

**Tuning regulators for tracking
MIMO positive LTI systems**

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Systems Control Group Report - No. 0701

January 10, 2007

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Abstract

In this paper we consider the study of the servomechanism problem for multi-input multi-output (MIMO) linear time-invariant (LTI) positive systems. In particular, we present results on the tracking problem of nonnegative constant reference signals for unknown stable MIMO positive LTI systems with nonnegative constant disturbances. Our results exploit the structure of tuning regulators and show that any unknown stable MIMO positive LTI system can track any nonnegative constant reference signal while maintaining *almost* state nonnegativity.

1 Introduction

In this paper we study the tracking problem for MIMO positive LTI systems. In particular, we consider the tracking problem of nonnegative constant reference signals for unknown stable MIMO positive LTI systems with nonnegative constant disturbances.

A positive linear system can be regarded as a linear system where the state variables are nonnegative for all time [1]. Positive systems are of great practical importance, as the nonnegative property occurs quite frequently in numerous applications and in nature. A special class of positive systems that appears quite frequently in the literature is the class of compartmental systems. Compartmental systems are composed of a finite number of storage devices or reservoirs. These type of systems frequently occur in hydrology where they are used to model natural and artificial networks of reservoirs [2]. Positive systems and their counterparts, compartmental systems, are visible in biology where they are used to describe the transportation, accumulation, and drainage processes of elements and compounds like hormones, glucose, insulin, metals, etc. Stocking and industrial systems which involve chemical reactions, heat exchangers, and distillation columns are also examples of positive systems [2]. It should be of no surprise that positive systems are so widely visible in applications and in nature, and as was pointed out in [1] "it is for positive systems that dynamic systems theory assumes one of its most potent forms".

Positive systems have been of great interest to numerous researchers over several decades. Some interesting results on positive systems are as follows: on reachability and controllability, see [3], [4],

[5], [6], [7], [8], on results re realization of positive systems, see [9], [10], [11], [12], and for results on 2D positive linear systems, see [13], [14]. Some other results on stability control, pole-assignment and some general stability feedback control can be found in [15], [16], [17], [18], [19], [20].

From the above references, we can deduce that interest in positive systems spans several decades and numerous disciplines. In this paper, we will present results that will add to positive system theory by introducing the concept of tracking reference signals under disturbances while maintaining *almost* nonnegativity of the states and outputs. Our interest will be to show that any unknown stable MIMO positive LTI system under nonnegative constant disturbances can attain both tracking and *almost* nonnegativity of states and outputs for all nonnegative constant references.

The paper is organized as follows. Preliminaries are given first, where the terminology, positive systems and compartmental systems, *almost* state positive systems, and tuning regulators are discussed. All assumptions on the system plant treated in the paper and the Problem Statement are described in Section 3. We then follow a discussion on Singular Perturbation modeling in Section 4. Section 5 provides the main result of the paper. A few examples illustrating the theory are given in Section 6, while all concluding remarks and discussion of future work complete the paper.

2 Background and Preliminaries

2.1 Terminology

Let the set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, the set $\mathbb{R}_+^n := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_+, \forall i = 1, \dots, n\}$. Similarly, let $\mathbb{R}_- := \{x \in \mathbb{R} \mid x \leq 0\}$, and the set $\mathbb{R}_-^n := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_-, \forall i = 1, \dots, n\}$. If exclusion of 0 from the sets will be necessary, then we'll denote the sets in the standard way $\mathbb{R}_+ \setminus \{0\}$ ($\mathbb{R}_+^n \setminus \{0\}$). The set of eigenvalues of a matrix \mathcal{A} will be denoted as $\sigma(\mathcal{A})$. The ij^{th} entry of a matrix \mathcal{A} will be denoted as a_{ij} . A matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$ is Hurwitz or stable when all the eigenvalues (λ) of \mathcal{A} are in the open left half plane of the complex plane \mathbb{C} , i.e. the real part of all eigenvalues is negative. A *nonnegative* matrix \mathcal{A} has all of its entries greater or equal to 0, i.e. $a_{ij} \in \mathbb{R}_+$. A *Metzler* matrix \mathcal{A} is a matrix for which all off-diagonal elements of \mathcal{A} are

nonnegative, i.e. $a_{ij} \in \mathbb{R}_+$ for all $i \neq j$. A *compartmental* matrix \mathcal{A} is a matrix that is Metzler, where the sum of the components within a column is less than or equal to zero, i.e. $\sum_{i=1}^n a_{ij} \leq 0$ for all $j = 1, 2, \dots, n$. A matrix \mathcal{A} that is compartmental, but also satisfies $\sum_{i=1}^n a_{ij} < 0$ for all $j = 1, 2, \dots, n$ will be referred to as a *strictly compartmental* matrix.

2.2 Positive Linear Systems and Compartmental Systems

In this section we give an overview of both *positive linear systems* [1], [2], and a very important subset of positive linear systems known as *compartmental systems* [2], [21]. The inclusion of compartmental systems within this subsection will be made because in general compartmental systems are stable, a property of great significance throughout the paper.

We first define a positive linear system [2] in the traditional sense.

Definition 2.1. A linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{r \times n}$, and $D \in \mathbb{R}^{r \times m}$ is considered to be a *positive linear system* if for every nonnegative initial state and for every nonnegative input the state of the system and the output remain nonnegative.

The above definition states that any trajectory starting at an initial condition $x_0 \in \mathbb{R}_+^n$ will not leave the positive orthant, and moreover, that the output also remains nonnegative. For convenience, if for all time a state x satisfies $x \in \mathbb{R}_+^n$, or the output y satisfies $y \in \mathbb{R}_+^r$, or the input u satisfies $u \in \mathbb{R}_+^m$, then we'll say that the state, output, or input maintains *nonnegativity*. Notice that Definition 2.1 states that the input to the system must be positive, a restriction that in applications is not always feasible; we'll return to this in the sequel.

It turns out that Definition 2.1 has a very nice interpretation in terms of the matrix quadruple (A, B, C, D) .

Theorem 2.1 ([2]). A linear system (1) is positive if and only if the matrix A is a Metzler matrix, and B , C , and D are nonnegative matrices.

Next we introduce compartmental systems.

A compartmental system consists of n interconnected compartments or reservoirs, see Figure 1 for a model of one reservoir.

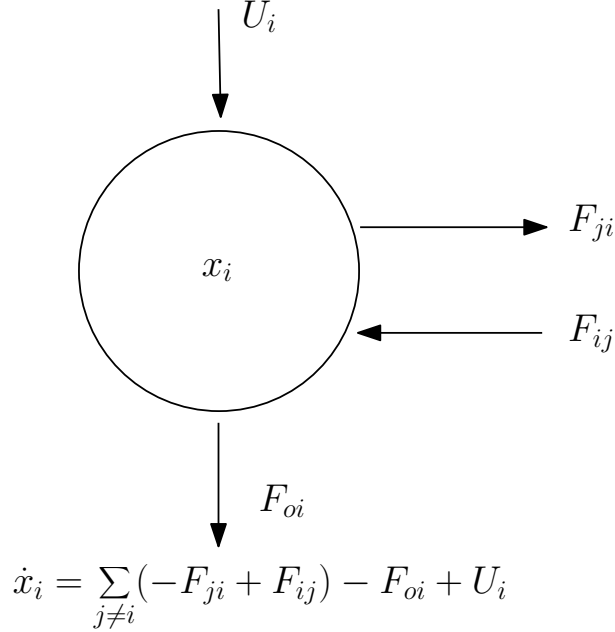


Figure 1: Model of one compartment.

