as well. Finally, by computing the time derivatives of y along the vector fields (1), (34), it is easy to show that

$$\xi_1 = \ddot{y}_1 - y_1^2 - \Delta_1,\ \xi_2 = \dddot{y}_1 + \ddot{y}_2 - \Delta_2,\ \xi_3 = -(\ddot{y}_2 + 2y_1\dot{y}_1 + \dot{\Delta}_1 - \Delta_2),$$

and thus the boundedness of y_{X_2} and that of Δ and its derivatives imply that ξ is bounded as well.

This idea may still be applied, in some cases, when the matching condition A1 does not hold. It is not difficult to see, for instance, that if in our example we replace (1) by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + u_1 + \Delta_1(t) \\ \dot{x}_3 &= x_4 - x_1^2 - u_1 \\ \dot{x}_4 &= u_2 + \Delta_2(t) \\ y &= \mathrm{col}(x_1, x_3), \end{aligned} \tag{36}$$

then the matching condition A1 is not satisfied and yet the same high-gain controller will solve the practical tracking problem. On the other hand, the approach illustrated in Section 3.1 cannot be applied to control (36). This difference between the two approaches originates from the fact that the idea outlined in this example does not rely on the estimation of Δ, while the idea used in Section 3.1 does. Despite this, when applicable the approach in Section 3.1 is preferable because it relies on a high-gain controller of degree one which does not induce peaking in the state (x, ξ) of the extended system, while the approach described in this section relies on a high-gain controller of order $\max_i\{k_i\} + 1$ which induces peaking in the state of the extended system.

$\triangle$

The simple idea illustrated in Example 4 is generalized in the following lemma.

Lemma 2 *Assume that (31) is dynamic feedback linearizable. If there exist a smooth function* φ *and a positive integer* n_Δ *such that*

$$\xi = \varphi(y_{X_2}, \Delta, \ldots, \Delta^{(n_\Delta)}) \tag{37}$$

and

$$y_i^{(k_i+1)} = w_i + g_i(x, \xi, \Delta, \ldots, \Delta^{(k_i)}), \quad i = 1, \ldots, m, \tag{38}$$

with smooth g_i *vanishing when* $(\Delta, \ldots, \Delta^{(k_i)}) = (0, \ldots, 0)$*, then the pair* $(c(\cdot,\cdot,\cdot), d(\cdot,\cdot))$ *satisfies A4.*

Proof. From [2], the pair $(c(\cdot,\cdot,\cdot), d(\cdot,\cdot))$ is a practical internal model satisfying A3 and thus $\bar{k}_i = k_i + 1$, $i = 1, \ldots, m$. Condition (38) readily implies that

$$y_{X_2} = \left(y_1, \ldots, y_1^{(k_1)}, \ldots, y_m, \ldots, y_m^{(k_m)}\right),$$

calculated along the vector field (33), does not depend on w, proving that part (i) of A4 is satisfied. As for part (ii), recall that

$$y_{X_1} = \mathrm{col}\left(r_1, \ldots, r_1^{(k_1)}, \ldots, r_m, \ldots, r_m^{(k_m)}\right),$$

let $e = y_{X_1} - y_{X_2}$, $g = \mathrm{col}(g_1, \ldots, g_m)$, and consider the e dynamics

$$\dot{e} = A_c e + B_c \left[\mathrm{col}\left(r_1^{(k_1+1)}, \ldots, r_m^{(k_m+1)}\right) - w - g(x, \xi, \Delta, \ldots, \Delta^{(k_i)})\right], \tag{39}$$

where (A_c, B_c) is in Brunovsky normal form. The high-gain controller

$$\bar{w}_i = \sum_{j=1}^{k_i+1} h_{i,j} \lambda^{k_i - j + 2} e_i, \quad i = 1, \ldots, m,$$

where $\lambda > 1$ is the high-gain parameter and the scalars $h_{i,j}$ are chosen so that the polynomials $s^{k_i+1} + h_{i,k_i+1} s^{k_i} + \ldots + h_{i,1}$, $i = 1, \ldots, m$, are Hurwitz, achieves practical stabilization of the origin $e = 0$. Recall that $y_{X_1} = \mathcal{H}_X(X_1)$ and $y_{X_2} = \mathcal{H}_X(X_2)$, where $X_1 = \mathrm{col}(x^r, \zeta^r)$ and $X_2 = \mathrm{col}(x, \zeta)$ and $\mathcal{H}_X$ is a diffeomorphism. The feedback controller $\bar{w}$ is then expressed as $\bar{w}(X_1, X_2) = K(\mathcal{H}_X(X_1) - \mathcal{H}_X(X_2))$, where

$$K = \begin{bmatrix} h_{1,1}\lambda^{k_1+1} & \ldots & h_{1,k_1}\lambda \\ \vdots & \vdots & \vdots \\ h_{m,1}\lambda^{k_m+1} & \ldots & h_{m,k_m}\lambda \end{bmatrix}.$$

