Synopsis

3L,1T : Wt. 12

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Text:	Notes provided
References:	See Bibliography

This course is an introduction to system and signal analysis via Fourier and Laplace transform methods in the continuous-time setting. Emphasis is on basic principles, but applications from control, communications and signal processing are provided for illustration.

Topics:

1. Fourier Series: Formal computation, Riemann-Lebesgue Lemma, Dirichlet's convergence theorem, Gibbs' phenomenon and Böcher's Theorem, orthogonality, mean-square convergence, Bessel's inequality, Parseval's Theorem, Riesz-Fischer Theorem. Applications to rectification, detection.

2. Fourier Integral: Formal computation, Riemann-Lebesgue Lemma, Fubini's Theorem, convolution, Jordan inversion theorem, inversion techniques, Schwarz inequality, Parseval's Theorem, mean-square convergence, Plancherel's Theorem. Applications to amplitude modulation, time-bandwidth duality, uncertainty principle, Nyquist-Shannon sampling.

3. Laplace Transform: Convolution, inversion integral, inversion techniques, initial-value and final-value theorems, transfer functions, significance of poles and zeros. Applications to ordinary and delay-differential equations with illustrations from control theory.

4. Linear Systems: Impulse and frequency response functions; time-invariance, stability and causality in time and frequency domains.

Tutorials:

These are mainly devoted to discussion of homework assignments, but include problem drills and one or two term tests.

Acknowledgement

I would like to acknowledge the critical advice and helpful comments of Engineering Science students over the years of my teaching this course, as well as the expert typing and editorial assistance of the Systems Control Group secretary, Linda Espeut.

To The Student

These notes have been produced to save you (and me) the trouble of writing everything down in class. The main thing left to do is add some pictures and work out some verifications in detail.

The material is standard, and basic to the engineering disciplines of control, communications and signal processing. For reasons of time, we study only the 'classical', continuous-time theory, leaving discrete-time and digital versions to later courses. You should find those versions relatively easy when you get to them.

The classical theory can be fairly hard. Proofs of the mathematical results are sometimes intricate; however, they can be followed with some effort. You won't be expected to reproduce the proofs, but rather to understand the underlying ideas and gain practice with the methods of calculation that are based on them. The computational exercises should help. Incidentally, our subject lends itself nicely to experimentation on a personal computer, and you are urged to develop your own software package as the course proceeds.

Fourier methods have been around for over 200 years, and the literature is enormous. A few of the books which you might find useful to browse in are listed in the bibliography. The ones I like best are Carslaw, Hamming, Dym-McKean, and Widom.

W.M.W.

96.01.01

Contents

Foreword

This course is devoted to Fourier analysis, Laplace transforms, and some of their applications to the representation and analysis of systems and signals.

The key idea here is *function representation.* A signal or system is described by a mathematical *function.* A given function may be representable in many different ways: e.g. as an infinite series, or a contour integral, or (approximately) as a table of numbers, a computer program, and so on: consider from this point of view the function $f(x) = \sin x$. In analysis, the first objective is to select a function representation which is conveniently matched to the problem at hand. Roughly speaking, Fourier methods are based on the representation of a function as a sum or integral of its 'components' at various frequencies. This kind of representation is very well matched to the problem of specifying the action on our function of *linear* devices, for instance, linear filters or electrical networks. For this reason, Fourier methods are central to the study of control, communications, and signal processing generally.

Chapter 1

Fourier Series

1.1 Introduction

In this course we define a *signal* as a real- or complex-valued function $f = f(t)$ of the real variable t, defined for t in some interval I, where e.g.

$$I = [-\pi, \pi] = \{t : -\pi \leq t \leq \pi\}$$

or

$$I = [0, \infty) = \{t : 0 \leq t < \infty\}$$

or

$$I = (-\infty, \infty) = \{t : -\infty < t < \infty\}.$$

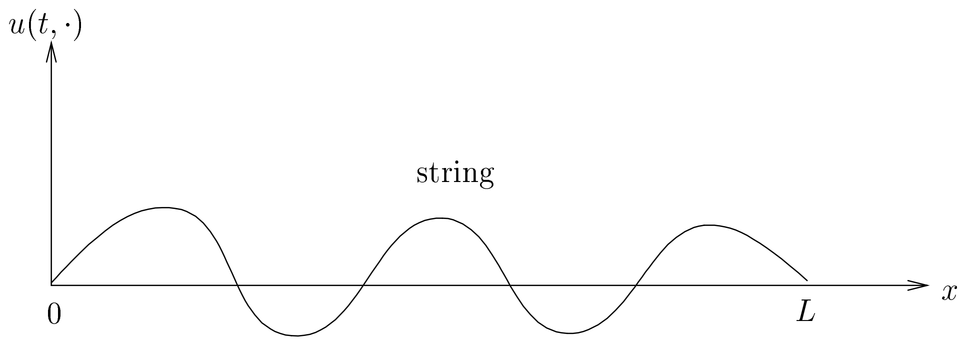
I is the *domain of definition* of the signal.

In other, related contexts signals may be defined on domains which are not intervals: e.g. on the integers (for a sampled-data, or digital signal), or on a square (e.g. in optical image-processing).

Fourier analysis is the representation of a signal as a sum or integral of its 'components' at various frequencies. Fourier *series* are infinite-series representations. They first arose around 1750, in studies of wave motion by D'Alembert, Euler and Daniel Bernoulli. Very briefly, here's how:

Problem of vibrating string

Stretch an elastic string under tension and fix the endpoints at $x = 0$, $x = L$.



Let $u(t,x)$ be the string's vertical displacement from the horizontal at time $t \geq 0$, and place x, $0 \leq x \leq L$. From physics you get that $u(\cdot,\cdot)$ is determined by the four relations:

$$\frac{\partial^2 u(t,x)}{\partial t^2} - c^2 \frac{\partial^2 u(t,x)}{\partial x^2} = 0, \quad t > 0, \quad 0 < x < L; \tag{PDE}$$

Here c^2 = tension/linear density.

$$\left.\begin{aligned} u(0,x) &= f(x) \\ \left.\frac{\partial u(t,x)}{\partial t}\right|_{t=0} &= g(x) \end{aligned}\right\}; \tag{IC}$$

$$u(t,0) = u(t,L) = 0. \tag{BC}$$

Here $f(\cdot)$ and $g(\cdot)$ are the initial displacement and velocity, assumed known.

The problem is to get a 'formula' for $u(t,x)$. To this end, Bernoulli noticed that the functions

$$u_n(t,x) := \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right)\right] \cdot \sin\left(\frac{n\pi x}{L}\right),$$

(where $n = 1, 2, \ldots$, and a_n, b_n are arbitrary constants) satisfy (PDE) and (BC), although not (IC) in general. So now you try the formula

$$u(t,x) \stackrel{?}{=} \sum_{n=1}^{\infty} u_n(t,x). \tag{S}$$

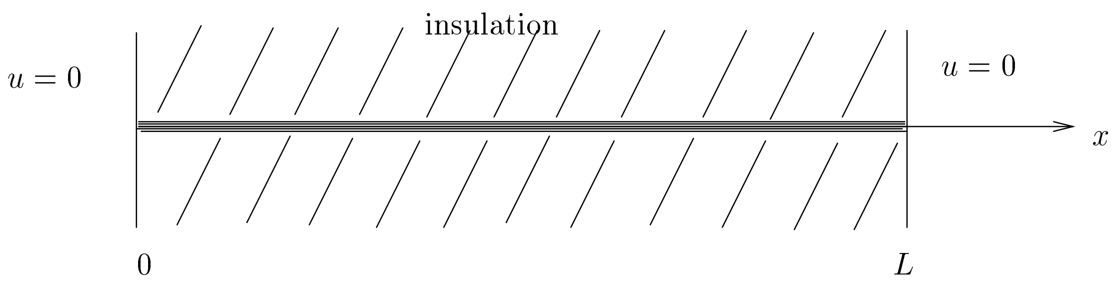
This will (formally) satisfy (PDE) and (BC), *and* (IC) *provided* a_n, b_n can be chosen such that

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \\ g(x) &= \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) \end{aligned}\right\} \tag{FS}$$

Thus our problem is reduced to *representing* the 'signals' $f(\cdot)$ and $g(\cdot)$, on the interval of definition $[0, L]$, by the trigonometric, infinite series (FS).

Of the same ilk was the problem considered by Joseph Fourier, c.1807:

Problem of the cooling bar



A bar of length L is heated to an initial temperature profile $f(x)$, $0 \leq x \leq L$, then allowed to cool by conduction through the ends, where the temperature is held at zero for $t > 0$. If the bar temperature is $u(t, x)$ then from physics,

$$\frac{\partial u(t,x)}{\partial t} - \kappa \frac{\partial^2 u(t,x)}{\partial x^2} = 0 \qquad (PDE)$$

(here κ depends on the bar material); and we have the auxiliary conditions

$$u(0,x) = f(x), \quad 0 \leq x \leq L, \qquad (IC)$$

$$u(t,0) = u(t,L) = 0, \quad t > 0. \qquad (BC)$$

Somewhat as before, you note that

$$u_n(t,x) := e^{-(n^2\pi^2\kappa/L^2)t} \sin\left(\frac{n\pi x}{L}\right)$$

satisfies (PDE) and (BC); you try

$$u(t,x) \stackrel{?}{=} \sum_{n=1}^{\infty} b_n u_n(t,x); \qquad (S)$$

and on substituting in (IC) are led to the representation problem

$$f(x) \stackrel{?}{=} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \qquad (FS)$$

for suitable constants b_n.

Finally, consider the

Electrical engineering problem

Let $x(t)$, $t \geq 0$, be a 60 Hz sinusoid of unit amplitude:

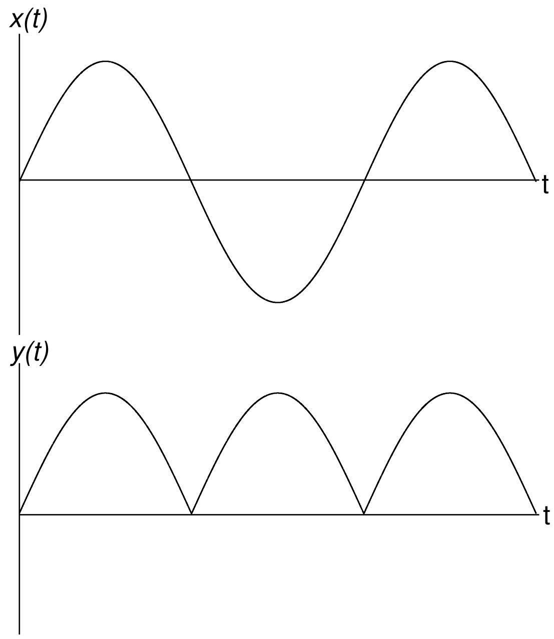
$$x(t) = \sin \omega_0 t; \quad t \geq 0, \quad \omega_0 = 2\pi 60$$

The output of a full-wave rectifier, with $x(\cdot)$ as input, would be

$$y(t) = |\sin(\omega_0 t)|, \quad t \geq 0.$$

To analyse the effect of a smoothing filter on $y(\cdot)$ one would try a representation

$$y(t) = \frac{a_0}{2} + a_2 \cos(2\omega_0 t) + a_4 \cos(4\omega_0 t) + ... \qquad (FS)$$



This problem will be studied in detail later. We turn now to the mechanics of calculating the coefficients in Fourier Series (FS).

1.2 Calculating Fourier Series

Given a signal $f(t)$, $t \in I := [-\pi, \pi]$, consider the

Problem: Represent $f(\cdot)$ as a series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \quad -\pi \leq t \leq \pi, \tag{1}$$

for suitable constants a_n $(n = 0, 1, 2, \ldots)$ and b_n $(n = 1, 2, \ldots)$.

The key to (1), if such a representation is possible at all, lies in the

Orthogonality relations:

For $m, n = 0, 1, 2, \ldots,$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \begin{cases} 1, & m = n \neq 0 \\ 0, & m \neq n \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = \begin{cases} 1, & m = n \neq 0 \\ 0, & m \neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0, \text{ all } m, n$$

So if (1) holds, then formally[1]

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(0.t)dt \\ &= a_0/2.\end{aligned}$$

In general, multiplying both sides of (1) by $\cos(mt)$ or $\sin\,(mt)$ and integrating on I, we get the formulas:[2]

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt; \quad n = 0,1,2,\ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt; \quad n = 1,2,\ldots \end{aligned}\right\} \tag{2}$$

All this suggests the

Definition. Given $f(t)$, $t \in [-\pi,\pi]$, the *Fourier Series* (FS) of $f(\cdot)$ is the series (1) with the coefficients a_n, b_n given by (2). In that case one writes

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

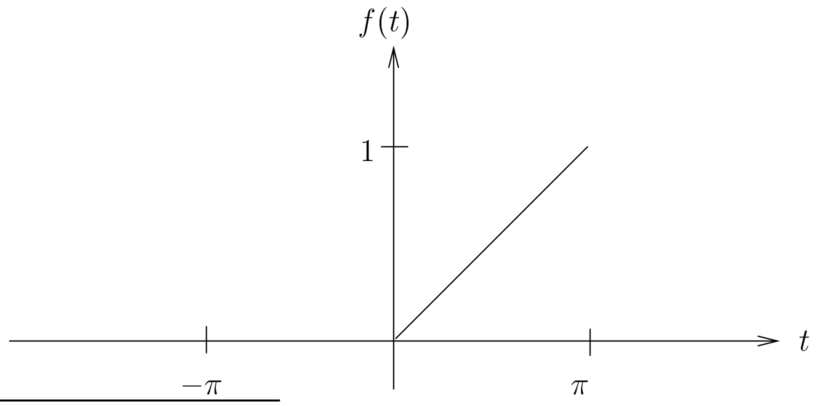
where $\sim$ means "the series on the right is associated with the function $f(\cdot)$, via (2)".

The FS is said to *exist* whenever all the integrals (2) exist. Then $\{a_n, b_n\}$ are the *Fourier coefficients* of $f(\cdot)$.

Warning: the definition does *not* say that the Fourier Series of f (when it exists) converges or, if it does converge, that its sum is f. This we investigate later.

Example

$$f(t) = \begin{cases} 0, & -\pi \le t \le 0 \\ t/\pi, & 0 \le t \le \pi \end{cases}$$



[1] i.e. without regard to mathematical technicalities

[2] due to L. Euler (1755) and J.L. Lagrange (1759).

$$
\begin{aligned}
\frac{a_0}{2} &= \frac{1}{2\pi}\int_0^\pi (t/\pi)dt = \frac{1}{4} \\
a_n &= \frac{1}{\pi}\int_0^\pi (t/\pi)\cos(nt)dt \\
&= \frac{1}{\pi^2}\int_0^\pi \frac{t}{n} \quad \frac{d}{dt}(\sin nt)dt \\
&= \frac{1}{n\pi^2}\left[t\sin nt|_0^\pi - \int_0^\pi \sin nt \cdot dt\right] \\
&= \frac{1}{n\pi^2}\left[\frac{1}{n}\cos nt\right]_0^\pi \\
&= \frac{1}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} -\dfrac{2}{n^2\pi^2}, & n \text{ odd} \\ \\ 0, & n \text{ even} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^\pi (t/\pi)\sin(nt)dt \\
&= -\frac{1}{n\pi^2}\int_0^\pi t\frac{d}{dt}(\cos nt)dt \\
&= -\frac{1}{n\pi^2}\left[t\cos nt|_0^\pi - \int_0^\pi \cos nt dt\right] \\
&= -\frac{1}{n\pi^2}[\pi(-1)^n] \\
&= (-1)^{n+1}/n\pi.
\end{aligned}
$$

So

$$
\begin{aligned}
f(t) \sim \frac{1}{4} &+ \left(-\frac{2}{1^2\pi^2}\cos t + \frac{1}{\pi}\sin t\right) - \frac{1}{2\pi}\sin 2t \\
&+ \left(-\frac{2}{3^2\pi^2}\cos 3t + \frac{1}{3\pi}\sin 3t\right) - \frac{1}{4\pi}\sin 4t + \ldots
\end{aligned}
$$

Exercise 1.1: Modify the foregoing example by defining

$$f(t) = \begin{cases} 1 + t/\pi, & -\pi \leq t < 0 \\ 0, & 0 \leq t \leq \pi \end{cases}$$

(i) Calculate the Fourier Series of f.