In this figure, x_i represents the current state of reservoir i , U_i is the external inflow coming into reservoir i ; F_{oi} is the flow exiting reservoir i , hence the notation to the outside (o) from reservoir (i) is used; F_{ji} is the outflow into reservoir j from reservoir i ; F_{ij} is the inflow into reservoir i from reservoir j . The assumptions made for linear compartmental systems are that the variables just described satisfy:

$$\begin{aligned} U_i &= b_{i1}u_1 + b_{i2}u_2 + \cdots + b_{im}u_m \\ F_{oi} &= \beta_{oi}x_i \\ F_{ji} &= \beta_{ji}x_i \\ F_{ij} &= \beta_{ij}x_j. \end{aligned} \tag{2}$$

where $u_i \in \mathbb{R}$, $i = 1, \dots, m$ denote the inputs that compose U_i .

Note that in a true compartmental system, by definition, all variables are nonnegative, i.e.

$$x_i, U, F_{oi}, F_{ij}, F_{ji} \in \mathbb{R}_+ \text{ or } b_{i1}, u_1, b_{i2}, u_2, \dots, b_{im}, u_m, \beta_{io}, \beta_{ij}, \beta_{ji} \in \mathbb{R}_+.$$

See [21] for a more in depth treatment of (2).

With the above description of one compartment, we can easily come up with the entire state space model for an overall system consisting of n interconnected compartments:

$$\begin{aligned} \dot{x}_i = & -(\beta_{oi} + \sum_{j \neq i} \beta_{ji})x_i + \sum_{j \neq i} \beta_{ij}x_j \\ & + b_{i1}u_1 + b_{i2}u_2 + \dots + b_{im}u_m. \end{aligned} \quad (3)$$

Setting

$$\alpha_i = (\beta_{oi} + \sum_{j \neq i} \beta_{ji}), \quad (4)$$

results in

$$\begin{aligned} \dot{x}_i = & \begin{bmatrix} -\alpha_1 & \beta_{12} & \beta_{13} & \dots & \beta_{1n} \\ \beta_{21} & -\alpha_2 & \beta_{23} & \dots & \beta_{2n} \\ & \vdots & & & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \dots & -\alpha_n \end{bmatrix} x \\ & + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} u, \end{aligned} \quad (5)$$

where $x = [x_1 \ x_2 \ \dots \ x_n]^T$ and $u = [u_1 \ u_2 \ \dots \ u_m]^T$. For convenience, a compartmental matrix will be denoted by

$$A_c = \begin{bmatrix} -\alpha_1 & \beta_{12} & \beta_{13} & \dots & \beta_{1n} \\ \beta_{21} & -\alpha_2 & \beta_{23} & \dots & \beta_{2n} \\ & \vdots & & & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \dots & -\alpha_n \end{bmatrix} \quad (6)$$

and a compartmental B matrix will be denoted by

$$B_c = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix}$$

as was done for positive linear systems. Notice that, by definition, the summation of all elements within a column of A is less than or equal to zero, i.e.

$$-\alpha_i + \sum_{j \neq i} \beta_{ji} = -\beta_{oi} \leq 0,$$

where equality holds if there is no outflow lost to the outside environment. Note that any matrix (6) that is strictly compartmental is stable due to its structure and Gerschgorin's Theorem.

Before we proceed, let's make a distinction, in the control systems sense, between *inflow*, *outflow* and *output*. From control theory, we are used to the term *output* to signify a measurement that can be obtained from our system in question, normally in the linear system sense, designated as $y = Cx + Du$. When dealing with compartmental systems, the term *inflow* designates the movement of material¹ into the system, the term *outflow* designates the movement of material out of the system, and the *outputs* of the system are measurements on some compartment or a combination of compartments, and may have nothing to do with the material outflows from the system. The above clarification has been nicely captured in [21].

With the above paragraph in mind, the output equation for compartmental systems

$$y = Cx + Du$$

can be arbitrarily specified; however, to satisfy our positive linear system definition we'll assume $C \in \mathbb{R}_+^{r \times n}$ and assume $D \in \mathbb{R}_+^{r \times m}$.

Finally, the above description of a compartmental system is not unique; in fact there are other

¹"material" here can designate anything; for example, liquid, voltage, current, hormones, glucose, etc.

descriptions in the literature, see for example [2], [21]. One common addition made to the above description, pointed out in [2], is that the summation of a column of the matrix B_c must equal one, i.e.

$$\sum_{j=1}^n b_{ji} = 1, \quad \forall i = 1, \dots, m,$$

where each $0 \leq b_{ji} \leq 1$. This assumption is very natural to make due to the fact that one expects the maximum inflow not to exceed u_i , i.e.

$$\sum_{j=1}^n b_{ji} u_i \leq u_i \quad \forall i = 1, \dots, m,$$

2.3 "Almost" Positive Systems

In this section, we introduce four definitions: *state positive linear systems*, *output positive linear systems*, *almost state positive linear systems*, and *almost output positive linear systems*.

In real life systems, nonnegativity of states occurs quite often; however, the need for the input u to be also nonnegative is not a necessity, as was also pointed out in [15]. For this reason we introduce the following definitions.

Definition 2.2. An arbitrary linear system is considered to be a *state positive linear system* if for a given nonnegative initial state x_0 and for a given input u , the state of the system remains nonnegative for all time.

An equivalent definition can be given for *output positive linear systems*.

Definition 2.3. An arbitrary linear system is considered to be an *output positive linear system* if for a given nonnegative initial state x_0 and for a given input u , the output of the system remains nonnegative for all time.

Next, we state two definitions that will lead us to the concept of *almost* positivity.

Definition 2.4. An arbitrary linear system is considered to be a δ -*state positive linear system* with respect to x_0 and u if for a given $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{R}_+^n \setminus \{0\}$, the state x of the system satisfies

$$x_i(t) \geq -\delta_i, \quad \forall i = 1, 2, \dots, n, \quad \forall t \in [0, \infty)$$

An equivalent definition can be given for δ -output positive linear system.

Definition 2.5. An arbitrary linear system is considered to be a δ -output positive linear system with respect to x_0 and u if for a given $\delta = (\delta_1, \delta_2, \dots, \delta_r) \in \mathbb{R}_+^r \setminus \{0\}$, the output y of the system satisfies

$$y_i(t) \geq -\delta_i, \quad \forall i = 1, 2, \dots, r, \quad \forall t \in [0, \infty)$$

In both definitions 2.4 and 2.5 there is no restriction on how small or large the components of δ should be. It will be of interest, however, to study the case when each component $\delta_i \rightarrow 0$, for all i .