By the fact that $\mathcal{H}_X$ is a diffeomorphism, the practical stability of $e = 0$ implies that the origin of the $\tilde{x}$ dynamics is practically stable. Further, by smoothness of φ, it also implies that $\varphi(y_{X_2}, \Delta, \ldots, \Delta^{(n_\Delta)}) - \varphi(y_{X_1}, \Delta, \ldots, \Delta^{(n_\Delta)})$ is practically stable which, by (37), is equivalent to saying that $\xi - \varphi(y_{X_1}, \Delta, \ldots, \Delta^{(n_\Delta)})$ is practically stable. Defining γ in A4 as

$$\gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)}) = \varphi(\mathcal{H}_X(X_1), \Delta, \ldots, \Delta^{(n_\Delta)})$$

we conclude the proof of the lemma.

■

4 Solution to the Practical Output Tracking Problem

In this section we solve Problem 1 using the separation principle in [8]. Consider the dynamic output feedback controller

$$\begin{aligned} \dot{\xi} &= c(\xi, \hat{x}^P, \bar{w}(\hat{X}_1^P, \hat{X}_2^P, \xi)) \\ u &= d(\xi, \hat{x}^P), \end{aligned} \tag{40}$$

where $X_1^P = \mathrm{col}((\hat{x}^r)^P, (\hat{\zeta}^r)^P)$, $X_2^P = \mathrm{col}(\hat{x}^P, \hat{\zeta}^P)$ are given by

$$\dot{\hat{X}}_i^P = \begin{cases} \left[\dfrac{\partial \mathcal{H}_X}{\partial \hat{X}_i^P}\right]^{-1} \left\{ (L_{\hat{F}}\, \mathcal{H}_X)^i - \Gamma^i \dfrac{N^i(\hat{y}_{X_i}) L_G\, g^i}{N^i(\hat{y}_{X_i})^\top \Gamma^i N^i(\hat{y}_{X_i})} \right\} & \text{if } L_G\, g^i \geq 0 \text{ and } \hat{y}_{X_i} \in \partial \mathcal{C}^i \\ \hat{F}(\hat{X}_i, y^i) = F(\hat{X}_i, 0) + \left[\dfrac{\partial \mathcal{H}_X(\hat{X}_i)}{\partial \hat{X}_i}\right]^{-1} (\mathcal{E}^i)^{-1} L^i \left(y^i - H(\hat{X}_i)\right) & \text{otherwise} \end{cases} \tag{41}$$

for $i = 1, 2$, where the various parameters are defined in Table 1. The estimator (41) incorporates a high-gain component to guarantee convergence, and a dynamic projection to avoid peaking and confine the estimator state to within the observable region $\mathcal{X}$ (see [8] for more details). We are now in the position to state the main result of this paper.

$\hat{y}_{X_i} = \mathcal{H}_X(\hat{X}_i^P)$	$(L_{\hat{F}}\, \mathcal{H}_X)^i = \dfrac{\partial \mathcal{H}_X}{\partial \hat{X}_i^P} \hat{F}(\hat{X}_i^P, y^i)$ $G(\hat{X}_i^P, v^i) = (L_{\hat{F}}\, \mathcal{H}_X)^i$	$L_G\, g^i = \dfrac{\partial g^i}{\partial \hat{y}_{X_i}} G(\hat{X}_i^P, v^i)$
$N^i(\hat{y}_{X_i}) = \left(\dfrac{\partial g^i(\hat{y}_{X_i})}{\partial \hat{y}_{X_i}}\right)^\top$	$\mathcal{E}^i = block\text{-}diag[\mathcal{E}_1^i, \ldots, \mathcal{E}_p^i]$ $\mathcal{E}_j^i = diag[\rho_i, \ldots, \rho_i^{\bar{k}_j}]$	$L^i = block\text{-}diag[L_1^i, \ldots, L_p^i]$ L_j^i Hurwitz $(\bar{k}_j \times 1)$
$\Gamma^i = (S^i \bar{\mathcal{E}}^i)^{-1} (S^i \bar{\mathcal{E}}^i)^{-\top}$	$\bar{\mathcal{E}}^i = block\text{-}diag[\bar{\mathcal{E}}_1^i, \ldots, \bar{\mathcal{E}}_p^i]$ $\bar{\mathcal{E}}_j^i = 1/\rho_i^{\bar{k}_j}\ \mathcal{E}_j^i$	$S^i = (P^i)^{1/2}$ P^i satisfies:
$A^{i\top} P^i + P^i A^i = -I_{(n+q)\times(n+q)}$ $A^i = \begin{bmatrix} 0_{(n+q-1)\times 1} & I_{(n+q-1)\times(n+q-1)} \\ & 0_{1\times(n+q)} \end{bmatrix} - L^i [1, 0_{1\times n+q-1}]$		

Table 1: Definitions of various parameters in observer (41).