(ii) Define the N^{th} partial sum of Fourier Series by

$$f_N(t) := a_0/2 + \sum_{n=1}^{N}(a_n \cos nt + b_n \sin nt)$$

For $N = 5, 10, 50, 100$ compute $f_N(t)$ and plot for $-\pi \leq t \leq \pi$.

(iii) Discuss the results in (ii), especially the behavior of the $f_N(t)$ near $t = 0$. Is f_N a 'good' approximation to f for large N?

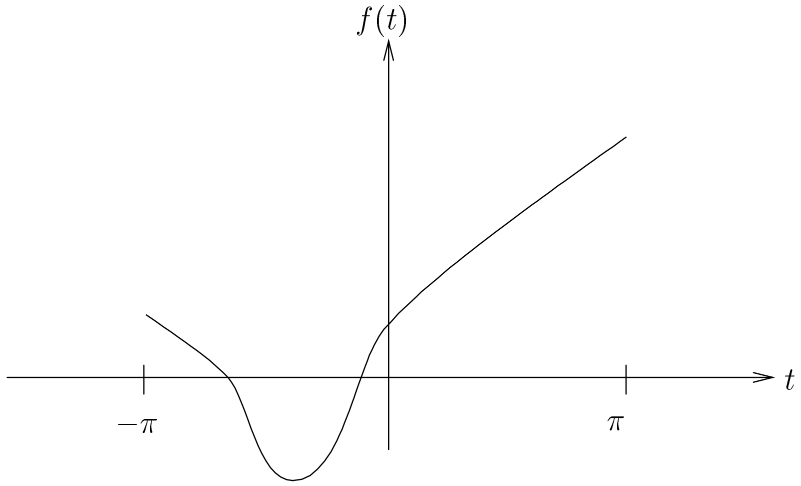
Fourier Series for a Periodic Function

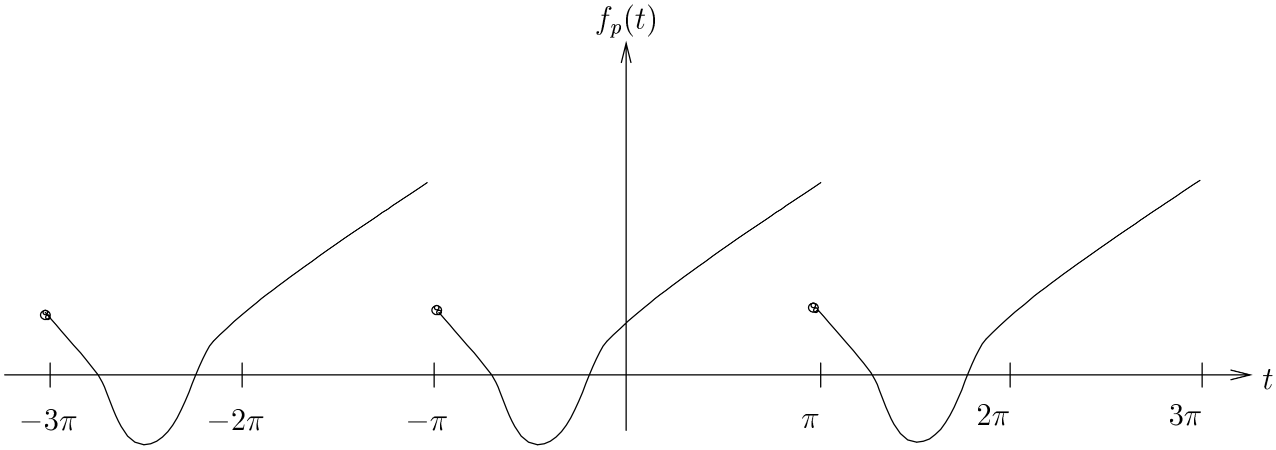
Definition. A signal $f(t)$, $-\infty < t < \infty$, is *periodic with period* T $(T > 0)$ if

$$f(t + T) = f(t), \quad -\infty < t < \infty.$$

Definition. Let $f(\cdot)$ have domain $[-\pi, \pi]$. The *periodic extension* of $f(\cdot)$ is the function

$$f_p(t) := \begin{cases} f(t), & -\pi \leq t < \pi \\ f(t - 2n\pi), & (2n-1)\pi \leq t < (2n+1)\pi, \\ & n = \pm 1, \pm 2, \ldots \end{cases}$$





Definition. The *Fourier series for* $f_p(\cdot)$ is the same as the Fourier Series for $f(\cdot)$, but now with extended domain $-\infty < t < \infty$.

Remark: Since each term of the Fourier Series is periodic with period 2π, so is each of the partial sums $f_N(\cdot)$. In particular the convergence behavior of Fourier Series for $f_p(\cdot)$ is no different from that of Fourier Series for $f(\cdot)$.

Example: Rectified sine wave

$$y(t) = |\sin \omega t|, \quad -\infty < t < \infty$$

$y(\cdot)$ has basic period π/ω. Put $t = \tau/\omega$,

$$y(t) = y(\tau/\omega) =: \tilde{y}(\tau) = |\sin \tau|, \quad -\pi \le \tau \le \pi.$$

As an exercise, verify that for the signal $\tilde{y}(\tau)$:

$$a_0/2 = 2/\pi$$

$$a_n = \begin{cases} 0, & n = 1, 3, \ldots \\ -\frac{4}{\pi}\frac{1}{n^2-1} & n = 2, 4, \ldots \end{cases}$$

$$b_n = 0.$$

i.e.

$$\tilde{y}(\tau) \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos(2n\tau),$$

$$\begin{aligned} y(t) &= \tilde{y}(\omega t) \\ &\sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos(2n\omega t). \end{aligned}$$

Here we have defined the Fourier Series of $y(\cdot)$ via an obvious scaling of the time axis, in order to revert to our standard situation.

Fourier series of a signal on an arbitrary finite interval

Suppose $f(\cdot)$ is defined on $[t_0, t_1]$. One 'natural' way to define Fourier Series for $f(\cdot)$ is just rescale $[t_0, t_1]$ to $[-\pi, \pi]$, by putting

$$\tau := -\pi + \frac{t - t_0}{t_1 - t_0} \cdot 2\pi, \quad t_0 \leq t \leq t_1,$$

then grind out the new formulas.

But it's equivalent, and more convenient, to develop orthogonality relations for the system of functions below, from scratch. Take

$$1, \cos\left(n \cdot \frac{2\pi}{(t_1 - t_0)} t\right), \quad \sin\left(n \cdot \frac{2\pi}{(t_1 - t_0)} t\right); \quad n = 1, 2, \ldots$$

on the interval $[t_0, t_1]$. Then *define* Fourier Series for $f(\cdot)$ on $[t_0, t_1]$ as

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi}{t_1 - t_0} t\right) + b_n \sin\left(\frac{2n\pi}{t_1 - t_0} t\right)\right]$$

Here (by the same reasoning as before) one *defines*

$$\left\{ \begin{array}{c} a_n \\ b_n \end{array} \right\} = \frac{2}{t_1 - t_0} \int_{t_0}^{t_1} f(t) \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} \left(\frac{2n\pi}{t_1 - t_0} t\right) dt, \quad n = 0, 1, 2, \ldots,$$

whenever the indicated integrals exist.

Exercise 1.2: Verify formally that the expressions just given for a_n, b_n are "correct", in the sense that if $\sim$ were replaced by $=$, the formulas would follow by the usual integration procedure.

Example: Sawtooth wave

Let

$$f(t) = \frac{1}{3}(t - t_0), \quad t_0 = 1 \leq t \leq 4 = t_1.$$

Define $f(\cdot)$ by periodic extension for $t < t_0$ and $t > t_1$. The result is a periodic 'sawtooth wave' or 'scanning signal'. One gets $a_0/2 = 1/2$, and for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{3} \int_1^4 \cos\left(\frac{2n\pi}{3} t\right) \frac{1}{3}(t - 1) dt \\ b_n &= \ldots \sin \ldots \end{aligned}$$

To wrap up the 'formalities', we discuss the Fourier Series for *even* and for *odd* functions, and bring in some extension tricks which are sometimes convenient.

Fourier Series for an even function

Let $f(\cdot)$ be defined on $[-\pi, \pi]$. $f(\cdot)$ is *even* if $f(-t) = f(t)$, $-\pi \le t \le \pi$. Check that $b_n = 0$, i.e.

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

Fourier Series for an odd function

Let $f(\cdot)$ be defined on $[-\pi, \pi]$. $f(\cdot)$ is *odd* if $f(-t) = -f(t)$, $-\pi \le t \le \pi$. Check that $a_n = 0 (n = 0, 1, \ldots)$, i.e.

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$$

Even and odd extensions

Let $f(\cdot)$ be defined initially for $0 \le t \le \pi$. [The story for $f(\cdot)$ on, say, $[0, L]$ goes by obvious rescaling].

The *even extension of* $f(\cdot)$ is

$$f_e(t) := \begin{cases} f(-t), & -\pi \le t \le 0 \\ f(t), & 0 \le t \le \pi \end{cases},$$

defined on $[-\pi, \pi]$. The *Fourier Cosine Series for* $f(\cdot)$ is defined to be the Fourier Series for $f_e(\cdot)$: by the above, this series just has a constant term, plus cosines. Clearly

$$\begin{aligned} \frac{a_0}{2} &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_e(t) dt = \frac{1}{\pi} \int_0^{\pi} f(t) dt, \\ a_n &:= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(t) \cos nt dt = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt dt, \quad n \ge 1. \end{aligned}$$

By the same token, the *odd extension of* $f(\cdot)$ is

$$f_0(t) := \begin{cases} -f(-t), & -\pi \le t < 0 \\ 0, & t = 0 \\ f(t), & 0 < t \le \pi \end{cases}$$

The *Fourier Sine Series for* $f(\cdot)$ is defined to be the Fourier Series for $f_0(\cdot)$: by the above, this series just has sine terms. Clearly

$$\begin{aligned} b_n &:= \frac{1}{\pi} \int_{-\pi}^{\pi} f_0(t) \sin nt dt \\ &= \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt dt. \end{aligned}$$

The point of introducing Fourier Cosine Series or Fourier Sine Series for a given $f(\cdot)$ is that you may thereby satisfy certain auxiliary conditions, and/or you may get better convergence. In the cooling bar problem (p.2) a sine series was used to satisfy (BC). As to convergence, consider the

Example: Comparison of three Fourier Series

Let

$$f(t) = t/\pi, \quad 0 \le t \le \pi.$$

Verify:

$$f(t) \sim \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin(2nt) \tag{FS}$$

$$f(t) \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)t \tag{FCS}$$

and

$$f(t) \sim \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin nt \tag{FSS}$$

Note that Fourier Series fails to converge to $f(t)$ at $t = 0$ and $t = \pi$, and that Fourier Sine Series fails to converge to $f(t)$ at $t = \pi$. However, we'll see later that Fourier Cosine Series converges to $f(t)$ for all $t \in [0, \pi]$, and in fact convergence is uniform. At this stage, you can see that Fourier Cosine Series converges (to *something*) faster than Fourier Series or Fourier Sine Series, since its terms go as $1/n^2$ as $n \to \infty$. So it's plausible that Fourier Cosine Series is 'better behaved' than Fourier Series, in this example.

Exercise 1.3: Do the verifications for example above.

Exercise 1.4: Calculate and compare the three Fourier Series for

$$f(t) = \sin t, \quad 0 \le t \le \pi.$$

Exercise 1.5: Fourier Series for a pulse train, and Poisson's formula

Let

$$0 < \varepsilon \ll \pi.$$

Suppose

$$f_\varepsilon(t) := \begin{cases} \varepsilon^{-1} - \varepsilon^{-2}|t|, & |t| \le \varepsilon \\ 0, & \varepsilon \le |t| \le \pi \end{cases}$$

and assume $f_\varepsilon(\cdot)$ has period 2π for $-\infty < t < \infty$.

(i) Work out Fourier Series for f_ε.

(ii) Calculate the limiting Fourier Series as $\varepsilon \to 0$.

(iii) Let $g(t)$ be defined and continuous for $-\infty < t < \infty$, with $g(t) \to 0$ fast as $t \to \pm\infty$. From the definition of f_ε, compute

$$s_1 = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f_\varepsilon(t) g(t) dt.$$

(iv) Let
$$\hat{g}(\lambda) := \int_{-\infty}^{\infty} g(t)\cos(\lambda t)dt, \quad -\infty < \lambda < \infty.$$
Using the result in (ii), find an alternative expression (s_2, say), in terms of $\hat{g}$, for the limit s_1.

(v) Now you can infer $s_1 = s_2$, which is *Poisson's summation formula.* Can you use this formula to sum the series
$$\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} \quad = \quad ?$$

Exercise 1.6: Half-wave vs. full-wave rectification

For a 60 Hz signal $f(t) = \sin\omega_0 t$ let

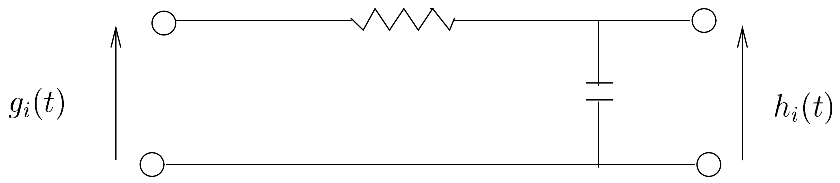
$$g_1(t) \quad := \quad |\sin\omega_0 t|, \quad -\infty < t < \infty$$

$$g_2(t) \quad := \quad \begin{cases} \sin\omega_0 t, & 0 \le t \le \pi/\omega_0 \\ 0, & \pi/\omega_0 \le t \le 2\pi/\omega_0 \end{cases}$$

and extend $g_2(\cdot)$ periodically to $(-\infty, \infty)$.

(i) Work out the Fourier Series for $g_2(\cdot)$ (for g_1 see p.8).

(ii) Suppose $g_i(\cdot)$ $(i = 1, 2)$ is a voltage signal maintained at the input of a smoothing network:

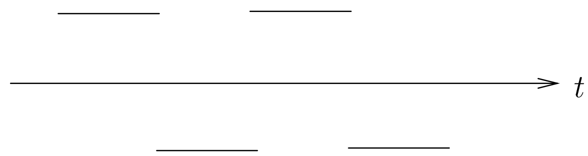


Compute the Fourier Series for $h_i(\cdot)$ in the steady state. [Hint: consider network response to a sinusoid, and use linearity].

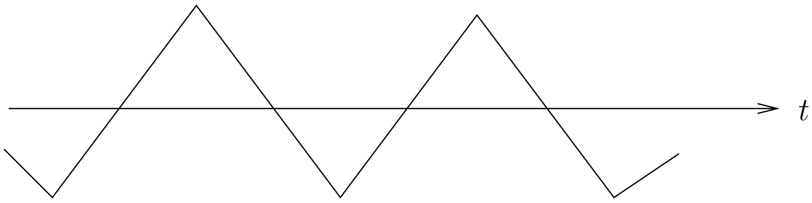
(iii) In each case, calculate the value of network time constant required to reduce the amplitude of first oscillatory component of the output to 10% of the 'd.c.' component of output.

Exercise 1.7: Integration of Fourier Series

Work out a Fourier Series for the periodic 'rectangular' wave shown, choosing your own scales for the axes:



Do the same for the triangular wave:



Could you get this Fourier Series easily from the first one? How?