Definition 2.6. An arbitrary linear system is considered to be a almost-state positive linear system with respect to x_0 and u if for any given $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{R}_+^n \setminus \{0\}$, the state x of the system satisfies

$$x_i(t) \geq -\delta_i, \quad \forall i = 1, 2, \dots, n, \quad \forall t \in [0, \infty)$$

In similar fashion to Definition 2.5, we can define almost-output positive linear systems.

Definition 2.7. An arbitrary linear system is considered to be a almost-output positive linear system with respect to x_0 and u if for any given $\delta = (\delta_1, \delta_2, \dots, \delta_r) \in \mathbb{R}_+^r \setminus \{0\}$, the output y of the system satisfies

$$y_i(t) \geq -\delta_i, \quad \forall i = 1, 2, \dots, r, \quad \forall t \in [0, \infty)$$

It is worth pointing out that in a practical setting, and even in the theoretical one, the notion of the number zero has been often questioned, i.e. when is a number small enough to consider it to be zero; with the above remark in mind it would appear that if δ can be made arbitrarily small, then one could argue that any almost state (output) positive system is actually a state (output) positive system.

2.4 Tuning Regulators

In this section we describe a particular compensator, known as the tuning controller or tuning regulator, which solves the tracking problem for *unknown*² stable linear systems under constant disturbances. The results of this section can be found in their entirety and in their general form in [22, 23]. The tuning regulator described within this subsection is nothing more but a generalization of the classical "on-line tuning" controller [24].

Consider the plant

$$\begin{aligned} \dot{x} &= Ax + Bu + E\omega \\ y &= Cx + Du + F\omega \\ e &:= y_{ref} - y \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, the disturbance vector $\omega \in \Omega \subset \mathbb{R}^{\bar{\Omega}}$, and $y_{ref} \in Y_{ref} \subset \mathbb{R}^r$ is a desired tracking signal. Assume that the output y is measurable, that the matrix A is Hurwitz, and that the disturbance vector and tracking signal satisfy:

$$\begin{aligned} \dot{z}_1 &= \mathcal{A}_1 z_1 \\ \omega &= \mathcal{C}_1 z_1, \end{aligned}$$

and

$$\begin{aligned} \dot{z}_2 &= \mathcal{A}_2 z_2 \\ \sigma &= \mathcal{C}_2 z_2 \\ y_{ref} &= G\sigma, \end{aligned}$$

respectively, where $z_i \in \mathbb{R}^{n_i}$, $(\mathcal{C}_i, \mathcal{A}_i)$ is observable for $i = 1, 2$, $z_1(0)$ may or may not be known, and $z_2(0)$ is known. Then in the case of constant disturbances (ω) and constant tracking (y_{ref}) signals, the tuning regulator that solves the "robust control of a general servomechanism problem", i.e.

²by unknown we mean that there is no knowledge of (A, B, C, D)

such that

- (i) the closed loop system is stable,
- (ii) for all tracking signals and disturbances $e \rightarrow 0$ as $t \rightarrow \infty$, and
- (iii) property (ii) occurs for all plant perturbations which maintain closed loop stability,

is given by:

$$\begin{aligned}\dot{\eta} &= \epsilon e \\ u &= (D - CA^{-1}B)^\dagger \eta,\end{aligned}\tag{8}$$

where $(\cdot)^\dagger$ represents the pseudo-inverse³ of (\cdot) , $\epsilon \in (0, \epsilon^*]$, $\epsilon^* \in \mathbb{R}_+ \setminus \{0\}$, and we assume that the initial condition $\eta_0 = 0$. The latter assumption of η_0 is justified by the study of perfect control, see [26]. The tuning regulator just presented serves the same purpose as the one given in [27]; however, its present form, as will be shown in the sequel, will allow us to encompass the problem within a singular perturbation framework.

We summarize the above discussion by a Theorem for the case of MIMO linear systems.

Theorem 2.2 ([22]). Consider the system (7), under the assumption that $y_{ref} \in \mathbb{R}^r$ and $\omega \in \mathbb{R}^{\bar{\Omega}}$ are constants. Then there exists an ϵ^* such that $\forall \epsilon \in (0, \epsilon^*]$ the tuning regulator (8) achieves robust control of a general servomechanism problem if and only if $rank(D - CA^{-1}B) = r$.

Remark 2.1. It is worth noting that in order to obtain the gain matrix

$$(D - CA^{-1}B)^\dagger$$

without the knowledge of (A, B, C, D) we only need the knowledge of the steady state values of the outputs and the ability to excite the inputs. For a complete procedure of finding the gain matrix we refer the interested reader to [22].

³for a discussion on the pseudo-inverse one can consult [25]

3 Problem Statement

In this section, we provide the details of the problem of interest with all the accompanying assumptions.

In this paper, we consider the following problem:

Problem 3.1. Consider the plant (7), where A is a stable Metzler matrix, and B , C , D are nonnegative matrices. Assume that $\text{rank}(D - CA^{-1}B) = r$ and that the sets Ω and Y_{ref} are chosen such that $E\omega \in \mathbb{R}_+^n$ and $F\omega \in \mathbb{R}_+^r$, and that the steady state values of the plant's states are nonnegative, i.e. for all tracking and disturbance signals in question, it's assumed that the steady-state of the system (7) is given by

$$-\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} E & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix} \quad (9)$$

and has the property that $x_{ss} \in \mathbb{R}_+^n$, $y_{ss} = Cx_{ss} + Du_{ss} + F\omega = y_{ref} \in \mathbb{R}_+^r$. It is to be noted that a solution to (9) exists if and only if $\text{rank}(D - (CA^{-1}B)) = r$.

Then, find a controller u that

- (a) guarantees closed loop stability,
- (b) ensures tracking of the reference signals, i.e. $e = y - y_{ref} \rightarrow 0$, as $t \rightarrow \infty$, $\forall y_{ref} \in Y_{ref}$ and $\forall \omega \in \Omega$, and
- (c) ensures the plant (7) can be made an *almost state* and an *almost output* positive system.
- (d) For all perturbations of the nominal plant model, which do not de-stabilize the open loop system and do not change the positivity of the system (i.e. A is still Metzler and B , E , C , $F\omega$, $E\omega$ are still nonnegative matrices), it is desired that the controller can still achieve asymptotic error regulation and *almost state* and *almost output* positivity, i.e. properties (a), (b) and (c) should still be attainable if the system model is unknown.

Remark 3.1. A remark is in order with respect to one of the assumptions; namely, the steady-state equation (9) is both necessary and sufficient when $m = r$ as we can solve it directly, i.e.

$$\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = - \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}^{-1} \begin{bmatrix} E & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix};$$

however, if $m > r$ equation (9) may only be sufficient. Thus, to also show necessity one can just solve the linear feasibility programming problem

$$\begin{aligned} -A^{-1}Bu_{ss} &\leq A^{-1}E\omega \\ (D - CA^{-1}B)u_{ss} &= (CA^{-1}E - F)\omega + y_{ref}, \end{aligned}$$

for u_{ss} . The existence of u_{ss} is both necessary and sufficient for the existence of an $x_{ss} \in \mathbb{R}_+^n$. The above linear programming problem results from manipulating equation (9).