Theorem 1 *Suppose that A1-A6 hold. Then, for any smooth bounded $\Delta(t)$ with bounded derivatives, (40), (41) solve Problem 1 on a compact set $\mathcal{A}$ whose size depends on c^* and the sets $\mathcal{C}_1$ and $\mathcal{C}_2$. If A4 holds for arbitrarily large c^* and a radially unbounded V, and A3 holds globally (i.e., $\mathcal{X} = \mathbb{R}^{n+q}$) with $\mathcal{H}_X(\mathbb{R}^{n+q})$ a convex set, then the solution of Problem 1 is semiglobal in that $\mathcal{A}$ can be chosen to be an arbitrarily large compact set.*

Sketch of the Proof. The idea used to solve Problem 1 in the presence of uncertainties is illustrated in Figure 4. If X_1 and X_2 are available for feedback, the dynamic controller (17)

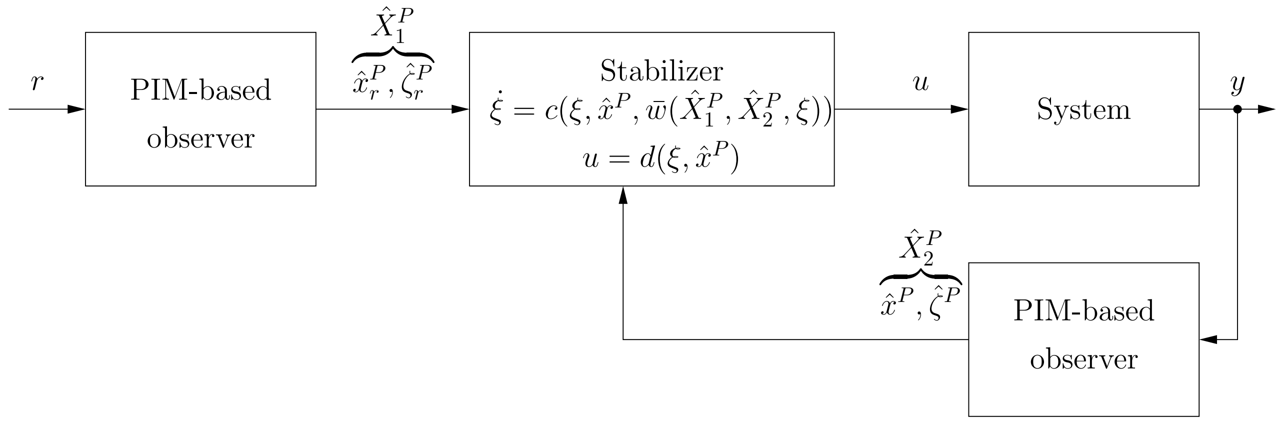

Figure 4: Robust tracking scheme.

yields the closed-loop system

$$
\begin{aligned}
&\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, m(x, d(\xi, x), \Delta)) \\
&\dot{\tilde{\xi}} = c(\xi, x, \bar{w}((x^r, \zeta^r), (x, \zeta), \xi)) - \dot{\gamma} \\
&\dot{x}^r = f(x^r, b(\zeta^r, x^r), 0) \\
&\dot{\zeta}^r = a(\zeta^r, x^r, v^r) \\
&\dot{\zeta} = a(\zeta, x, v) \quad \text{constraint: } b(\zeta, x) = m(x, d(\xi, x), \Delta) \\
&y = h(x) \quad r = h(x^r).
\end{aligned}
\tag{42}
$$

By A2, $x^r(t)$ is uniformly bounded for all $t \geq 0$ and, by the regularity property of the compensator $(a(\cdot, \cdot, \cdot), b(\cdot, \cdot))$ (see A3), ζ^r is uniformly bounded as well. Recall that ζ is related to ξ and x by

$$b(\zeta, x) = \tilde{u} = m(x, u, \Delta) = m(x, d(\xi, x), \Delta),$$

which expresses the constraint that the output of the practical internal model, $b(\zeta, x)$, be equal to the input to the disturbance-free plant, $\tilde{u}$, induced by the dynamic extension (17). By this constraint and the regularity of $(a(\cdot, \cdot, \cdot), b(\cdot, \cdot))$, $\zeta(t)$ is uniformly bounded as long as $x(t)$ and $\xi(t)$ are uniformly bounded. Since, by A4, $(\tilde{x}, \tilde{\xi}) = (0, 0)$ is practically stable, we conclude that all states in (42) are uniformly bounded. Thus, if X_1 and X_2 are available for feedback, practical tracking is achieved for all $(\tilde{x}(0), \tilde{\xi}(0)) \in \{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q'} \,|\, V(\tilde{x}, \tilde{\xi}) \leq c^*\}$. Next, we show that by virtue of the estimators (41) and the separation principle in [8] the dynamic output feedback controller (40), obtained from (17) by replacing X_1 and X_2 by their estimates, solves Problem 1. Consider the X_1, X_2 dynamics in (16), obtained by augmenting (12) and (9) with two practical internal models. By A3, we have