Exercise 1.8: Let $f(\cdot)$ be periodic with period T, for $-\infty < t < \infty$. Show that for arbitrary $t_0 \in (-\infty, \infty)$,

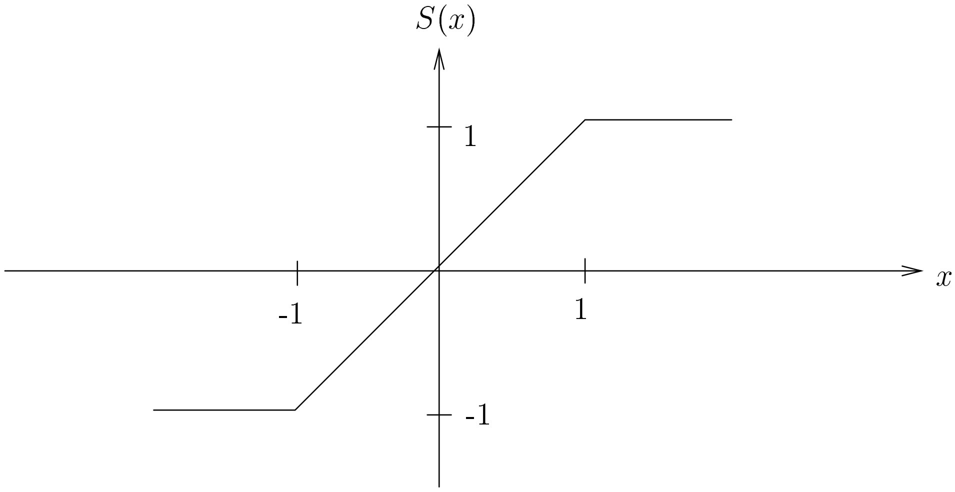
$$\int_{t_0}^{t_0+T} f(t)dt = \int_{-T/2}^{T/2} f(t)dt.$$

Thus the integral of a periodic function, over an interval of length one period, is independent of where the interval is located.

Exercise 1.9: 'Describing function' of a saturating amplifier

Consider the saturating amplifier with input-output characteristic:

$$S(x) = \begin{cases} 1, & x \geq 1 \\ x, & |x| \leq 1 \\ -1, & x \leq -1 \end{cases}$$



If the input is $x(t) = a \sin t$, $-\infty < t < \infty$, where $a \geq 0$ is a parameter, calculate the first two nonzero Fourier coefficients of the output $y(t) = S[x(t)]$. Call the first coefficient $\beta = \beta(a)$ and sketch the graph of $\gamma(a) := \beta(a)/a$ for $0 < a < \infty$. [In control theory, $\gamma(\cdot)$ is the 'describing function' of the nonlinear function $S(\cdot)$. How would you interpret it?]

Exercise 1.10: Compute the standard Fourier Series for

$$f(t) = t^2, \quad -1 \leq t \leq 1$$

Assuming the series converges to $f(t)$ at all values of t, evaluate the sum

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Exercise 1.11: Let $f(t)$ be periodic with period 2π, on $-\infty < t < \infty$. Using the formulas on p.9, compute the Fourier coefficients of f on an arbitrary interval $t_0 \leq t \leq t_1$, where $t_1 - t_0 = 2\pi$. Show that, for each n, the general term of this series is equal to the general term of the standard series computed on $[-\pi, \pi]$.

1.3 Pointwise and Uniform Convergence

Before studying the convergence of Fourier series, we shall review some general concepts to do with convergence of sequences and series of functions. The fundamental case is that of numerical sequences and series.

Convergence concepts

Consider an infinite sequence of complex numbers, say

$$Seq \sim \{c_N | N = 1, 2, ...\}, \qquad c_N \in \mathbb{C}$$

In practice, the sequence Seq is often 'given' by specifying c_N as a definite function of N, for instance $c_N = \exp(iN)/N$. The practical role of Seq may be to provide successively better numerical approximations to some number of interest that has no 'exact' rational representation, the number π, for example, or the complex cube roots of 2. What a convergence concept expresses is the sense in which Seq does indeed provide successive approximations to some complex number, say c. The usual definition is the following.

Definition (convergence of an infinite sequence of complex numbers):
The sequence $Seq \sim \{c_N\}$ *converges to the limit* $c \in \mathbb{C}$ *as* $N \to \infty$ if, for every positive real number ε, there exists an integer M such that, for all $N > M$, $|c_N - c| < \varepsilon$.
In that case one writes

$$\lim c_N = c \ (N \to \infty) \quad \text{or} \quad \lim_{N\to\infty} = c \quad \text{or} \quad c_N \to c \ (N \to \infty)$$

Intuitively, the definition says that, if you go sufficiently far out along the sequence, then you can get, and remain, as close to the number c as you please. Familiar examples are provided by the real-valued sequences $\{1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,...\}$ and $\{3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,...\}$ that develop the decimal expansions of the numbers $\sqrt{2}$ and π respectively. For instance, taking $\varepsilon = 10^{-6}$ and $M = 7$, then in the first of these sequences we clearly have $|c_N - \sqrt{2}| < \varepsilon$ for all $N > M$ (because all terms c_N of the sequence with $N > 7$ approximate $\sqrt{2}$ to better than 6 decimal places accuracy). On replacing $\sqrt{2}$ with π, the same statement is true of the second sequence.

Now consider an infinite *series* of complex numbers c_n, written

$$Ser \sim \sum_{n=1}^{\infty} c_n$$

Without further definition, such a symbol is meaningless, since there is no way to perform an infinite number of additions. One therefore brings in the *sequence of partial sums* $\{s_N | N = 1, 2, ...\}$, defined by

$$s_N = \sum_{n=1}^{N} c_n$$

Since each sum s_N is finite, it can in principle be calculated provided the c_n (for $n = 1, 2, ..., N$) are given. The infinite sequence $Seq \sim \{s_N\}$ is then well-defined, and we define convergence of Ser in terms of that of Seq:

Definition (convergence of an infinite series of complex numbers):

The series $Ser \sim \sum_{n=1}^{\infty} c_n$ *converges to the limit* $s \in \mathbb{C}$ provided the sequence of partial sums

$$Seq \sim \{s_N := \sum_{n=1}^{N} c_n \mid N = 1, 2, ...\}$$

converges to s (as $N \to \infty$).

In that case one writes, of course,

$$s = \sum_{n=1}^{\infty} c_n$$

A convergent numerical series may be regarded as an 'abstract algorithm' for computing an approximation to the number declared to be its limit. To compute s to within a desired accuracy ε, one must first pick M so large that $|s_N - s| < \varepsilon$ for all $N > M$, and then calculate any such finite sum s_N (e.g. take $N = M + 1$). Unfortunately, the definition itself provides no hint whatsoever as to how M might be selected after ε is specified; in that sense the algorithm is merely 'abstract'. In practice one tries to estimate a suitable M by analyzing the case at hand.

Example 1 (convergence of a numerical sequence):

Consider the sequence $c_N = \cos(\pi - 2/N)$ $(N = 1, 2, ...)$. By inspection we'd guess that $\lim c_N = c := \cos\pi = -1$ $(N \to \infty)$. To confirm this using the definition, we try to show that $|c_N - c|$ becomes, and remains, as small as we please when N is taken sufficiently large. Thus

$$\begin{aligned}
|c_N - c| &= |\cos(\pi - 2/N) - (-1)| \\
&= |\cos(\pi)\cos(2/N) + \sin(\pi)\sin(2/N) + 1| \\
&= |1 - \cos(2/N)| = |\int_0^{2/N} \sin(x)dx| \\
&\leq \int_0^{2/N} |\sin(x)|dx \leq 2/N
\end{aligned}$$

Thus $|c_N - c| < \varepsilon$ if $2/N < \epsilon$, namely $N > M = [2/\varepsilon] + 1$. As one expects, the smaller ε is specified to be, the larger M must be taken to achieve the required accuracy of approximation to c by all terms c_N with $N > M$.

Example 2 (Tail estimation of an infinite series):

An example that we shall find very useful later on is the series

$$\sum_{n=1}^{\infty} n^{-\alpha}$$

where α is a real number. It can be shown that this series converges to a finite limit, say $s(\alpha)$, if and only if $\alpha > 1$. To approximate the number $s(\alpha)$ within accuracy ε, by use of the series, we try to pick N so large that

$$|s(\alpha) - s_N(\alpha)| < \varepsilon,$$

where $s_N(\alpha) = \sum_{n=1}^{N} n^{-\alpha}$. Now $s(\alpha) - s_N(\alpha)$ is just the 'tail' series

$$s(\alpha) - s_N(\alpha) = \sum_{n=N+1}^{\infty} n^{-\alpha}$$

We can find an upper bound on this infinite sum (strictly speaking, on the limit of this infinite series) as follows. Consider the real variable x on the interval $1 \leq x < \infty$, and note that, with $\alpha > 1$, $n^\alpha \geq x^\alpha$ if $n - 1 \leq x \leq n$ $(n \geq 2)$. Therefore $n^{-\alpha} \leq x^{-\alpha}$ if $n - 1 \leq x \leq n$. It follows that

$$n^{-\alpha} \leq \int_{n-1}^{n} x^{-\alpha}dx, \qquad n = 2, 3, ...$$

So, if $\alpha > 1$ then

$$\sum_{n=N+1}^{\infty} n^{-\alpha} \leq \sum_{n=N+1}^{\infty} \int_{n-1}^{n} x^{-\alpha}dx = \int_N^{\infty} x^{-\alpha}dx = \frac{1}{\alpha - 1}\frac{1}{N^{\alpha-1}}$$

Applying this result to estimate the tail series above, we get that

$$|s(\alpha) - s_N(\alpha)| \leq (\alpha - 1)^{-1} N^{-(\alpha-1)}$$

and therefore the desired inequality $|s(\alpha) - s_N(\alpha)| < \varepsilon$ will certainly be satisfied provided we pick N so large that

$$(\alpha - 1)^{-1} N^{-(\alpha-1)} < \varepsilon$$

namely

$$N > [1/(\alpha - 1)\varepsilon]^{1/(\alpha-1)}$$

Thus if $\varepsilon = 0.001$ and $\alpha = 2$, it suffices to pick $N > 1/\varepsilon = 1000$ (i.e. $N \geq 1001$); if $\alpha = 3$, $N > (1/2\varepsilon)^{1/2} = 22.4$ (i.e. $N \geq 23$); if $\alpha = 4$, $N > 6.93$ (i.e. $N \geq 7$). Notice that the larger α, the faster the series converges, namely the fewer terms one needs for a specified approximation to the limit.

The definition of convergence of an infinite sequence, and more complicated definitions that we shall meet later on, are expressed most succinctly in the notation of predicate logic. In such statements "for all" is written using the *universal* quantifier symbol $\forall$, and "there exists" using the *existential* quantifier $\exists$. The statement "if P then Q" may be written $P \Rightarrow Q$. So, the full definition of the property that $s_N \to s$ $(N \to \infty)$ takes the concise form

$$(\forall \varepsilon > 0)(\exists M)(\forall N) N > M \Rightarrow |s_N - s| < \varepsilon$$

You always read such statements from left to right: "For all $\varepsilon > 0$, there exists M, such that for all N, if $N > M$ then $|s_N - s| < \varepsilon$."

Sequences and series of complex-valued functions

We generalize from numerical sequences to functional sequences in the natural way: a functional sequence is just a family of numerical sequences. Let $\{f_N | N = 1, 2, ...\}$ be a family of complex valued functions f_N all defined on a common (real) interval of definition I. Thus $f_N : I \to \mathbb{C}$, or $f_N(t) \in \mathbb{C}$ for $t \in I$. Let $f : I \to \mathbb{C}$ be a new function defined on I. We make the definition that *the sequence f_N converges to the limit (function) f pointwise on I as $N \to \infty$* if, for every fixed $t \in I$, the (complex-valued) numerical sequence $\{f_N(t) | N = 1, 2, ...\}$ converges to the numerical limit $f(t)$ as $N \to \infty$. In logic notation,

$$(\forall t \in I)[\lim f_N(t) = f(t) \text{ as } N \to \infty]$$

On writing out the detailed definition of the limit property at fixed t, the full statement becomes

$$(\forall t \in I)(\forall \varepsilon > 0)(\exists M)(\forall N) N > M \Rightarrow |f_N(t) - f(t)| < \varepsilon$$

It is important to realize that, in this definition, the choice of M is conditioned on the prior choice of both ε *and* t: that is, in a loose sense, M can be thought of as dependent on, or as a function of, the variables ε and t that precede it in the logic expression. Intuitively, how far you have to go out along the sequence $f_N(t)$ to get your desired ε-approximation to the limit $f(t)$ may very well depend, not only on ε, but also on the fixed t-value at which you are stationed.

Example 3 (pointwise convergence of a sequence of functions):

As a generalization of Example 1 above, let

$$f_N(t) = \cos[(\pi - 2/N)t], \qquad 0 \le t < \infty, \quad N = 1, 2, ...$$

Let $f(t) := \cos(\pi t)$, $0 \le t < \infty$. We claim that $f_N \to f$ pointwise for $t \in I := [0, \infty)$. Indeed, using the inequalities $1 - \cos(x) \le x^2/2$ and $|\sin(x)| \le x$ valid for $0 \le x < \infty$, it is not difficult to show that

$$|f_N(t) - f(t)| \le (1/2)(2t/N)^2 + 2t/N, \qquad 0 \le t < \infty, \quad N = 1, 2, ...$$

and the right side clearly tends to 0, for any fixed t, as $N \to \infty$. Notice that, by this estimate, how large M must be taken, to guarantee that $|f_N(t) - f(t)| < \varepsilon$ for all $N > M$, will clearly depend on t as well as on ε: roughly, one must take M at least as large as $[2t/\varepsilon]$, if ε is small. This means that the desired ε-approximation is difficult to achieve (i.e. M is very large) if the location t is far out. By looking at the graphs of $f(t)$ and $f_N(t)$, try to see why this is really so (and is not just a byproduct of our rather crude inequalities).

In connection with pointwise convergence, you sometimes have the following situation: the given sequence $f_N(t)$ converges to $f(t)$ at each fixed t in I *except*, perhaps, at a finite or countably infinite set of points, say $t_1, t_2, ...$ At such exceptional points t_i the sequence $\{f_N(t_i) | N = 1, 2, ...\}$ may or may not converge at all. In that case one can hedge, by saying that f_N converges to f $(N \to \infty)$ *pointwise almost everywhere on* I, or briefly $f_N \to f$ a.e. on I. The standard term "almost everywhere" actually has a more general meaning in integration theory, but we shall (correctly) use it only in the special sense just mentioned. If a sequence converges "pointwise", then it certainly converges "pointwise almost everywhere", since the exceptional set is then null.

Example 4 (convergence pointwise almost everywhere):

Let $I = [0, 1]$, and $f_N(t) := \begin{cases} 1 - Nt, & 0 \le t \le 1/N \\ 0, & 1/N \le t \le 1 \end{cases}$

So

$$\left.\begin{array}{l} f_N(t) \to 0, \quad 0 < t \le 1 \\ \\ f_N(0) \to 1. \end{array}\right\} \text{ as } N \to \infty,$$

i.e.

$$f_N \to 0 \text{ a.e. on } I, \text{ as } N \to \infty.$$

A stronger mode of convergence than pointwise convergence is *uniform convergence*, defined as follows. With the same setup as before, the sequence f_N *converges to* f *uniformly on* I as $N \to \infty$ if, for every real $\varepsilon > 0$, there exists M such that, for all $t \in I$ and all $N > M$, $|f_N(t) - f(t)| < \varepsilon$. In logic notation,

$$(\forall \varepsilon > 0)(\exists M)(\forall N > M)(\forall t \in I)|f_N(t) - f(t)| < \varepsilon$$

Since the order of two successive quantifiers of the same type is immaterial, we could write "$(\forall t \in I)(\forall N > M)$" in this expression without changing the meaning. Furthermore,

"$(\forall N > M)$..." is equivalent to "$(\forall N)\ N > M \Rightarrow$ ...". So an equivalent expression of uniform convergence is the statement

$$(\forall \varepsilon > 0)(\exists M)(\forall t \in I)(\forall N)\ N > M \Rightarrow |f_N(t) - f(t)| < \varepsilon$$

This expression clearly brings out the important fact that, with uniform convergence, the choice of M now depends, not on the location t, but only on the prior specification of ε. Once ε has been specified, M can be selected so that, with $N > M$, the ε-approximation by $f_N(t)$ to $f(t)$ is valid for *all* t in the interval I. In other words the approximation can indeed be made "uniformly" good on I by proper choice of M. On comparing the logic expression above with the counterpart expression for pointwise convergence, you see that the only formal difference is that, with uniform convergence, the quantification "$(\forall t \in I)$" has been shifted from the left side of "$(\exists M)$" to the right side. But this has the crucial effect of now making the choice of M not dependent on t but only on ε. The logic notation exhibits this subtle distinction visually in a clear way.