To show that equation (9) is indeed only sufficient, an example is provided.

Example 3.1. Consider the following system

$$\begin{aligned} \dot{x} &= -x + [1 \ 1]u \\ y &= x + [1 \ 3]u \end{aligned}$$

with $y_{ref} = 1$. Notice that with $u_{ss} = [-1 \ 0.75]$ the steady state of $x_{ss} = -0.25$, which would not satisfy the assumption of Problem 3.1, yet there exists a solution to the above linear feasibility programming problem, and hence the assumptions of Problem 3.1 are met. That is, from above we obtain

$$\begin{aligned} [1 \ 1]u_{ss} &\leq 0 \\ [2 \ 4]u_{ss} &= 1 \end{aligned}$$

which can be satisfied with $u_{ss} = [-0.5 \ 0.5]^T$, for example.

In Section 5, we will solve the above problem with the tuning regulator that was introduced in Section 2.4.

4 Singular Perturbation

This section has been added for completeness and covers singular perturbation results needed in order to prove the main results of this paper. The following discussion has been taken from [29], Chapter 11 and Chapter 4.

The standard singular perturbation model can be described as

$$\begin{aligned} \dot{q} &= f(t, q, z, \epsilon), & q(t_0) &= q_0 \\ \epsilon \dot{z} &= g(t, q, z, \epsilon), & z(t_0) &= z_0 \end{aligned} \tag{10}$$

where the functions f and g are continuously differentiable in their arguments $(t, q, z, \epsilon) \in [0, \infty) \times D_q \times D_z \times [0, \epsilon_0]$, with $D_q \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^s$ being open and connected sets. By setting $\epsilon = 0$, we obtain

$$0 = g(t, q, z, 0), \tag{11}$$

where we designate the real root ⁴ of (11) as

$$z = h(t, q). \tag{12}$$

To obtain a reduced model, we substitute (12) into (10) resulting in

$$\dot{q} = f(t, q, h(t, q), 0), \quad q(t_0) = q_0. \tag{13}$$

The reduced model is sometimes referred to as the *slow* model, while (11) is referred to as the *quasi-steady-state model*, because z may rapidly converge to a root of (11).

⁴without loss of generality, we assume there is only one root

Now denote the solution of (13) by $\bar{q}(t)$ and define

$$\bar{z}(t) = h(t, \bar{q}(t)),$$

which describes the quasi-steady-state behavior of z when $q = \bar{q}$.

In order to present a very important result on singular perturbations, we need to perform a change of variables first

$$p = z - h(t, q), \tag{14}$$

which shifts the quasi-steady-state of z to the origin. In the new variables (q, p) the full problem is

$$\begin{aligned} \dot{q} &= f(t, q, p + h(t, q), \epsilon), & q(t_0) &= q_0 \\ \epsilon \dot{p} &= g(t, q, p + h(t, q), \epsilon) - \epsilon \frac{\partial h}{\partial t} \\ &\quad - \epsilon \frac{\partial h}{\partial q} f(t, q, p + h(t, q), \epsilon), \\ p(t_0) &= z_0 - h(t_0, q_0) \end{aligned} \tag{15}$$

Next, we set

$$\epsilon \frac{dp}{dt} = \frac{dp}{d\tau}; \quad \text{hence, } \frac{d\tau}{dt} = \frac{1}{\epsilon}$$

and use $\tau = 0$ as the initial value at $t = t_0$. In the new time scale, (15) becomes

$$\begin{aligned} \dot{q} &= f(t, q, p + h(t, q), \epsilon), & q(t_0) &= q_0 \\ \frac{dp}{d\tau} &= g(t, q, p + h(t, q), \epsilon) - \epsilon \frac{\partial h}{\partial t} \\ &\quad - \epsilon \frac{\partial h}{\partial q} f(t, q, p + h(t, q), \epsilon), \\ p(t_0) &= z_0 - h(t_0, q_0) \end{aligned} \tag{16}$$

By setting $\epsilon = 0$, the latter equation reduces to

$$\frac{dp}{d\tau} = g(t, q, p + h(t, q), 0), \quad p(t_0) = z_0 - h(t_0, q_0), \tag{17}$$

which is commonly referred to as *boundary-layer model*.

We will also make use of the autonomous system

$$\frac{dp}{d\tau} = g(t_0, q_0, p + h(t_0, q_0), 0), \quad p(t_0) = z_0 - h(t_0, q_0) \quad (18)$$

which has an equilibrium at $p = 0$, and has been derived from (17) by setting $t = t_0$ and $q = q_0$. Define the solution of (18) as $\hat{p}(\tau)$.

Before we state the singular perturbation result on an infinite interval of time, we recall the following theorem on Lyapunov stability:

Theorem 4.1 ([29] pg.152). Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(t, x)$$

and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \quad (19)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \quad (20)$$

$\forall t \geq 0$ and $\forall x \in D$, where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D . Then, $x = 0$ is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < \min_{\|x\|=1} W_1(x)$, then every trajectory starting in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies

$$\|x\| \leq \beta(\|x(t_0)\|, t - t_0), \quad t \geq t_0 \geq 0$$

for some class KL function⁵ β .

The following theorem presents the singular perturbation result of interest in this paper.

⁵see [29] for an overview of class KL functions

Theorem 4.2 ([29] pg.439). Consider the singular perturbation problem of (10). Assume that the following conditions are satisfied for all

$$[t, q, z - h(t, q), \epsilon] \in [0, \infty) \times D_q \times D_p \times [0, \epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^s$, which contain their respective origins:

1. On any compact subset of $D_x \times D_y$, the functions f, g , their first partial derivatives with respect to (q, z, ϵ) , and the first partial derivative of g with respect to t are continuous and bounded, $h(t, q)$ and $[\partial g(t, q, z, 0)/\partial z]$ have bounded first partial derivatives with respect to their arguments, and $[\partial f(t, q, h(t, q), 0)/\partial q]$ is Lipschitz in q , uniformly in t ;
2. the origin is an exponentially stable equilibrium point of the reduced system (13); i.e. there is a Lyapunov function $V(t, x)$ that satisfies the conditions of Theorem 4.1 for (13) for $(t, q) \in [0, \infty) \times D_q$ and $\{W_1(q) \leq c\}$ is a compact subset of D_q ;
3. the origin is an exponentially stable equilibrium point of the boundary-layer model (17), uniformly in (t, q) . Let $\mathcal{R}_p \subset D_p$ be the region of attraction of (18) and Γ_y be a compact subset of \mathcal{R}_y .