$$y_{X_1} = \mathcal{H}_X(X_1), \quad y_{X_2} = \mathcal{H}_X(X_2),$$

where $\mathcal{H}_X$ is a diffeomorphism of $\mathcal{X}$ onto its image. Thus the observability assumption A1 in [8] is satisfied for (16). Assumptions A5 and A6 imply that the sets $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy assumption A3 in [8]. The following result is a straightforward consequence of Theorem 1 and Lemma 1 in [8] and so its proof is omitted.

Lemma 3 *Consider (16) and (41), and assume that A3 and A5 ($i = 1$) or A6 ($i = 2$) hold. Then the estimates $\hat{X}_i^P$ enjoy the following properties*

(i) Positive Invariance of $\mathcal{H}_X^{-1}(\mathcal{C}^i)$: *if $\hat{X}_i^P(0) \in \mathcal{H}_X^{-1}(\mathcal{C}^i)$, then $\hat{X}_i^P(t) \in \mathcal{H}_X^{-1}(\mathcal{C}^i)$ for all $t \geq 0$.*

(ii) Uniform Ultimate Boundedness of the Estimation Error: *For all $\delta > 0$, there exist $\bar{\rho}_i \in (0, 1)$ and $T(\rho_i) > 0$ such that $\hat{X}_i^P(t) - X_i(t) \leq \delta$ for all $t \geq T(\rho_i)$, whenever $\rho_i \in (0, \bar{\rho}_i)$.*

(iii) Arbitrarily fast rate of convergence: *$T(\rho_i)$ in part (ii) has the property that $T(\rho_i) \to 0$ as $\rho_i \to 0$.*

For the estimator obtained setting $i = 2$, parts (ii) and (iii) hold provided that $X_2(t) \in \Sigma_{\bar{c}}$ ($\bar{c}$ is defined in A6), for all $t \geq 0$.

This result, together with a relatively standard Lyapunov analysis similar to the one used in the proof of Lemma 2 in [8] allows us to conclude that, for any $\underline{c} \in (0, \bar{c})$, there exist sufficient small values of ρ_1 and ρ_2 such that for all $(\tilde{x}(0), \tilde{\xi}(0), \hat{X}_1^P(0), \hat{X}_2^P(0)) \in \mathcal{A}$,

$$\mathcal{A} = \left\{ (\tilde{x}, \tilde{\xi}, \hat{X}_1^P, \hat{X}_2^P) \in \mathbb{R}^{3n+q'+2q} \,|\, V(\tilde{x}, \tilde{\xi}) \leq \underline{c}, \hat{X}_1^P \in \mathcal{H}_X^{-1}(\mathcal{C}_1), \hat{X}_2^P \in \mathcal{H}_X^{-1}(\mathcal{C}_2) \right\},$$

$(\tilde{x}(t), \tilde{\xi}(t), \hat{X}_1^P(t), \hat{X}_2^P(t)) \in \mathcal{A}$ for all $t \geq 0$ and $(\tilde{x}, \tilde{\xi}) = (0, 0)$ is practically stable. Since $\mathcal{A}$ is compact, we have that $(\tilde{x}(t), \tilde{\xi}(t))$ are uniformly bounded which, from the boundedness of X_1 and the definition of $\tilde{\xi}$ in A4, implies that x and ξ are bounded as well.

If c^* in A4 can be chosen arbitrarily large and V is radially unbounded, then for any compact set $D \subset \mathbb{R}^n \times \mathbb{R}^{q'}$, there exists $\underline{c} > 0$ such that $D \subset \{(\tilde{x}, \tilde{\xi}) \,|\, V(\tilde{x}, \tilde{\xi}) \leq \underline{c}\}$. Further, if $\mathcal{X} = \mathbb{R}^{n+q}$ and $\mathcal{H}_X(\mathbb{R}^{n+q})$ is convex A6 is satisfied by *any* $\bar{c} > 0$ and a sufficiently large convex compact set $\mathcal{C}_2$ (see [8]). Choose $\bar{c} > \underline{c}$. Finally, we have that for any bounded reference trajectory with bounded derivatives (i.e., y_{X_1} is bounded), there exists a sufficiently large set $\mathcal{C}_1$ satisfying A5. By the first part of this theorem we conclude that there exist sufficiently small values of ρ_1 and ρ_2 such that (40), (41) solve Problem 1 on $\mathcal{A}$ which, now, can be chosen arbitrarily large.

■

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