Example 5 (uniform convergence):

Consider Example 3, with the sole difference that now the interval I is declared to be $[0, T]$, where $T > 0$ is fixed and finite (although it can be arbitrarily large). Using the same estimates as before, we see that $|f_N(t) - f(t)| < \varepsilon$ provided

$$(1/2)(2t/N)^2 + 2t/N < \varepsilon$$

Certainly this will be true for all t with $0 \leq t \leq T$, and all $N > M$, provided M is chosen so large that

$$(1/2)(2T/M)^2 + 2T/M < \varepsilon$$

and this can always be done, because T is finite; i.e. with T fixed, such M exists for every $\varepsilon > 0$. We have proved

$$f_N(t) \to f(t) \quad \text{uniformly on } I := [0, T] \text{ as } N \to \infty$$

Of course, for given ε, the larger T happens to be the larger M will have to be; there's a price to be paid. Reverting to the case where $I = [0, \infty)$, we have an instance of the situation where $f_N \to f$ pointwise on I, but *not* uniformly on I. Uniform convergence always implies pointwise convergence, but not the other way around. Another example is the following.

Example 6 (pointwise convergence without uniform convergence): Let $I = [0, 1]$, and $f_N(t) := \frac{Nt}{1+N^2t^2}$. So

$$f_N(t) \to 0 (N \to \infty), \text{ all } t \in I.$$

But

$$f_N\left(\frac{1}{N}\right) = \frac{1}{2}, \quad N = 1, 2, \ldots$$

So if you pick $\varepsilon = 1/4$ then for each N (no matter how large!) you can find a place ($t = 1/N$) where $|f_N(t) - 0| > \varepsilon$: your approximation is *bad, somewhere*. It is *not* good everywhere. Convergence is *not uniform*.

Look at the graphs of $f_N(\cdot)$ and notice the spike of height $1/2$ which moves to the left as $N \to \infty$. Did you see anything like this in your computation, Exercise 1.1?

Miscellaneous facts from calculus

The following facts will play a role in proving convergence of Fourier series.

1. If $f(t)$ is continuous on the (closed, finite) interval $[t_0, t_1]$, then f is bounded there, i.e.

$$(\exists K < \infty)(\forall t) : t_0 \leq t \leq t_1 \Rightarrow |f(t)| < K.$$

2. If f is continuous on $[t_0, t_1]$ then

$$\left| \int_{t_0}^{t_1} f(t)dt \right| \leq \int_{t_0}^{t_1} |f(t)|dt \leq \left(\max_{t_0 \leq t \leq t_1} |f(t)| \right) (t_1 - t_0).$$

3. Fact 2 is just a 'continuous version' of the 'triangle inequality'

$$|c_1 + c_2 + \ldots + c_k| \leq |c_1| + |c_2| + \ldots + |c_k|,$$

for real or complex numbers. Why?

1.4 Dirichlet's Theorem on Convergence of Fourier Series

Now we can turn to the hard question of when Fourier series converge. The first correct proof of convergence was done by P.L. Dirichlet in 1829, almost 80 years after 'Fourier' series first came in! It's a long story (adapted from [Widom, 1969]), which we break up into short sections.

1. Dirichlet kernel

Suppose $f(\cdot)$ is defined on $I = [-\pi, \pi]$, with Fourier Series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Let

$$f_N(t) := \frac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt).$$

Recalling how the a_n, b_n are defined, we can write

$$\begin{aligned} f_N(t) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{N} (\cos nt' \cos nt + \sin nt' \sin nt) \right] f(t')dt' \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{N} \cos n(t' - t) \right] f(t')dt' \end{aligned}$$

To simplify this expression, bring in the *Dirichlet kernel*, defined by

$$D_N(x) := \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{N} \cos nx\right] \tag{1}$$

Then

$$f_N(t) = \int_{-\pi}^{\pi} D_N(t'-t) f(t') dt' \tag{2}$$

From this you see that the convergence properties of Fourier Series will strongly depend on the behavior of the function $D_N(\cdot)$, which we now examine.

(i) $D_N(\cdot)$ is periodic, with period 2π

(ii) $D_N(-x) = D_N(x)$

(iii) $D_N(0) = (2N+1)/2\pi$

(iv) $\int_{-\pi}^{\pi} D_N(x)dx = 1$

(v) $$D_N(x) = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x} \tag{3}$$

provided $\sin(x/2) \neq 0$.

{To verify, substitute

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$

in (1) and sum the geometric series. Also, calculate the limit

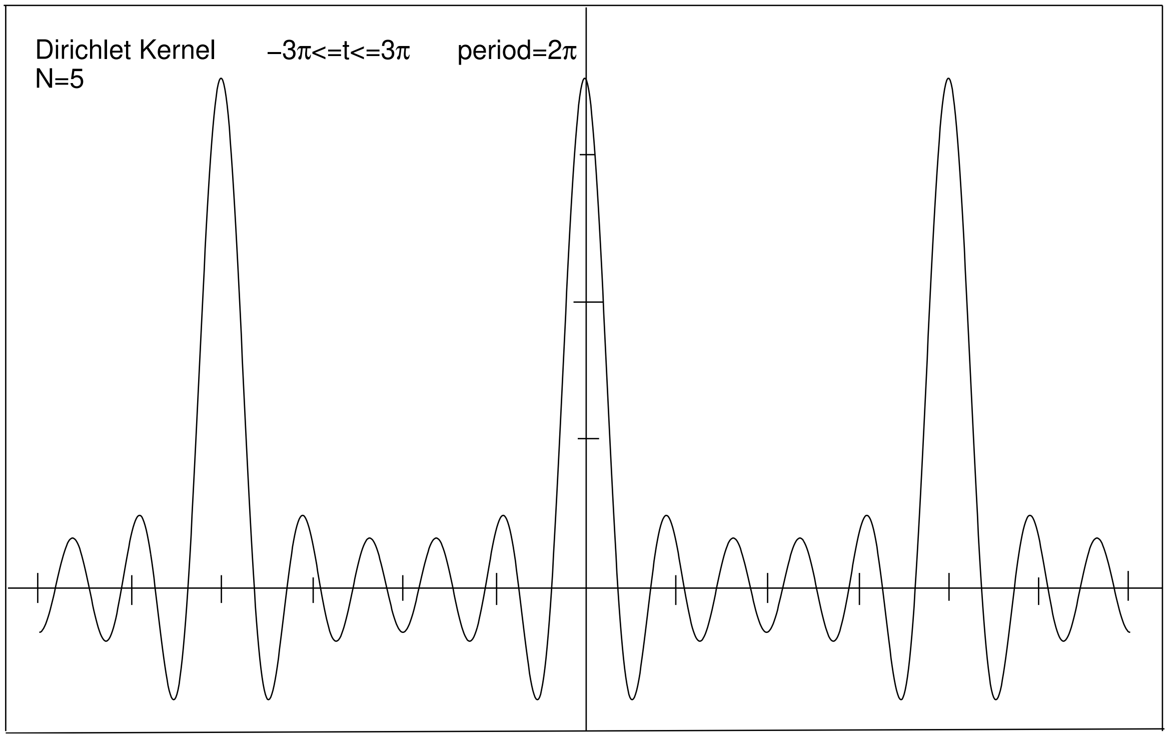
$$\lim_{x\to 0} D_N(x)$$

from (3) and check that you get $D_N(0)$.}

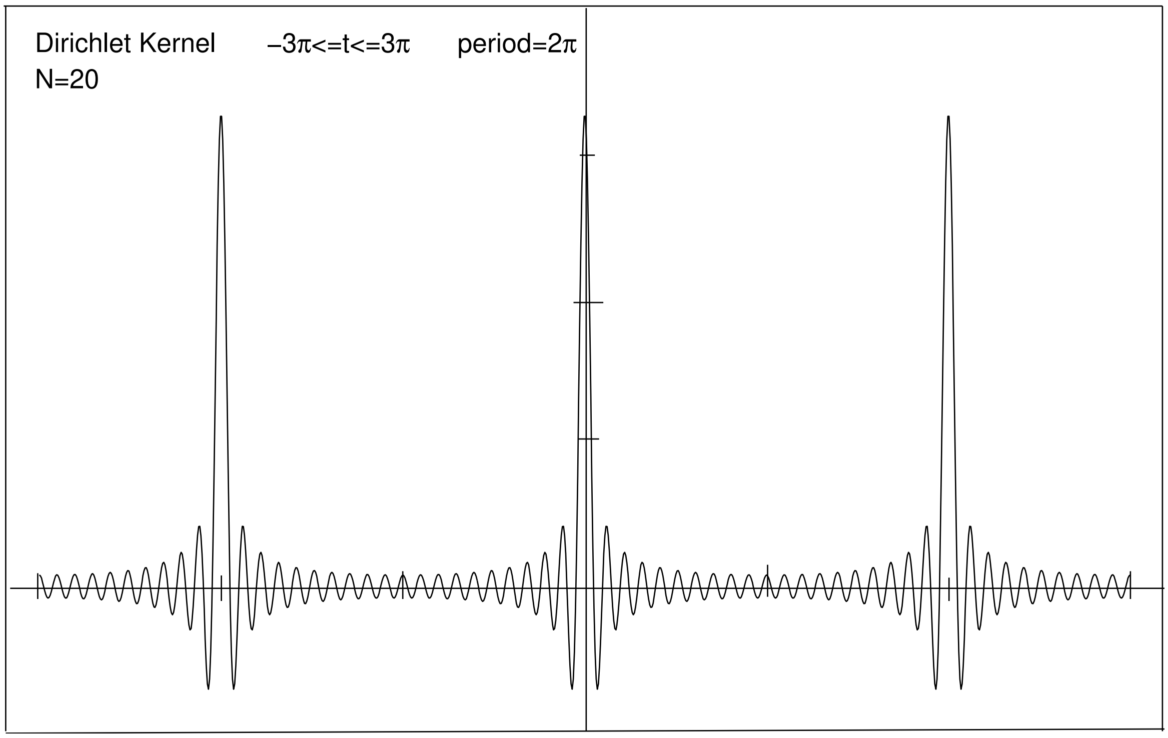
(vi) From (3), for $-\pi \leq x \leq \pi$,

$$D_N(x) = 0 \text{ at } x = \frac{\pm k\pi}{N+\frac{1}{2}}, \quad k = 1, 2, \ldots, N.$$

The graph looks like:



Now you can see why Fourier Series ought to work: As $N \to \infty$, $D_N(\cdot)$ begins to resemble a string of δ-functions!



But then (2) would give, for $t \in (-\pi, \pi)$,

$$f_N(t) \approx \int_{-\pi}^{\pi} f(t')\delta(t'-t)dt' = f(t).$$

So the job is to make this heuristic argument precise.

2. Piecewise smooth functions

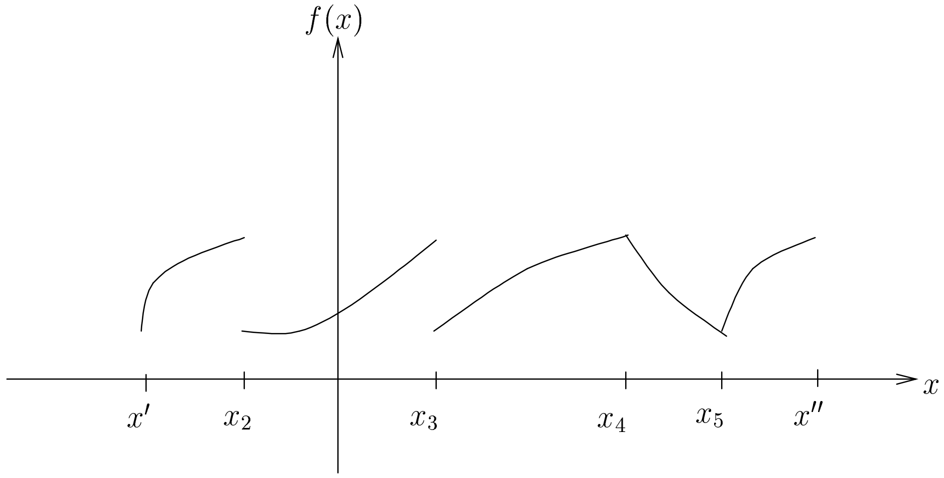
We'll have to confine attention to functions which are reasonably well-behaved, essentially those which are not too jumpy or oscillatory. This is no real restriction in engineering practice.

Definition. Let $f(x)$ be defined for $x' \le x \le x''$. f is *piecewise smooth* ($f \in PS$) on $[x', x'']$ if there exists a finite set of points $\{x_i\}$,

$$x' = x_1 < x_2 < \ldots < x_k = x'',$$

such that, for $x_i < x < x_{i+1}$ $(i = 1, \ldots, k-1)$, $f(x)$ and $df(x)/dx$ are continuous, bounded, and have finite limits as $x \to x_i$ and $x \to x_{i+1}$.

In plain English, "$f(\cdot) \in PS$ if it has (at worst) a finite number of jumps and 'corners' in its (finite) interval of definition, and is smooth otherwise". Typically the graph might look like this:



[Here x_2, x_3 are jump points; x_4, x_5 are corner points.]

Intuitively, "a PS function has small high-frequency components". The following makes this statement precise.