Then, for each compact set $\Gamma_q \subset \{W_2(x) \leq \xi c, 0 < \xi < 1\}$ there is a positive constant ϵ_1 such that for all $t_0 \geq 0$, $q_0 \in \Gamma_q$, $z_0 - h(t_0, q_0) \in \Gamma_p$, and $0 < \epsilon < \epsilon_1$, the singular perturbation problem (15) has a unique solution $q(t, \epsilon), z(t, \epsilon)$ on $[t_0, \infty)$, and

$$q(t, \epsilon) - \bar{q}(t, \epsilon) = O(\epsilon)$$

$$z(t, \epsilon) - h(t, \bar{q}(t)) - \hat{p}(\tau) = O(\epsilon)$$

We will also make use of the standard theorem on the continuity of solutions in terms of parameters, which we recall below.

Theorem 4.3 ([29] pg.97). Let $f(t, x, \lambda)$ be continuous in (t, x, λ) and locally Lipschitz in x (uniformly in t and λ) on $[t_0, t_1] \times D \times \{\|\lambda - \lambda_0\| \leq c\}$, where $D \subset \mathbb{R}^n$ is an open connected set. Let $y(t, \lambda_0)$

be a solution of $\dot{x} = f(t, x, \lambda_0)$ with $y(t_0, \lambda_0) = y_0 \in D$. Suppose $y(t, \lambda_0)$ is defined and belongs to D for all $t \in [t_0, t_1]$. Then, given $\epsilon_2 > 0$, there is a $\delta > 0$ such that if

$$\|z_0 - y_0\| < \delta \text{ and } \|\lambda - \lambda_0\| < \delta$$

then there is a unique solution $z(t, \lambda)$ of $\dot{x} = f(t, x, \lambda)$ defined on $[t_0, t_1]$, with $z(t_0, \lambda) = z_0$, and $z(t, \lambda)$ satisfies

$$\|z(t, \lambda) - y(t, \lambda_0)\| < \epsilon_2, \quad \forall t \in [t_0, t_1]$$

5 Main Result

In this section, we outline the solution to Problem 3.1. In particular, we show that the tuning regulator presented in Section 2.4 will suffice. Thus, we encompass the main result of this paper as a theorem, which is presented next.

Theorem 5.1. The tuning regulator (8) solves Problem 3.1.

Proof. In order to show that the tuning regulator (8) solves Problem 3.1, we must show that under the given assumptions all four conditions of Problem 3.1 hold. We'll show this in a sequential form.

(i) With the assumption that $\text{rank}(D - CA^{-1}B) = r$, we can apply Theorem 2.2 which will guarantee (a) closed loop stability and (b) ensure tracking of the reference signals, i.e. $e = y - y_{ref} \rightarrow 0$, as $t \rightarrow \infty$.

(ii) To ensure (c), i.e. that the plant (7) can be made an *almost state* and an *almost output* positive system, we must show that for any given $\delta > 0$ Definition 2.4 and Definition 2.5 hold true. In order to prove the latter, we can break down the proof into three parts, i.e. we will show that

(1) there exists an ϵ_1 and a time $t_{O(\epsilon)}$ such that for all $t \geq t_{O(\epsilon)}$

$$\|x(t) - \bar{x}(t)\| = O(\epsilon) \Rightarrow |x_i(t) - \bar{x}_i(t)| \leq \delta_i,$$

$$\|y(t) - \bar{y}(t)\| = O(\epsilon) \Rightarrow |y_j(t) - \bar{y}_j(t)| \leq \delta_j,$$

$\forall i = 1, \dots, n, \forall j = 1, \dots, r$ where $\bar{x}(\bar{y})$ monotonically approaches a nonnegative value for $\epsilon \in (0, \epsilon_1]$; and

(2) that there exists an ϵ_2 such that for all time $t \in [0, t_{O(\epsilon)}]$

$$x_i(t) \geq -\delta_i, \quad \forall i = 1, \dots, n$$

(3) finally by choosing $\epsilon_\delta = \min[\epsilon_1 \epsilon_2]$ the result follows.

Point (2) and (3) are clear, with (2) being a consequence of Theorem 4.3; thus, it only leaves (1) to prove. In order to show (1), we will use the results of Theorem 4.2. The closed loop system with the tuning regulator in place is of the form:

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & BK \\ -\epsilon C & -\epsilon DK \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} E & 0 \\ -\epsilon F & \epsilon I \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix}$$

where $K = (D - CA^{-1}B)^\dagger$. First, let's show that the equilibrium of the closed loop system is independent of ϵ . This is easily seen by noticing

$$\begin{aligned} \dot{\eta} = 0 &= \epsilon(-Cx - DK\eta - F\omega + y_{ref}) \\ &= -Cx - DK\eta - F\omega + y_{ref} \end{aligned}$$

Now, since the equilibrium (x_{ss}, η_{ss}) is independent of ϵ and invariant (note that this was not the case in [27]), we can transform the system as needed:

$$\begin{bmatrix} z \\ q \end{bmatrix} = \begin{bmatrix} x \\ \eta \end{bmatrix} - \begin{bmatrix} x_{ss} \\ \eta_{ss} \end{bmatrix}$$

resulting in the new system

$$\begin{bmatrix} \dot{z} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & BK \\ -\epsilon C & -\epsilon DK \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix}. \quad (21)$$

For convenience, rewrite (21) as

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\epsilon DK & -\epsilon C \\ BK & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}. \quad (22)$$

Next, let's scale the derivatives (i.e. scaling of time) by

$$\epsilon dt = d\tau$$

resulting in the transformed system

$$\begin{bmatrix} \overset{\circ}{q} \\ \epsilon \overset{\circ}{z} \end{bmatrix} = \begin{bmatrix} -DK & -C \\ BK & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \quad (23)$$

with $\epsilon \overset{\circ}{q} = \dot{q}$ and $\epsilon \overset{\circ}{z} = \dot{z}$. We have now transformed our model into that of the singular perturbation model (10). In order to use Theorem 4.2, we must show that all assumptions of Theorem 4.2 hold true. However, as (23) is linear and time invariant it suffices to show that the reduced model and the boundary layer model indeed yield exponential stability; all other assumptions clearly hold. Let's begin with the reduced model. By setting $\epsilon = 0$ we obtain

$$z = h(q) = -A^{-1}BKq,$$

as A is Hurwitz, $h(q)$ exists and is unique. Next by substituting $h(q)$ into $\overset{\circ}{q}$ we obtain the

reduced model:

$$\begin{aligned}
\overset{\circ}{q} &= -DKq + CA^{-1}BKq \\
&= -(D - CA^{-1}B)Kq \\
&= -(D - CA^{-1}B)(D - CA^{-1}B)^\dagger q \\
&= -q,
\end{aligned}$$

clearly exponentially stable. Proceeding to the boundary layer model, we obtain:

$$\begin{aligned}
p &= z - h(q) \\
p &= z + A^{-1}BKq \\
\dot{p} &= \dot{z} + A^{-1}BK\dot{q} \\
\dot{p} &= Az + BKq + A^{-1}BK(-\epsilon Cz - \epsilon DKq) \\
\dot{p}|_{\epsilon=0} &= A(p - A^{-1}BKq) + BKq \\
\dot{p}|_{\epsilon=0} &= Ap,
\end{aligned}$$

now since A is Hurwitz, the boundary layer model must be exponentially stable, as desired⁶.