Lemma 1: Special case of Riemann-Lebesgue lemma

Let $f \in PS$ on $[x', x'']$. Then

$$\int_{x'}^{x''} f(x) \left\{ \begin{array}{c} \sin \lambda x \\ \cos \lambda x \end{array} \right\} dx \to 0 \text{ as } \lambda \to \pm\infty.$$

Proof: Let $\{x_i\}$ be as in the definition of PS, and write

$$\int_{x'}^{x''} = \int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \ldots + \int_{x_{k-1}}^{x_k}$$

Since $f(\cdot)$ is well-behaved on each open interval (x_i, x_{i+1}) we define[3]

$$f_i(x) := \begin{cases} f(x), & x_i < x < x_{i+1} \\ f(x_i + 0), & x = x_i \\ f(x_{i+1} - 0), & x = x_{i+1} \end{cases}$$

So

$$\begin{aligned} \int_{x'}^{x''} f(x) \sin \lambda x dx &= \int_{x_1}^{x_2} f_1(x) \sin \lambda x dx + \ldots \\ &+ \int_{x_{k-1}}^{x_k} f_{k-1}(x) \sin \lambda x dx \end{aligned}$$

Taking the first integral on the right, we have

$$\begin{aligned} \left| \int_{x_1}^{x_2} f_1(x) \sin \lambda x dx \right| &= \left| -\frac{1}{\lambda} \int_{x_1}^{x_2} f_1(x) \frac{d(\cos \lambda x)}{dx} dx \right| \\ &= \frac{1}{|\lambda|} \left| f_1(x_2) \cos \lambda x_2 - f_1(x_1) \cos \lambda x_1 - \int_{x_1}^{x_2} \frac{df_1(x)}{dx} \cos \lambda x dx \right| \\ &\leq \frac{1}{|\lambda|} \left\{ 2 \max_x |f_1(x)| + (x_2 - x_1) \max_x \left| \frac{df_1(x)}{dx} \right| \right\} \end{aligned}$$

[using the Facts, p.20, applied to f_1, f_1']

$$= \frac{\text{const.}}{|\lambda|} \to 0 \text{ as } \lambda \to \pm\infty.$$

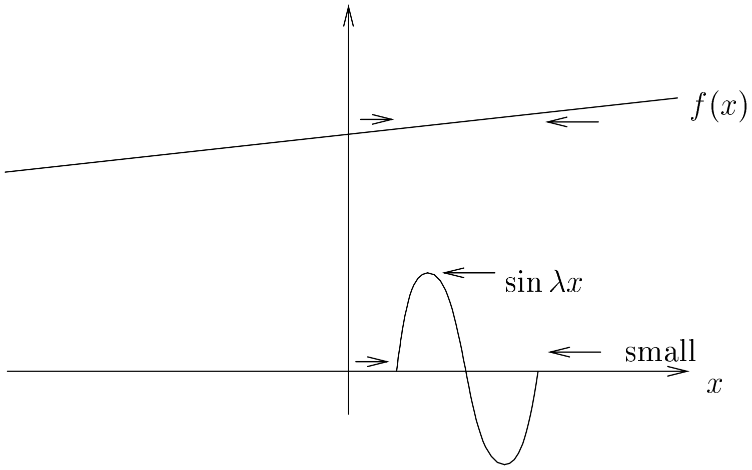
The other integrals behave similarly, so

$$\left| \int_{x'}^{x''} -- \right| \leq \left| \int_{x_1}^{x_2} -- \right| + \ldots + \left| \int_{x_{k-1}}^{x_k} -- \right| \to 0, \quad \lambda \to \infty.$$

The story for $\cos \lambda x$ is similar. Q.E.D.

To see what's going on, just note that for large λ, fixed, $f(\cdot)$ hardly changes over an interval of one period (length $2\pi/\lambda$), so the positive and negative contributions to the integral practically "knock each other out" [Dym-McKean].

[3] Notation: $f(a \pm 0) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(a \pm \varepsilon)$

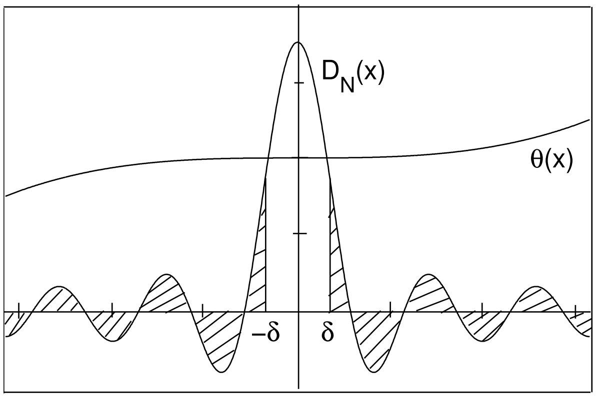


has not changed very much, graph almost parallel to x axis

Corollary: If $f \in PS$ on $[t_0, t_1]$ (say), the Fourier coefficients a_n, b_n of f always $\to 0$ as $n \to \infty$.

3. More on the Dirichlet Kernel

We want to show that "$D_N(\cdot)$ behaves like $\delta(\cdot)$ for large N". We already have (p.21, (iv)) the unit area property. Now we show that the 'tails' of the $D_N(\cdot)$ function contribute negligibly to an integral, for large N.



Lemma 2: Let $0 < \delta < \pi$, and $\theta(\cdot) \in PS$ on $[\delta, \pi]$. Then

$$\lim_{N\to\infty} \int_\delta^\pi D_N(x)\theta(x)dx = 0.$$

Proof:

$$D_N(x)\theta(x) = \left[\frac{\theta(x)}{2\pi \sin(\frac{1}{2}x)} \right] \cdot \sin(N + \frac{1}{2})x$$

Call the factor in [ ], $\psi(x)$. Clearly $\psi(\cdot) \in PS$ on $[\delta, \pi]$. Apply Lemma 1. Q.E.D.

Next we need a standard integral.

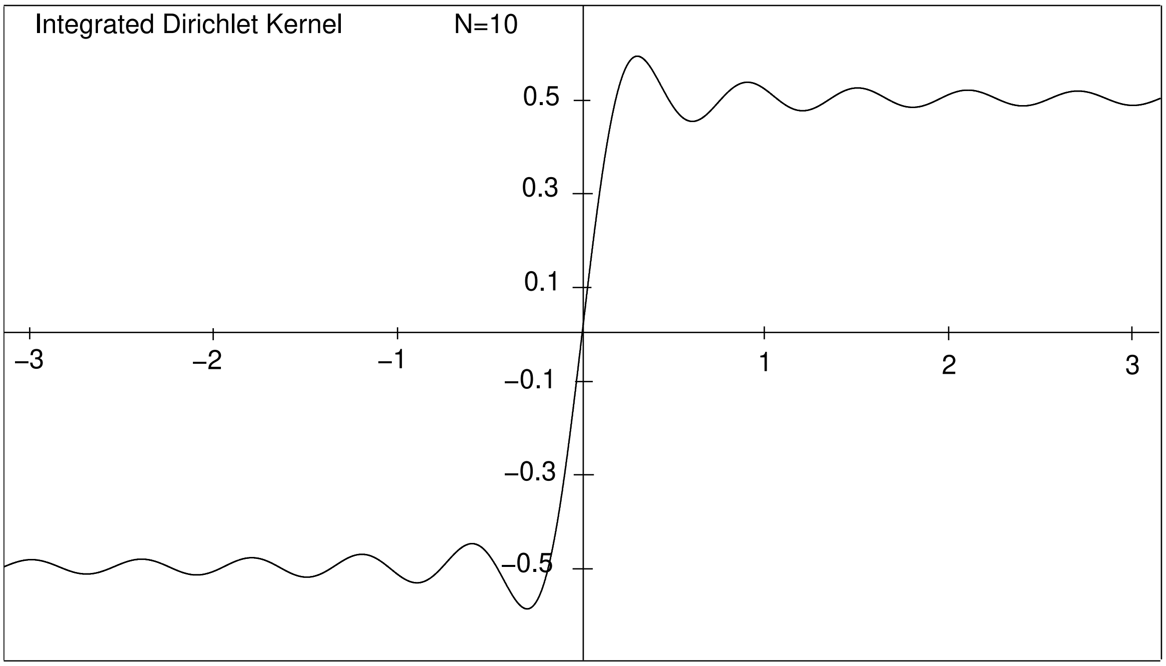
Lemma 3:
$$\int_0^\infty \frac{\sin x}{x} dx := \lim_{T\to\infty} \int_0^T \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Proof: See any book on complex variables.

Corollary:
$$\gamma(T) = \int_0^T \frac{\sin x}{x} dx$$
is a bounded, continuous function of T on $0 \le T < \infty$.

Now define
$$\Delta_N(x) := \int_0^x D_N(u)du, \quad |x| \le \pi.$$

$\Delta_N(\cdot)$ looks like this:



The next result says that the $\Delta_N(\cdot)$ are not only bounded in x (obvious), but that a single bound can be chosen that will do for all N.

Lemma 4. There is a constant K such that $|\Delta_N(x)| < K$ for all $x \in [-\pi, \pi]$ and all $N = 1, 2, \ldots$

Proof: Using (3), p.21,
$$\pi\Delta_N(x) = \pi \int_0^x D_N(u)du$$

$$= \int_0^x \frac{\sin(N+\frac{1}{2})u}{u}du + \int_0^x \sin(N+\frac{1}{2})u \cdot \left[\frac{1}{2\sin\frac{1}{2}u} - \frac{1}{u}\right] du$$

$$= \int_0^{(N+\frac{1}{2})x} \frac{\sin t}{t}dt + \int_0^x \sin(N+\frac{1}{2})u \cdot \psi(u)du,$$

where $\psi := [\ldots]$. Now observe that

(i) $\psi(u)$ is continuous for $0 < u \leq \pi$, and

$$\psi(u) \to 0 \text{ as } u \to 0 \text{ (verify!)}$$

Therefore $\exists K_1$ such that

$$|\psi(u)| < K_1, \quad 0 \leq u \leq \pi.$$

(ii) By Lemma 3 (Corollary), $\exists K_2$ such that

$$\left|\int_0^T \frac{\sin t}{t}dt\right| < K_2, \qquad \text{all } T \in [0, \infty).$$

So by (i) and (ii), for $0 \leq x \leq \pi$ and $N = 1, 2, \ldots$

$$\pi|\Delta_N(x)| \leq K_2 + \pi K_1$$

and this bound does not depend on N. The story for $x \leq 0$ is the same. Q.E.D.

Now we can state and prove the main theoretical result in this course.

4. THEOREM (Dirichlet convergence theorem)

Let $f \in PS$ on $-\pi \leq t \leq \pi$. Let the Fourier Series of f have partial sums f_N (see p.20). Then

(i) If $-\pi < t^* < \pi$ and $f(\cdot)$ is continuous at $t = t^*$, then

$$f_N(t^*) \to f(t^*), \quad N \to \infty$$

(ii) If $-\pi < t^* < \pi$ and $f(\cdot)$ has a jump at t^*, then

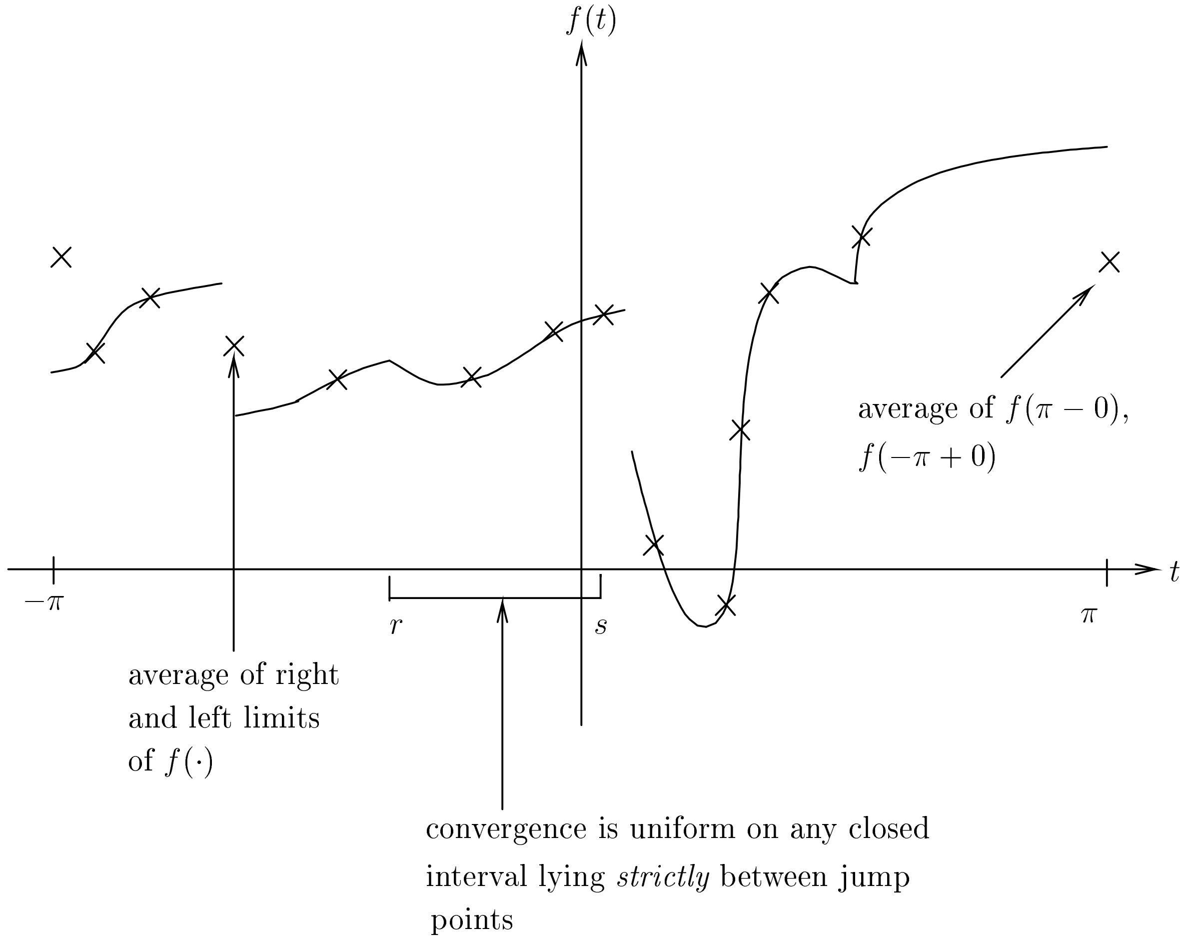
$$f_N(t^*) \to \frac{1}{2}[f(t^* - 0) + f(t^* + 0)], \quad N \to \infty.$$

(iii) $f_N(\pm\pi) \to \frac{1}{2}[f(-\pi + 0) + f(\pi - 0)]$, $N \to \infty$.

(iv) If $-\pi < r < s < \pi$ and $[r, s]$ includes no jump point of $f(\cdot)$, then $f_N \to f$ uniformly with respect to $t \in [r, s]$.

(v) If $f(\cdot)$ has no jump points for $-\pi < t < \pi$ and if $f(-\pi) = f(\pi)$, then $f_N \to f$ uniformly with respect to $t \in [-\pi, \pi]$.

Here's the picture, which is easy to remember: The values $\times$ are the limits of Fourier Series.



Proof:

Extend f so as to be periodic on $-\infty < t < \infty$. Then for fixed t, $f(x)D_N(x-t)$ is periodic in x with period 2π. Therefore

$$\begin{aligned}
f_N(t) &= \int_{-\pi}^{\pi} f(x)D_N(x-t)dx \\
&= \int_{-\pi-t}^{\pi-t} f(x'+t)D_N(x')dx' \qquad (x' = x-t) \\
&= \int_{-\pi}^{\pi} f(x+t)D_N(x)dx \quad \text{(by periodicity)} \\
&= \int_{-\pi}^{0} f(x+t)D_N(x)dx + \int_{0}^{\pi} f(x+t)D_N(x)dx \\
&= \int_{0}^{\pi} f(t-x)D_N(x)dx + \int_{0}^{\pi} f(t+x)D_N(x)dx
\end{aligned}$$

(replacing x by $-x$ in the first integral
and using $D_N(-x) = D_N(x)$)

$$= \int_0^\pi D_N(x)[f(t+x) + f(t-x)]dx.$$

Recalling $\int_0^\pi D_N(x)dx = \frac{1}{2}$, we get

$$\begin{aligned} & f_N(t) - \frac{1}{2}[f(t-0) + f(t+0)] \\ &= \int_0^\pi D_N(x)[f(t+x) + f(t-x) - f(t-0) - f(t+0)]dx \\ &= \int_0^\pi D_N(x) g_t(x) dx, \end{aligned}$$

$$\text{where} \qquad g_t(x) := f(t+x) + f(t-x) - f(t-0) - f(t+0)$$

$$=: J_N(t), \quad \text{say.}$$

We must show that $J_N(t) \to 0$ as $N \to \infty$, i.e. $(\forall \varepsilon > 0)(\exists N')(\forall N) N > N' \Rightarrow |J_N(t)| < \varepsilon$.
So

Fix t.
Fix $\varepsilon > 0$.

Choose δ, $0 < \delta < \pi$, and write

$$J_N(t) = \int_0^\delta D_N g_t dx + \int_\delta^\pi D_N g_t dx.$$

We'll establish:

Claim 1: $\delta > 0$ can be chosen so small that

$$(\forall N) \left| \int_0^\delta D_N g_t dx \right| < \varepsilon/2; \tag{1}$$

and then

Claim 2: With δ fixed, $\exists N'$ so large that

$$\left| \int_\delta^\pi D_N g_t dx \right| < \varepsilon/2, \quad \text{all} \quad N > N'. \tag{2}$$

From Claims 1 and 2 will follow $|J_N(t)| < \varepsilon$, as desired.

Proof of Claim 1:

Let's be more precise about $g_t(\cdot)$, and define

$$g_t(x) := \begin{cases} f(t+x) + f(t-x) - f(t+0) - f(t-0), & x > 0 \\ 0, & x = 0. \end{cases}$$

Since $f \in PS, \exists\delta > 0$ such that $g_t(x)$ and $g_t'(x)$ are continuous for $0 \leq x \leq \delta$. Choose such $\delta > 0$ initially. Then, integrating by parts,

$$\int_0^\delta D_N g_t dx = g_t(\delta)\Delta_N(\delta) - \int_0^\delta \Delta_N(x) g_t'(x) dx.$$

Now Lemma 4 $\Rightarrow |\Delta_N(x)| < K$ (say), all N, and $x \in [0, \pi]$.

Also:

$$\begin{cases} |g_t'(x)| \leq \max\limits_{0 \leq x \leq \delta} |g_t'(x)| =: M(\delta), \quad \text{say}, \\ \\ g_t(\delta) \to 0 \quad \text{as} \quad \delta \to 0. \end{cases}$$

So

$$\left| \int_0^\delta D_N g_t dx \right| \leq |g_t(\delta)| K + \delta K M(\delta) < \varepsilon/2$$

for $\delta > 0$ sufficiently small. Now you *fix* such δ and go on to

Proof of Claim 2:

Here Lemma 2 $\Rightarrow \displaystyle\int_\delta^\pi D_N g_t dx \to 0,\ N \to \infty$

i.e.

$$\exists N' : \left| \int_\delta^\pi D_N g_t dx \right| < \varepsilon/2, \quad \text{all} \quad N > N'.$$

We have now proved Claims 1 and 2, hence statements (i) - (iii) of THEOREM.

To prove (iv), you show that δ and N' can be chosen so that (1) and (2) hold for all t in $[r, s]$ simultaneously. Here are the details:

For fixed δ, $0 < \delta \leq \pi$, the proofs of Lemmas 1 and 2 guarantee that the estimate (2) holds uniformly for $r \leq t \leq s$. In particular, choose $\delta > 0$ such that $f(t)$ has no jump in $r - \delta \leq t \leq s + \delta$: this can always be done. Then $g_t(x)$ is continuous and bounded, and $\partial g_t(x)/\partial x$ is piecewise continuous and bounded, for $r \leq t \leq s$ and $0 \leq x \leq \delta$. Moreover, $g_t(x)$ is actually uniformly continuous for $r \leq t \leq s$ and $0 \leq x \leq \delta$ (since it is continuous there, and the domain is closed and bounded), and $g_t(0) = 0$. So $g_t(\delta) \to 0$ as $\delta \to 0$, uniformly for $r \leq t \leq s$. Therefore, given $\varepsilon > 0$, you can (and do) choose $\delta > 0$ so small that (1) holds for all $t \in [r, s]$ and then N' so large that (2) holds for all $t \in [r, s]$. But now you know:

$$(\forall\varepsilon > 0)(\exists N')(\forall N > N' \text{ and } t \in [r, s]) : |J_N(t)| < \varepsilon.$$

This is exactly the statement that

$$\lim_{N\to\infty} J_N(t) = 0 \text{ unif. for } t \in [r, s].$$

Finally, to prove (v) observe that (the periodic extension of) f is now continuous and piecewise smooth for all t, $-\infty < t < \infty$. In particular the argument employed in proving (iv) is now valid on setting $r = -\pi$, $s = \pi$. Q.E.D.

As an immediate consequence of Dirichlet's theorem, we also have a convergence result for complex exponential Fourier Series.

Definition: Complex exponential Fourier Series (CEFS)

Let $f(\cdot)$ be defined on $[-\pi, \pi]$. The *complex exponential Fourier Series* of f is the series

$$\sum_{n=-\infty}^{\infty} c_n e^{int}, \quad t \in [-\pi, \pi], \tag{$CEFS$}$$

where

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t') e^{-int'} dt'.$$

Exercise 1.12: Convergence of CEFS

Show that Dirichlet's theorem holds for CEFS, provided the N^{th} partial sum is taken to be

$$f_N(t) := \sum_{n=-N}^{N} c_n e^{int}.$$

Hint: Just write out the real and imaginary parts of the partial sum, and apply the original theorem to these components.

Exercise 1.13: CEFS on an arbitrary finite interval

If $f(\cdot)$ is defined on $[t_0, t_1]$, show that an appropriate definition for CEFS is

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n \exp\left(i \frac{2\pi n}{t_1 - t_0} t\right)$$

where

$$c_n = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f(t') \exp\left(\frac{-2\pi i n t'}{t_1 - t_0}\right) dt'$$

Example: Calculation of a CEFS

We revisit the example on p. 5:

$$f(t) = \begin{cases} 0, & -\pi \le t \le 0 \\ t/\pi, & 0 \le t \le \pi \end{cases}$$

For the CEFS, with $n = 0, \pm 1, \pm 2, ...$, we have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)dt \\ &= \frac{1}{2\pi^2}\int_{0}^{\pi} te^{-int}dt \\ &= \begin{cases} 1/4, & n = 0 \\ \frac{1}{2\pi^2 n^2}[(-1)^n(1+in\pi)-1], & n \neq 0 \end{cases} \end{aligned}$$

Note that the cases $n = 0$ and $n \neq 0$ should be calculated separately; and that $c_{-n} = \overline{c_n}$ since f is real-valued. For instance, setting $n = 1$,

$$\begin{aligned} c_1 &= (1/2\pi^2)(-2 - i\pi) \\ c_{-1} &= (1/2\pi^2)(-2 + i\pi) \end{aligned}$$

and

$$\begin{aligned} c_1 e^{it} + c_{-1}e^{-it} &= c_1 e^{it} + \overline{c_1 e^{it}} \\ &= 2 \text{ Re}(c_1 e^{it)} \\ &= 2[(\text{Re } c_1)\cos t - (\text{Im } c_1)\sin t] \\ &= -(2/\pi^2)\cos t + (1/\pi)\sin t \end{aligned}$$

in agreement with the result on p. 6.

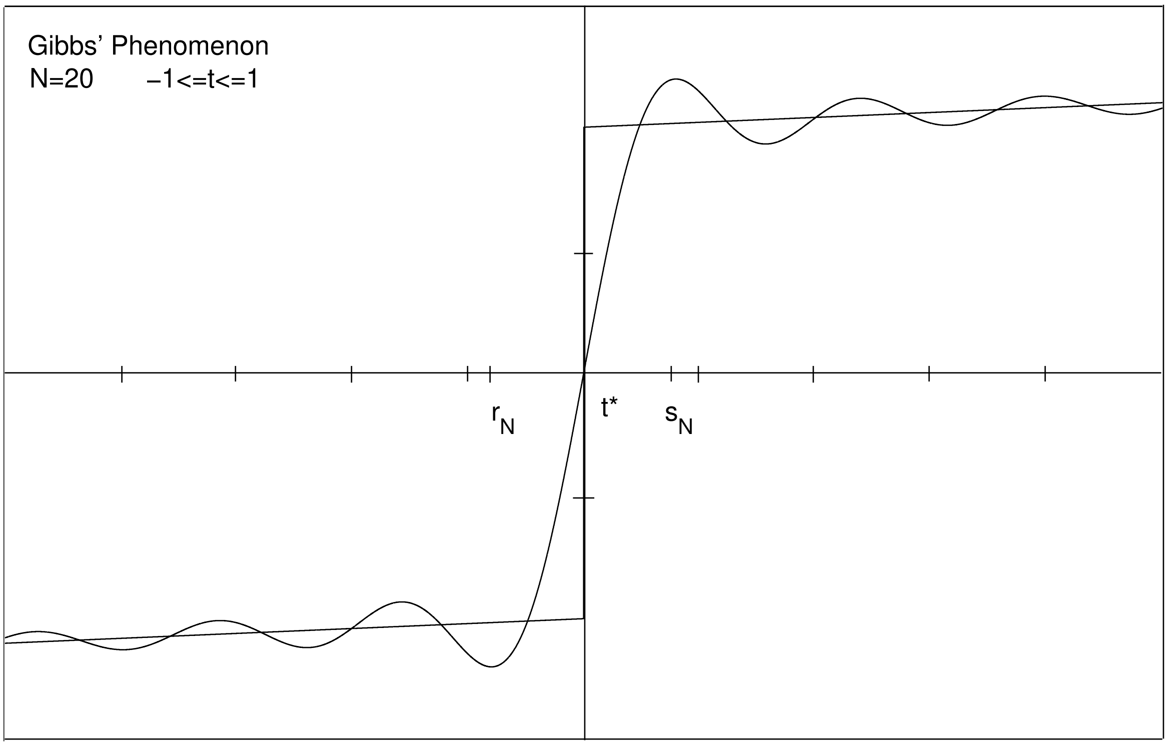
Remark: FS vs. CEFS

There is no 'rule' which dictates which of FS or CEFS is to be preferred, even when the target function f is real-valued: this is largely a matter of taste and/or convenience. Normally FS is more direct for numerical computation, while CEFS may be notationally more compact for theoretical calculations. Mathematically, the two representations are equivalent.

1.5 Gibbs' Phenomenon

As reported by J.W. Gibbs in 1899, 'Gibbs' phenomenon' is the tendency of the partial sums of a Fourier Series to peak sharply near a discontinuity of the original function. The effect is associated with the nonuniform convergence of the Fourier Series of f in any neighborhood of a jump point of f, combined with the oscillatory character of the Dirichlet kernel. For a simple instance of 'peaking', without reference to Fourier Series, review Example 6, p.19.

Let $f(\cdot) \in PS$, $-\pi \leq t \leq \pi$, and suppose f has a jump at $t = t^*$. The picture would appear as displayed below.



At the jump point t^*, we know

$$f_N(t^*)\underset{N\to\infty}{\longrightarrow}\frac{1}{2}[f(t^*+0)+f(t^*-0)].$$

Suppose for definiteness that $f(t^*+0) > f(t^*-0)$, as in the picture. Let $t = r_N$ be the location of the first *minimum* of $f_N(\cdot)$ for $t < t^*$, and $t = s_N$ the first *maximum* of $f_N(\cdot)$ for $t > t^*$: thus $r_N < t^* < s_N$. Then we have

THEOREM (Böcher, 1906).

1. For N sufficiently large:

$$r_N < r_{N+1} < \ldots, \text{ and } r_N \to t^* \text{ as } N \to \infty; \quad t^* - r_N \sim \pi/(N+\frac{1}{2})$$

$$s_N > s_{N+1} > \ldots, \text{ and } s_N \to t^* \text{ as } N \to \infty; \quad s_N - t^* \sim \pi/(N+\frac{1}{2})$$

2. $\lim_{N\to\infty} f_N(r_N) = f(t^*-0) - \gamma[f(t^*+0) - f(t^*-0)]$;

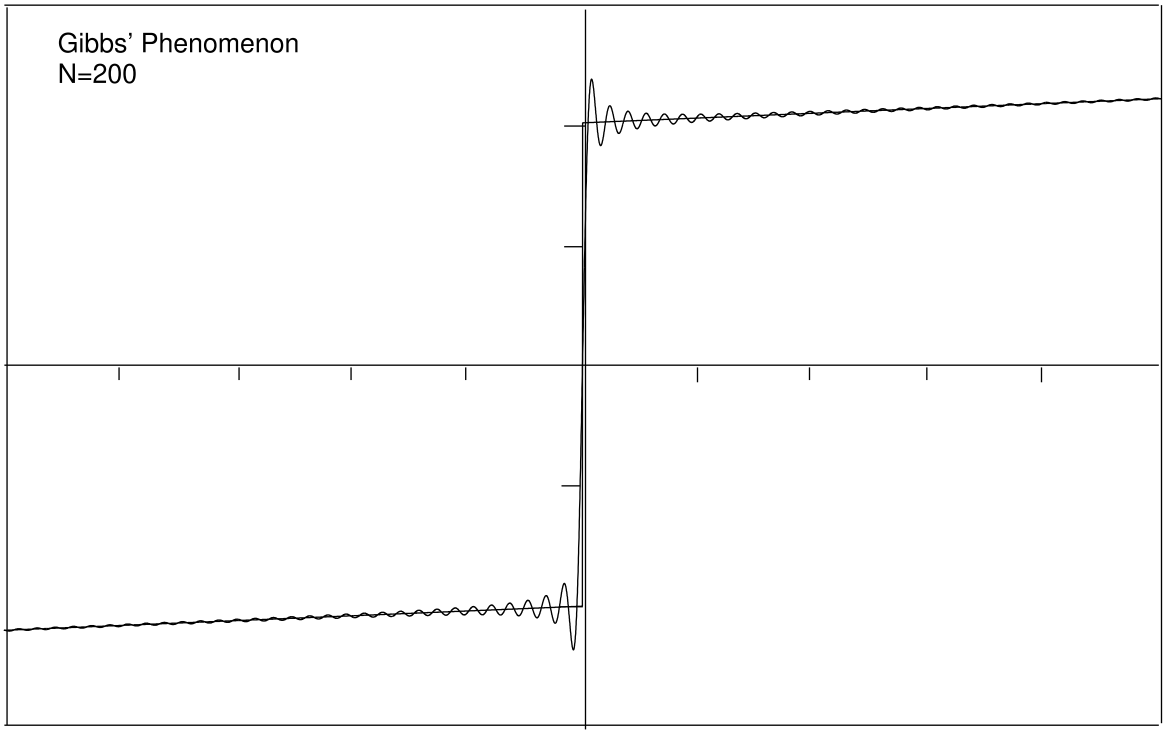
   $\lim_{N\to\infty} f_N(s_N) = f(t^*+0) + \gamma[f(t^*+0) - f(t^*-0)]$;
   and here

$$\gamma \ := \ \frac{1}{\pi}\left|\int_{\pi}^{\infty}\frac{\sin x}{x}dx\right|$$

$$\approx \frac{1}{\pi} \times 0.2811$$

$$= 0.0895.$$

In plain language, the graph of $f_N(\cdot)$ exhibits for large N an overshoot or undershoot (relative to f) near t^*, equal in magnitude to about 9% of the jump of f at t^*. For sufficiently large N, these would appear as two little spikes, as displayed below.