Next, from the boundary layer model

$$\dot{p}|_{\epsilon=0} = Ap$$

we can obtain

$$\begin{aligned}
\hat{p} &= e^{At}p_0 \\
&= e^{At}(z_0 + A^{-1}BKq_0).
\end{aligned}$$

⁶note that in our proof and Section 4, we have swapped the time variables t and τ

Now, since we have met all the assumptions of Theorem 4.2 we can conclude that

$$z - \bar{z} - \hat{p} = O(\epsilon) \quad \forall \tau \geq 0 \Rightarrow \forall t \geq 0$$

which implies that there exists a time $t_{O(\epsilon)}$ such that

$$z - \bar{z} = O(\epsilon) \quad \forall t \geq t_{O(\epsilon)}$$

i.e.

$$x - \bar{x} = O(\epsilon) \quad \forall t \geq t_{O(\epsilon)},$$

since \hat{p} decays exponentially independent of ϵ . Note that \bar{x} is monotonic and since $x_{ss} \in \mathbb{R}_+^n$, the trajectory \bar{x} will tend toward a nonnegative value monotonically. The result for y follows the same argument. This completes the proof of (1).

- (d) Finally, since we can assume that there is no knowledge of the system matrices in constructing of the servo-compensator (see Section 2.4), the result for properties (a), (b) and (c) of Problem 3.1 clearly holds despite an unknown system model.

By setting $\epsilon^* = \epsilon_\delta$ in Theorem 2.2 we have found our control u . Now since δ is arbitrary the result follows. □

6 Examples

In this section, we present several examples which illustrate the results of the paper.

Example 6.1. Consider the two-input two-output compartmental system:

$$\dot{x} = \begin{bmatrix} -1 & 0.5 & 0.1 \\ 0.1 & -1 & 0.2 \\ 0.3 & 0.1 & -1 \end{bmatrix} x + \begin{bmatrix} 0.2 & 0.5 \\ 0.7 & 0.3 \\ 0.1 & 0.2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x.$$

Let $y_{ref} = [1 \ 2]^T$ and $\omega = 0$. With the given reference and disturbance, the assumption of nonnegative steady-state and stability indeed hold true:

$$x_{ss} = [1 \ 2 \ 0.541]^T \quad \sigma(A) = \{-0.583 \ -1.209 \pm 0.175i\}.$$

Using the servo compensator presented in Section 2.4 we obtain the closed loop system:

$$\dot{\tilde{x}} = \begin{bmatrix} -1 & 0.5 & 0.1 & -0.935 & 0.495 \\ 0.1 & -1 & 0.2 & 0.229 & -1.011 \\ 0.3 & 0.1 & -1 & -0.347 & 0.153 \\ -\epsilon & 0 & 0 & 0 & 0 \\ 0 & -\epsilon & 0 & 0 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \epsilon \\ 2\epsilon \end{bmatrix} y_{ref}$$

with $\tilde{x} = [x \ \eta]^T$. The simulated result for the states, with $\epsilon = 0.1$, is shown in Figure 2. For this system, regardless of ϵ , x_1 initially always dips below zero. However, as $\epsilon \rightarrow 0$ so does δ . This is shown in Figure 3.

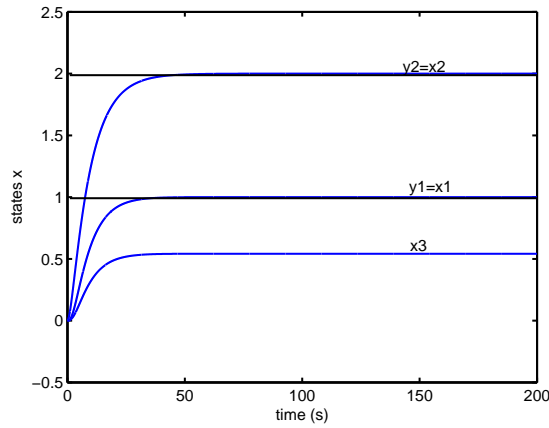


Figure 2: Total response of the states x for Example 6.1.

The systems under consideration in the next two examples have been taken from the literature.

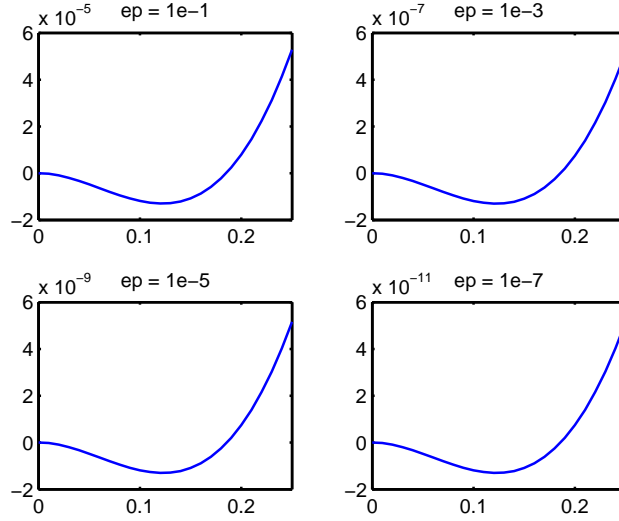


Figure 3: Depiction of $\epsilon \rightarrow 0$ for x_1 in Example 6.1.

The first example is of an electrically heated oven and the second is of a fluid 6-compartmental system.

Example 6.2. The system considered in this example has been taken from [30].

The interior temperature of an electrically heated oven is to be controlled by varying the heat input u to the jacket. Let the heat capacities of the oven interior and of the jacket be c_2 and c_1 , respectively, let the interior and exterior jacket surface areas be a_1 and a_2 , and let the radiation coefficient of the interior and exterior jacket surfaces be r_1 and r_2 . If the external temperature is T_0 , the jacket temperature T_1 and the oven interior temperature is T_2 , then the behaviour for the jacket is described by:

$$c_1 \dot{T}_1 = -a_2 r_2 (T_1 - T_0) - a_1 r_1 (T_1 - T_2) + u$$

and for the oven interior:

$$c_2 \dot{T}_2 = a_1 r_1 (T_1 - T_2)$$

By setting the state variables to be the excess of temperature over the exterior

$$x_1 := T_1 - T_0$$

$$x_2 := T_2 - T_0$$

results in the system:

$$\dot{x} = \begin{bmatrix} \frac{-(a_2 r_2 + a_1 r_1)}{c_1} & \begin{pmatrix} a_1 r_1 \\ c_1 \end{pmatrix} \\ \begin{pmatrix} a_1 r_1 \\ c_2 \end{pmatrix} & \begin{pmatrix} -a_1 r_1 \\ c_2 \end{pmatrix} \end{bmatrix} x + \begin{bmatrix} 1/c_1 \\ 0 \end{bmatrix} u.$$

Assume that the values of the constants above are chosen such that $c_1 = c_2 = 1$, $a_1 = a_2 = 1$ and $r_1 = r_2 = 1$, and that the disturbance vectors are $E = [1 \ 1]^T$ and $F = 0$ with $\omega = 1$ then:

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \omega$$

and the output equation is

$$y = [1 \ 0]x.$$

We will now show that if the temperature excess in the jacket is initially zero, then we can increase arbitrarily x_1 and maintain nonnegativity of the states in the process, using the tuning regulator. In this example we'll assume that we want to regulate y to $y_{ref} = 5$, i.e. the desired temperature excess is 5°C higher in the jacket than the outside.