We shall present a heuristic calculation to make Böcher's theorem plausible (for a rigorous discussion see e.g. Carslaw pp.293ff). Let $f(\cdot)$ be periodic with period 2π, and

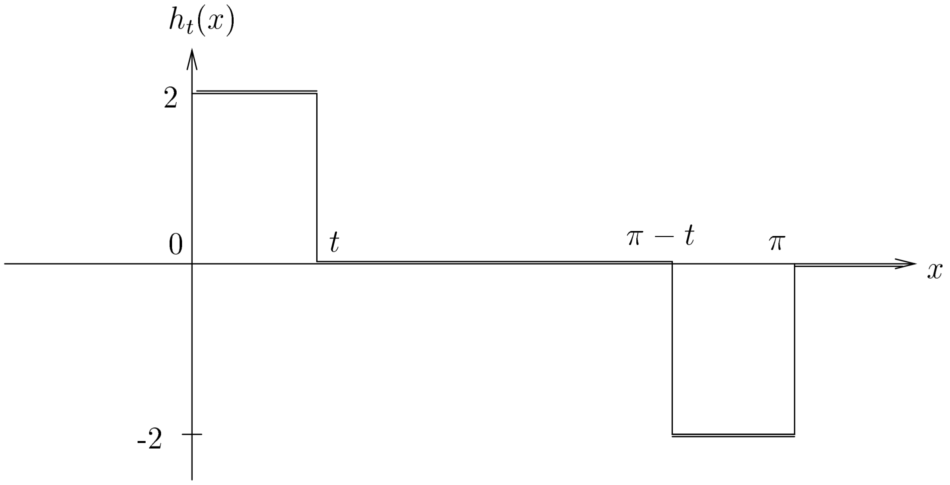
$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

As usual

$$\begin{aligned} f_N(t) &= \int_{-\pi}^{\pi} D_N(x-t) f(x) dx \\ &= \int_0^{\pi} D_N(x)[f(t+x) + f(t-x)] dx \\ &\quad \text{(as on pp. 28-29)} \\ &= \int_0^{\pi} D_N(x) h_t(x) dx, \quad \text{say} \end{aligned}$$

Taking $0 < t < \pi/2$, we have from the definition of f,

$$h_t(x) = \begin{cases} 2, & 0 < x < t \\ 0, & t < x < \pi - t \\ -2, & \pi - t < x < \pi \end{cases}$$



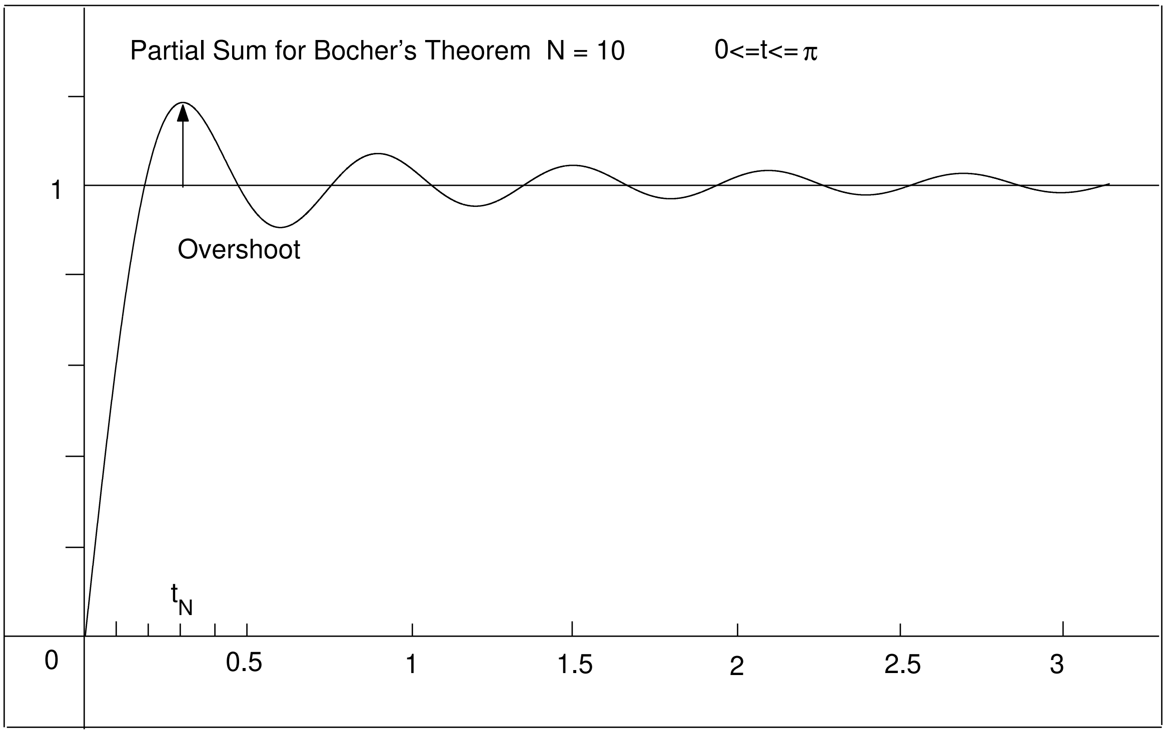
Assume $t > 0$ small and N large. On the basis of Lemma 2, p.25, we shall neglect the contribution to the integral for $\pi - t < x < \pi$:

$$\int_{\pi - t}^{\pi} D_N(x) h_t(x) dx \simeq 0$$

Therefore

$$\begin{aligned} f_N(t) &\simeq 2\int_0^t D_N(x)dx \\ &= \frac{1}{\pi}\int_0^t \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{1}{2}x)}dx \\ &\simeq \frac{2}{\pi}\int_0^t \frac{\sin(N+\frac{1}{2})x}{x}dx \qquad (*) \end{aligned}$$

since $0 \leq x \leq t$ implies x small. The graph of f_N is displayed below.



For $0 < t < \pi$, f_N has maxima and minima located where $f_N'(t) = 0$, i.e. approximately at

$$\left(N + \frac{1}{2}\right) t = \pi, 2\pi, 3\pi, \ldots, N\pi;$$

namely maxima at

$$t = \frac{\pi}{N + \frac{1}{2}}, \quad \frac{3\pi}{N + \frac{1}{2}}, \ldots$$

and minima at

$$t = \frac{2\pi}{N + \frac{1}{2}}, \quad \frac{4\pi}{N + \frac{1}{2}}, \ldots$$

At the first maximum $t_N \simeq \pi/(N + 1/2)$, we have

$$\begin{aligned} f_N(t_N) &\simeq \frac{2}{\pi} \int_0^{\pi/(N+1/2)} \frac{\sin(N + \frac{1}{2})x}{x} dx \\ &= \frac{2}{\pi} \int_0^{\pi} \frac{\sin u}{u} du \end{aligned}$$

Using the fact that $\int_0^\infty (\sin u/u) du = \pi/2$, we get, at the point $t = t_N$ at which the greatest overshoot occurs,

$$\frac{\text{overshoot}}{\text{jump}} \simeq \frac{\frac{2}{\pi} \int_0^{\pi} \frac{\sin u}{u} du - 1}{2}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{2}{\pi}\int_0^{\pi} - \frac{2}{\pi}\int_0^{\infty}\right)\frac{\sin u}{u}du \\
&= \frac{1}{\pi}\int_{\infty}^{\pi}\frac{\sin u}{u}du \quad (>0) \\
&= \frac{1}{\pi}\left|\int_{\pi}^{\infty}\frac{\sin u}{u}du\right|
\end{aligned}
$$

in agreement with the theorem of Böcher.

The nicest way to display Gibbs' phenomenon on a graph is first to normalize the time scale by setting $\tau = \left(N + \frac{1}{2}\right)t$, and then plot

$$g_N(\tau) := f_N\left(\frac{\tau}{N + \frac{1}{2}}\right)$$

for your actual partial sum f_N. According to the above analysis, $g_N(\tau)$ will be nearly independent of N if N is sufficiently large and $t = \tau/(N + \frac{1}{2})$ is small, say for $0 \le \tau \le 5\pi$. Thus

$$g_N(\tau) \simeq \frac{2}{\pi}\int_0^{\tau}\frac{\sin u}{u}du, \tag{†}$$

and the graph of g_N will display maxima very nearly at

$$\tau = \pi, 3\pi, 5\pi$$

along with minima at

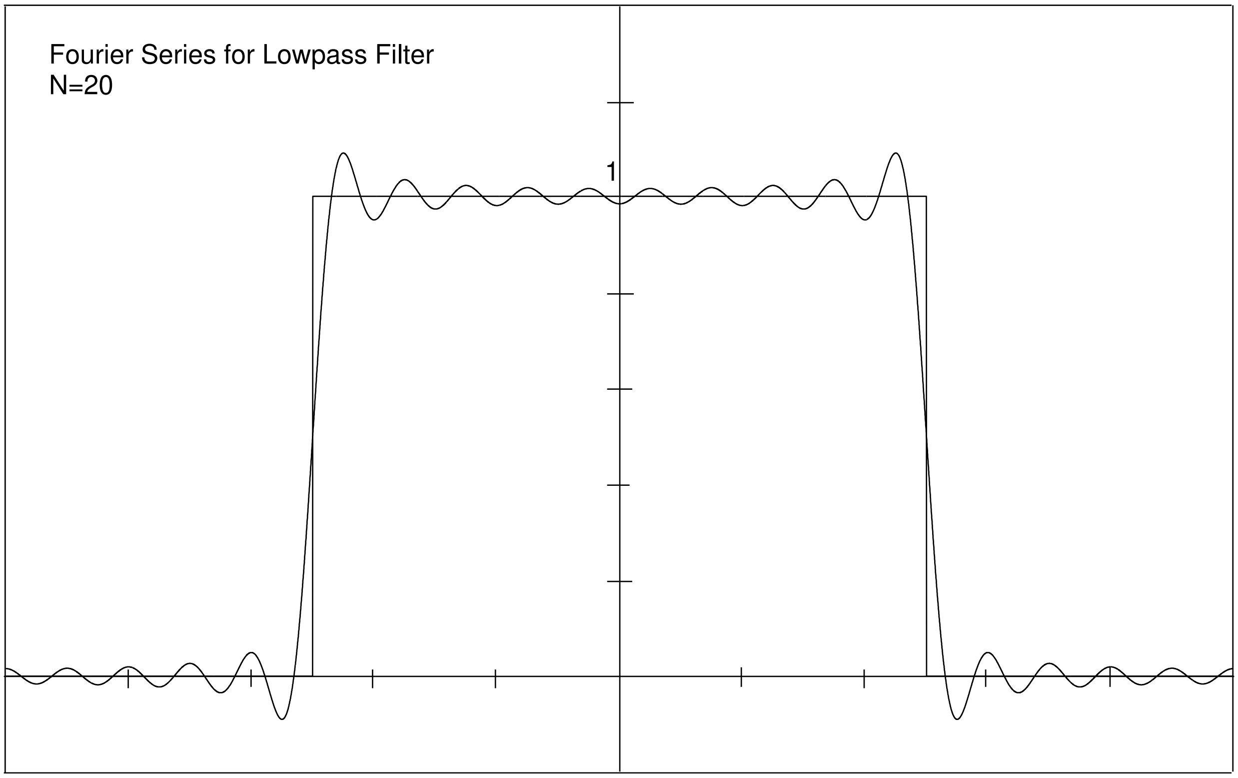
$$\tau = 2\pi, 4\pi.$$

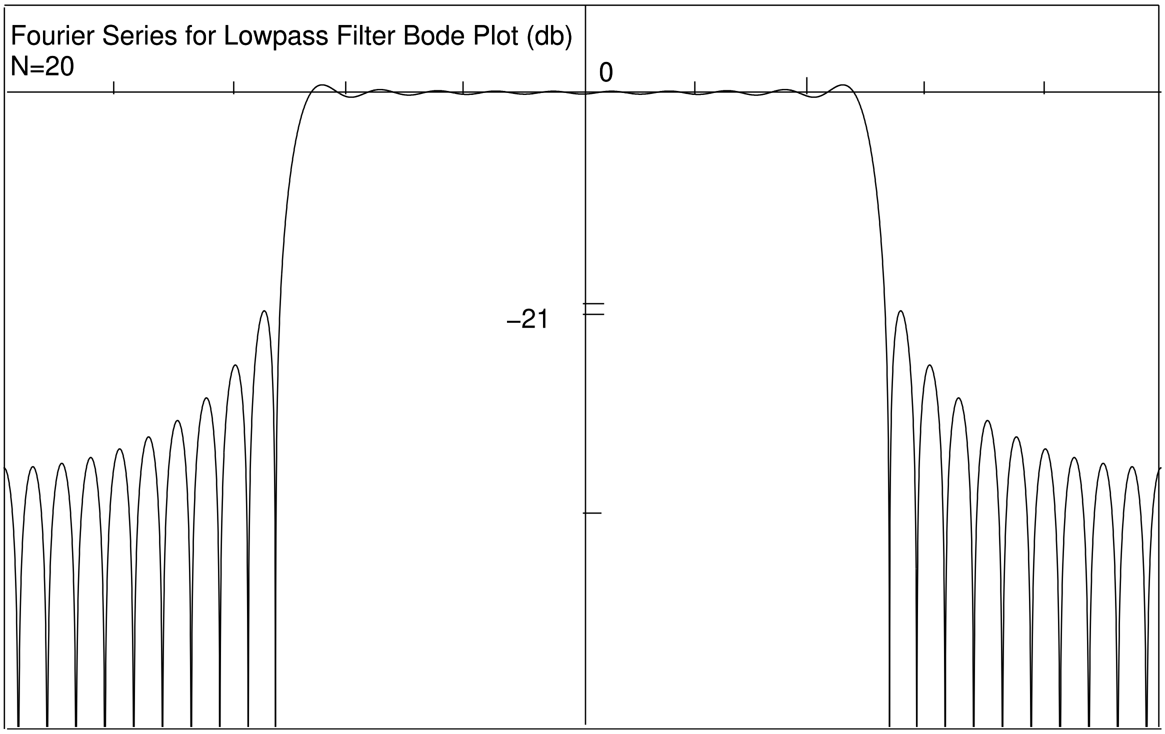
Furthermore the graph will be nearly the same (provided N is sufficiently large) for the Fourier series of any PS function f with the specified jump at $t = 0$. The results for other jump locations and magnitudes, and other base periods, are obtained by linear scaling in a straightforward way.

Exercise 1.14: For the signal f of Exercise 1.1, compute and plot the time-scaled functions $g_N(\tau)$ for $|\tau| \le 5\pi$ and $N = 100, 500, 800, 1000$. Discuss the results. Does the time-scaled picture stabilize in the predicted way? For each value of N, determine the location and magnitude of the Gibbs overshoot and undershoot: the magnitude should be calculated with respect to the value of f at the corresponding location. How do these location and magnitude parameters compare with the theoretical limits?

1.6 Cesàro Summation and Fejér Series

In the synthesis of digital filters, one may wish to approximate an ideal lowpass frequency characteristic (say) by a finite trigonometric sum (for ease of synthesis).





If you approximate with a *Fourier* partial sum, you will get the annoying Gibbs peaking effect at the pass-band edges. To some extent, this ripple can be smoothed by summing the terms of the Fourier series in a special way, namely in the *Cesàro* sense.

For this, let f_N be the N^{th} partial sum of the Fourier Series of f. Consider the modified functions

$$\hat{f}_N(t) := \frac{f_0(t) + f_1(t) + \ldots + f_{N-1}(t)}{N}, \quad N = 1, 2, \ldots$$

where

$$f_0(t) := a_0/2.$$

After a little algebra (verify!) you can write this as

$$\hat{f}_N(t) = \int_{-\pi}^{\pi} f(t')F_N(t'-t)dt' \qquad (*)$$

where

$$F_N(x) := \frac{\sin^2\left(\frac{1}{2}Nx\right)}{2\pi N \sin^2\left(\frac{1}{2}x\right)}.$$

F_N is the *Fejér kernel*, and $\hat{f}_N$ is the N^{th} *Cesàro sum* of Fourier Series.

Exercise 1.15: Show that, if a_n, b_n are the Fourier coefficients of f, then

$$\hat{f}_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N-1}(a_{N,n}\cos nt + b_{N,n}\sin nt)$$

where, for $N \geq 2$,

$$a_{N,n} := \left(1 - \frac{n}{N}\right)a_n, \quad n = 1, 2, \ldots, N-1$$

$$b_{N,n} := \left(1 - \frac{n}{N}\right)b_n, \quad n = 1, 2, \ldots, N-1$$

Now verify the formula (*) for $\hat{f}_N$.

The Fejér kernel is a little better behaved than Dirichlet's, essentially because it is always nonnegative, and thus does not depend for its 'delta-function action' on 'cancellation by oscillation'. On the other hand, the Fejér center-lobe width $4\pi/N$ is about twice the Dirichlet value $2\pi/(N + \frac{1}{2})$, suggesting that the partial sums $\hat{f}_N$ converge more slowly than the f_N: this tends to be true in practice.