First, it is easy to show that the A matrix is stable, i.e. $\sigma(A) = \{-2.618, -0.382\}$, and that the assumptions of Problem 3.1 hold, $x_{ss} \geq 0$. In this case, the application of controller (8) with $\epsilon = 0.1$ will suffice for any δ (even for $\delta = 0$). Figure 4 and Figure 5 illustrate the states x and the input u .

Example 6.3. The following system, which is compartmental, has been taken from [2] pg.105. This example additionally possesses a random set of constant disturbances and an additional input and output. Consider the reservoirs network of Figure 6; note that each reservoir is identified by a number (1, 2, ..., 6) where the water storage level (x_1, x_2, \dots, x_6) is a state of the system. Also γ and ϕ are the splitting coefficients of the flows at the branching points. The system is of order 6, as we assume the pump dynamics can be neglected. As pointed out in [2], the dynamics of each

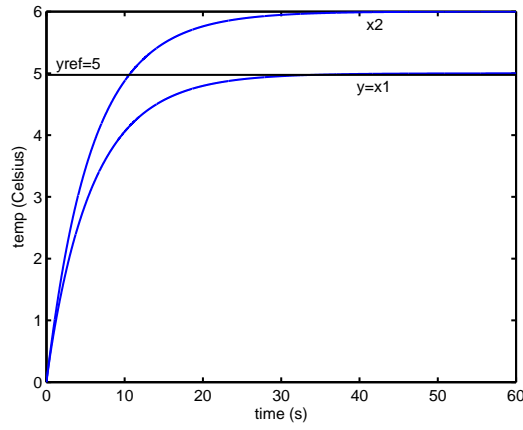


Figure 4: State response for Example 6.2.

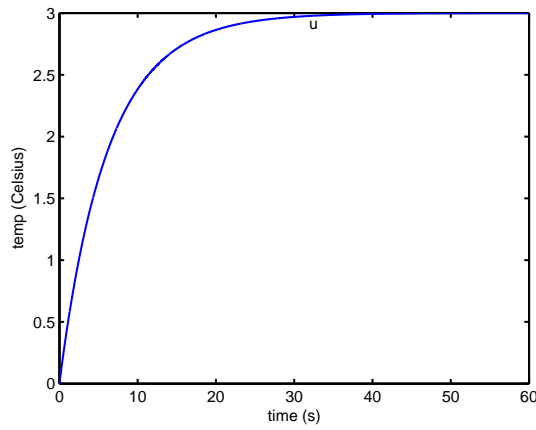


Figure 5: Input response for Example 6.2.

reservoir can be captured by a single differential equation:

$$\dot{x}_i = -\alpha_i x_i + v$$

$$z = \alpha_i x_i$$

for all $i = 1, \dots, 6$, with $\alpha > 0$, and where x_i is the water storage (assume in L) and α is the ratio between outflow rate z and storage. The input into the reservoir is designated by v (L/s).

Consider the case where $\gamma = 0.5$, $\phi = 0.7$, $\alpha_1 = 0.8$, $\alpha_2 = 0.7$, $\alpha_3 = 0.5$, $\alpha_4 = 1$, $\alpha_5 = 2$, and $\alpha_6 = 0.8$. Also, the rates are measured in L/s . This results in the following system:

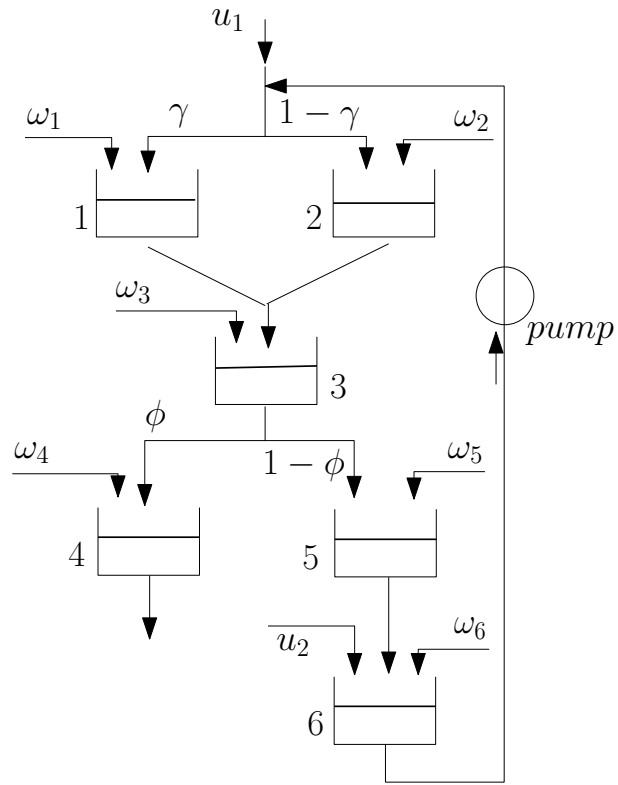


Figure 6: System set up for Example 6.3.

$$\dot{x} = \begin{bmatrix} -0.8 & 0 & 0 & 0 & 2 & 0 \\ 0 & -0.7 & 0 & 0 & 0 & 0 \\ 0.8 & 0.7 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0.35 & 0 & 0 & -0.8 \end{bmatrix} x$$

$$+ \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + [0.25 \ 0.3 \ 0.15 \ 0.35 \ 0.1 \ 0.05]^T \omega$$

Also, assume the output y is of the form

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} x$$

It is easy to verify that the above compartmental system is stable, as

$$\sigma(A) = \{-0.8, -0.2112, -0.9924 \pm 0.5249i, -2.1039, -0.7000\}$$

and

$$x_{ss} = [1.9167 \ 0.6905 \ 4.3333 \ 1.0000 \ 0.5500 \ 0.4500]^T.$$

Assume now that the initial condition $x_{i0} = 0 \ \forall i = 1, \dots, 6$, $\omega = 1$. Additionally, assume that we would like to track the reference input $y_{ref} = [1 \ 1]^T$. The assumptions of Problem 3.1 all hold, thus we can proceed to use the result of the previous section. The application of controller (8) with $\epsilon = 0.1$, solves the tracking problem, even for $\delta_i = 0, \forall i = 1, \dots, 6$. Figure 7 illustrates the simulated state response, while Figure 8 shows the output y .

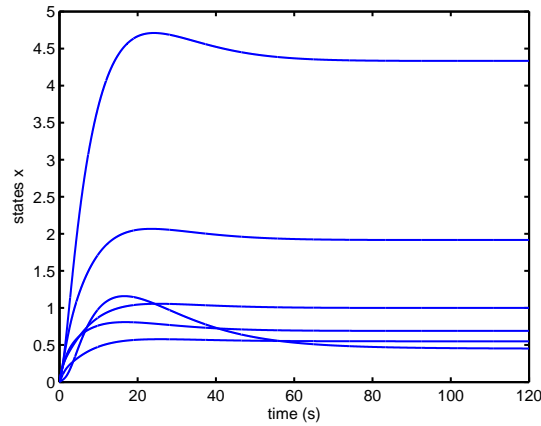


Figure 7: State response for Example 6.3.

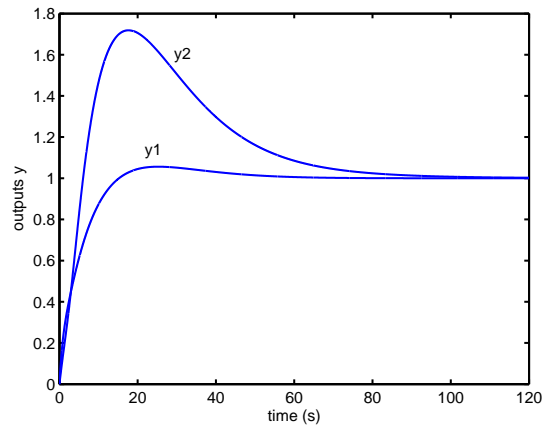


Figure 8: Output response for Example 6.3.

7 Conclusion

In this paper we have discussed a variation of the servomechanism problem for stable unknown MIMO positive linear systems. In particular, we have shown that tracking a nonnegative constant reference is possible while maintaining *almost* state and *almost* output nonnegativity with the use of a tuning regulator!

References

- [1] D. Luenburger, *Introduction to Dynamic Systems: Theory, Models and Applications*, New York:Wiley, 1979.
- [2] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, John Wiley & Sons, Inc, New York 2000
- [3] P.G. Coxon and H. Shapiro, "Positive input reachability and controllability of positive systems", *Linear Algebra and its Applications*, vol.94, 1987, pp.35-53.
- [4] M.P. Fanti, B. Maione, and B. Turchiano, "Controllability of linear single-input positive discrete-time systems", *International Journal of Control*, vol.50, 1989, pp. 2523-2542.

- [5] L. Farina and L. Benvenuti, "Polyhedral reachable set with positive controls", *Math. control Signal Systems*, vol.10, 1997, pp.364-380.
- [6] D.N.P. Murthy, "Controllability of a linear positive dynamic system", *International Journal of Systems Science*, vol. 17, 1986, pp.49-54.
- [7] V.G. Rumchev and D.J.G. James, "Controllability of positive linear discrete-time systems", *International Journal of Control*, vol.50, 1989, 845-857.
- [8] Y. Ohta, H. Maeda, and S. Kodama, "Reachability, observability and realizability of continuous-time positive systems", *SIAM J. Control Optim*, vol.22, 1984, pp.171-180.
- [9] B.D.O. Anderson, M. Deistler, L. Farina and L. Benvenuti, "Nonnegative realization of a linear system with nonnegative impulse respons", *IEEE Trans. Circuits and Syst.-I: Fundamental Theory and Applications*, vol.43, 1996, 134-142.
- [10] L. Benvenuti, L. Farina, B.D.O. Anderson and F. De Bruyne, "Minimal discrete-time positive realizations of transfer functions with positive real poles", *IEEE Trans. Circuits and Syst.-I: Fundamental Theory and Applications*, vol.47, 2000, pp. 13701377.
- [11] L. Benvenuti and L. Farina, "A note on minimality of positive realizations", *IEEE Trans. Circuits and Syst.-I: Fundamental Theory and Applications*, vol.45, 1998, pp. 676677.
- [12] J.M. van den Hof, "Realization of positive linear systems", *Linear Algebra and its Applications*, vol.256, 1997, pp. 287-308.
- [13] T. Kaczorek, "Reachability and controllability of 2D positive linear systems with state feedback", *Control and Cybernetics*, vol.29, 2000, pp.141-151.
- [14] T. Kaczorek, *1D and 2D systems*, Springer, New York, 2002.
- [15] P. De Leenheer and D. Aeyels, "Stabilization of positive linear systems", *Systems and Control Letters*, vol.44, 2001, pp.259-271.
- [16] T. Kaczorek, "Stabilization of Positive Linear Systems", in *Proceedings of the 37th IEEE Conference on Decision and Control*, December 1998, pp.620-621.

- [17] D.J.G. James, S.P. Kostova, and V.G. Rumchev, "Pole assignment for a class of positive linear systems", *International Journal of Systems Science*, vol.32, 2001, 1377-1388.
- [18] A. Berman, M. Neumann, and R.J. Stern, *Nonnegative Matrices in Dynamic Systems*, New York:Wiley 1989.
- [19] V.G. Rumchev and D.J.G James, "Spectral characterization and pole assignment for positive linear discrete-time systems", *International Journal of Systems Science*, vol.26, 1995, 295-312.
- [20] W.P.M.H. Heemels, S.J.L. van Eijninghoven, and A.A. Stoorvogel, "Linear quadratic regulator problem with positive control", *International Journal of Control*, vol.70, 1998, pp.551-578.
- [21] J.A. Jacquez and C.P. Simon, "Qualitative Theory of Compartmental Systems", *Society for Industrial and Applied Mathematics* , vol. 35(1), pp. 43-79, March 1993.
- [22] E.J. Davison, "Multivariable Tuning Regulators: The Feedforward and Robust Control of a General Servomechanism Problem", *IEEE Transactions on Automatic Control*, vol.AC-21, Feb. 1976, pp.35-47.
- [23] D.E. Miller and E.J. Davison, "The Self-Tuning Robust Servomechanism Problem", *IEEE Transactions on Automatic Control*, vol.34, May 1989, pp.511-523.
- [24] F.G. Shinskey, *Process Control Systems*, New York : McGraw-Hill, 1967, p.101.
- [25] S.L. Campbell and C.D. Meyer Jr., *Generalized Inverses of Linear Transformations*, New York, Dover, 1991.
- [26] E.J. Davison and B.M. Scherzinger, "Perfect Control of the Robust Servomechanism Problem", *IEEE Transactions on Automatic Control*, vol.AC-32, August 1987, pp.689-702.
- [27] B. Roszak and E.J. Davison, "Tuning regulators for tracking SISO positive linear systems", *European Control Conference 2007*, 8 pages, submitted October 2006.
- [28] A. Graham, *Nonnegative Matrices and Applicable Topics in Linear Algebra*, Ellis Horwood Limited, New York, 1987.

- [29] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, New Jersey 2002
- [30] S. Barnett and R.G. Cameron, *Introduction to Mathematical Control Theory* 2nd Ed., Clarendon Press, Oxford, 1985.
- [31] G.W. Stewart and Ji-guang Sun, *Matrix Perturbation Theory*, Academic Press, Inc., Boston, 1990.
- [32] D.H. Anderson, *Compartmental Modeling and Tracer Kinetics, Lecture Notes in Biomathematics 50*, Springer-Verlag, New York, 1983.