If $f(\cdot) \in PS$ and has a jump at t^* then $\hat{f}_N(t^*) \to (\frac{1}{2})[f(t^*+0) + f(t^*-0)]$, just as in the Fourier case. However, near t^*, $\hat{f}_N$ behaves more smoothly than the Fourier sum f_N, and the Gibbs 'spike' is now absent.[4] The best way to see this is to do

Exercise 1.16: Compute and plot some of the Cesàro sums for the function of Exercise 1.1, p. 7, and compare the corresponding results with the Fourier case.

[4]Nevertheless, the sequence $\{\hat{f}_N(t)\}$ *cannot* converge *uniformly* in any neighborhood of a jump point t^*, since the uniform limit of a sequence of continuous functions is necessarily continuous.

An intuitive explanation for the smoother behavior of $\hat{f}_N(\cdot)$ versus $f_N(\cdot)$ can be based on the formulas for $a_{N,n}$ and $b_{N,n}$, p. 39. You can see that the factor $(1-n/N)$ assigns relatively *small* weight to the *high* frequency components of $f(\cdot)$, as represented by the corresponding Fourier coefficients a_n, b_n for $n \lesssim N$.

This raises the interesting question of whether weighting functions other than $(1-n/N)$ might be even 'better', in some sense. The answer is 'yes'. In fact for the lowpass function (p. 38) the first amplitude peak outside the passband is at about -21 db. in the Fourier approximation (relative to a transmission level of 0 db. in the passband; -21 db. corresponds to the 9% 'spike'), at about -26 db. in the Cesàro approximation, but about -80 db. in the most refined approximations.

To complete this section we summarize some facts about the Fejér sums, in

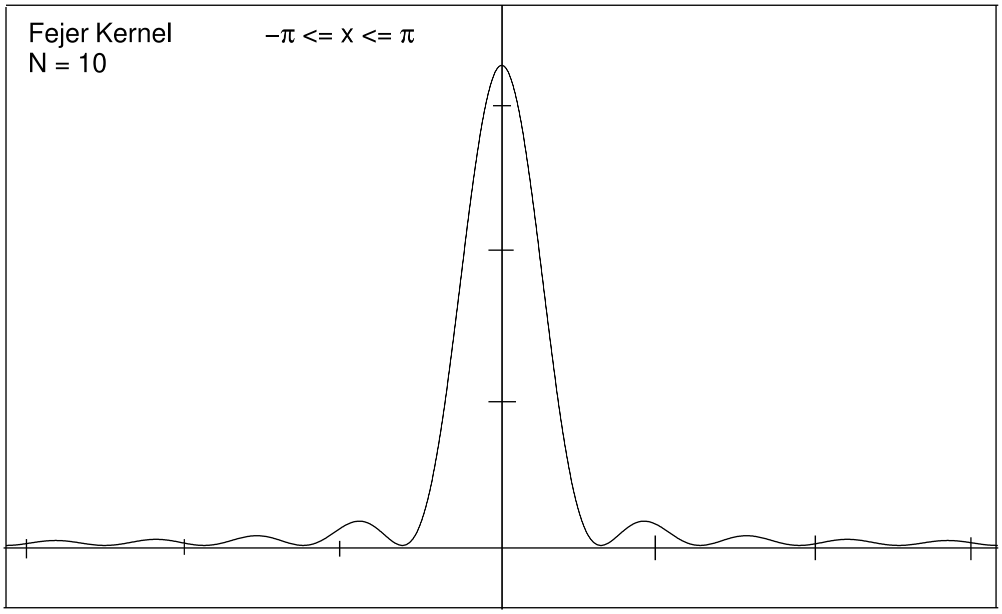
THEOREM (essentially due to Fejér - 1904)

(i) Let $f(\cdot)$ be an arbitrary function whose Fourier series exists. If $f_N(t)$ converges as $N \to \infty$, then so does $\hat{f}_N(t)$, and to the same value. In particular,

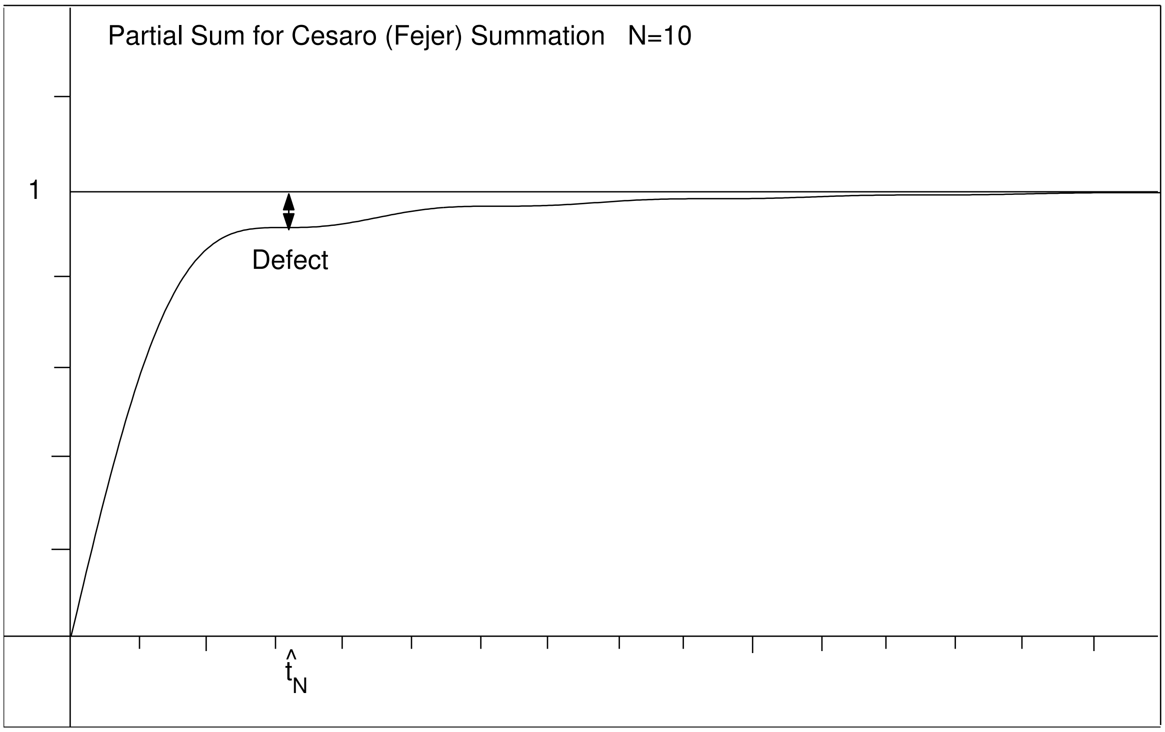
(ii) The main convergence theorem (p. 27) remains true if $f_N(\cdot)$ is replaced by $\hat{f}_N(\cdot)$ throughout.

(iii) If $f(\cdot)$ is merely continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, then $\hat{f}_N(t) \to f(t)$ as $N \to \infty$, uniformly for $t \in [-\pi, \pi]$.

We remark that (i) is true because 'ordinary' convergence always implies Cesàro convergence to the same value; and (iii) is *not* true (without added hypotheses such as $f \in PS$) for the Fourier sums f_N.



Exercise 1.17: Carry out an approximate analysis for large N of the behavior of Fejér's series near a jump point, obtaining counterparts of the results (*) (p. 35) and (†) on (p. 37). In this case the graph displays no overshoot, but instead a 'defect', as shown below.



Evaluate $\hat{t}_N$ approximately, and the ratio defect/jump, for large N.

1.7 Mean Square Convergence, Energy, and Power

The next major topic in Fourier Series we shall consider is the concept of mean square convergence, the geometry of mean square approximations, and their relation to energy and power calculations in engineering.

Let I be a real interval, $I \subseteq (-\infty, \infty)$. Let $L_2(I)$ be the class of all real- or complex-valued functions $f = f(t)$ defined for $t \in I$, and having the property

$$\int_I |f(t)|^2 dt < \infty. \qquad (*)$$

Namely $f \in L_2(I)$ iff it is finitely square-integrable on I. You can think of (*) as a condition of 'finite energy' on $f(\cdot)$.

Technically, f must first have the regularity property of being 'Lebesgue-measurable' on I, and the integral in (*) is supposed to be a 'Lebesgue integral'. But these tacit assumptions are certainly satisfied by the functions and integral we'll employ in concrete examples, e.g. PS functions and their 'ordinary' (Riemann) integrals.

By appeal to properties of measurability, and the Schwarz inequality (to be introduced later), it can be shown that $L_2(I)$ is a linear vector space (over the scalar field $\mathbb{C}$).

Exercise 1.18: Which of the functions below belong to $L_2(I)$?

1. $f(t) = 0, \quad t \in I = (-\infty, 0]$

2. $f(t) = 1, \quad t \in I = [0, 1]$

3. $f(t) = 1, \quad t \in I = [0, \infty)$

4. $f(t) = \sin t, \quad t \in I = [-\pi, \pi]$

5. $f(t) = \cos t, \quad t \in I = (-\infty, \infty)$

6. $f(t) = t^{-1/4}, \quad t \in I = (0, 1]$

7. $f(t) = t^{-1/4}, \quad t \in I = (0, \infty)$

8. $f(t) = (1 + t)^{-1/2}, \quad t \in I = [0, \infty)$

For $f \in L_2(I)$ the *norm* of f is

$$\|f\| := \left(\int_I |f(t)|^2 dt \right)^{1/2} \geq 0.$$

The norm is analogous to Euclidean length of a vector in $\mathbb{C}^n$: for if

$$x = (x_1 \dots x_n) \in \mathbb{C}^n, \quad x_i \in \mathbb{C},$$

then

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

Here's a subtlety: if $x \in \mathbb{C}^n$ and $\|x\| = 0$, then clearly $x = 0$. But if $f \in L_2(I)$ and $\|f\| = 0$, the most you can say in general is

$$f(t) = 0 \quad \text{for almost all } t$$

or

$$f = 0 \quad \text{a.e. (almost everywhere).}$$

Technically, a.e. means 'except possibly on a subset of I of Lebesgue measure zero'. For instance, all finite or countably infinite subsets are of measure zero. Thus if $I = [-1, 1]$ and

$$f(t) = \begin{cases} 17, & t = -1 \\ -3\pi^2, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

then $\|f\| = 0$, but it's *not* (quite) true that $f = 0$, i.e. $f(t) \equiv 0$, as there are two points t where $f(t) \neq 0$.

If $f, g \in L_2(I)$ their *scalar product* (or *inner product*) is

$$\langle f, g \rangle := \int_I f(t)\overline{g(t)} dt.$$

Here the overbar means complex conjugate. Scalar product is analogous to that for Euclidean n-vectors: if $x = (x_1 \dots x_n) \in \mathbb{C}^n$ and $y = (y_1 \dots y_n) \in \mathbb{C}^n$, recall

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \bar{y}_i.$$

Let f, g etc. $\in L_2(I)$, and $c \in \mathbb{C}$. Note the easy relations

$$\begin{aligned}
\langle f, g_1 + g_2 \rangle &= \langle f, g_1 \rangle + \langle f, g_2 \rangle \\
\langle f_1 + f_2, g \rangle &= \langle f_1, g \rangle + \langle f_2, g \rangle \\
\langle cf, g \rangle &= c\langle f, g \rangle \\
\langle f, cg \rangle &= \bar{c}\langle f, g \rangle \\
\langle f, g \rangle &= \overline{\langle g, f \rangle} \\
\langle f, f \rangle &= \|f\|^2
\end{aligned}$$

Now recall that two Euclidean vectors x, y are 'orthogonal' or 'perpendicular' if $\langle x, y \rangle = 0$. Analogously, $f, g \in L_2(I)$ are *orthogonal* if $\langle f, g \rangle = 0$. Furthermore, if $\{f_n\} := \{f_n : n = 1, 2, 3, \dots\}$ is a sequence of functions in $L_2(I)$, then it is an *orthogonal sequence* if

$$\langle f_n, f_m \rangle = 0 \quad \text{whenever } n \neq m.$$

Examples of orthogonal sequences

1. Trigonometric system:

$$1, \sin x, \cos x, \sin 2x, \cos 2x, ..., \quad x \in I = [-\pi, \pi].$$

2. Exponential system:

$$1, e^{ix}, e^{-ix}, e^{2ix}, e^{-2ix}, ..., \quad x \in I = [-\pi, \pi]$$

3. Legendre polynomials:

$$1, x, 3x^2 - 1, 5x^3 - 3x, ..., \quad x \in I = [-1, 1];$$

the sequence is manufactured by applying Gram-Schmidt orthogonalization to the nonorthogonal sequence

$$1, x, x^2, x^3, ...$$

4. Any subset in the above lists.

Suppose $\{\phi_n : n = 1, 2, \ldots\}$ is orthogonal on I. Given $f \in L_2(I)$, suppose that

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

with the $c_n \in \mathbb{C}$. Then, formally,

$$\begin{aligned} \langle f, \phi_m \rangle &= \langle \sum_{n=1}^{\infty} c_n \phi_n, \phi_m \rangle \\ &= \sum_{n=1}^{\infty} c_n \langle \phi_n, \phi_m \rangle \\ &= c_m \langle \phi_m, \phi_m \rangle \\ &= c_m \|\phi_m\|^2 \end{aligned}$$

or

$$c_m = \langle f, \phi_m \rangle / \|\phi_m\|^2 \qquad (*)$$

This motivates the following generalization of the idea of 'Fourier Series':

Definition. The series

$$\sum_{n=1}^{\infty} c_n \phi_n$$

with coefficients given by (*), is the *Fourier Series of* f *with respect to the (orthogonal) sequence* $\{\phi_n\}$. The c_n are the *Fourier coefficients of* f *with respect to* $\{\phi_n\}$.

Of course, there is as yet no implication whatever that the Fourier Series of f actually converges (in any sense) to f. Namely we do not assume that f 'equals' its Fourier series.

The formula (*) would be simplified if instead of $\{\phi_n\}$, we used the sequence $\{\psi_n\}$, where

$$\psi_n(x) := \phi_n(x) / \|\phi_n\|.$$

[To avoid fuss we assume $\|\phi_n\| > 0$, all n]. Then $\langle \psi_n, \psi_m \rangle = \delta_{nm}$. The system ψ_n is said to be *orthonormal* (ON).

From now on we assume the $\{\phi_n\}$ are ON.

Geometrically, think of the ϕ_n as unit orthogonal vectors in the (infinite-dimensional) linear space of functions $L_2(I)$. Then $c_n \phi_n$ is just the orthogonal projection of f on the ϕ_n 'axis', and $\sum_1^N c_n \phi_n$ is the orthogonal projection of f on the subspace spanned by $\{\phi_1, \ldots, \phi_N\}$.

Thinking of this projection as an 'approximation' to f, what is the approximation error? Simply,

$$e_N := f - f_N, \quad \text{where} \quad f_N := \sum_{n=1}^{N} c_n \phi_n.$$

Using the ON property of $\{\phi_n\}$ and expanding the scalar product, it's easy to verify

$$\|f_N\|^2 := \langle f_N, f_N\rangle = \sum_{n=1}^{N} |c_n|^2$$

Now we have

$$\begin{aligned}
\langle e_N, f_N\rangle &= \langle f - f_N, f_N\rangle \\
&= \langle f, f_N\rangle - \|f_N\|^2 \\
&= \langle f, \sum_{n=1}^{N} c_n\phi_n\rangle - \|f_N\|^2 \\
&= \sum_{n=1}^{N} \bar{c}_n\langle f, \phi_n\rangle - \|f_N\|^2 \\
&= \sum_{n=1}^{N} \bar{c}_n c_n - \|f_N\|^2 \\
&= \sum_{n=1}^{N} |c_n|^2 - \sum_{n=1}^{N} |c_n|^2
\end{aligned}$$

or

$$\langle e_N, f_N\rangle = 0. \tag{1}$$

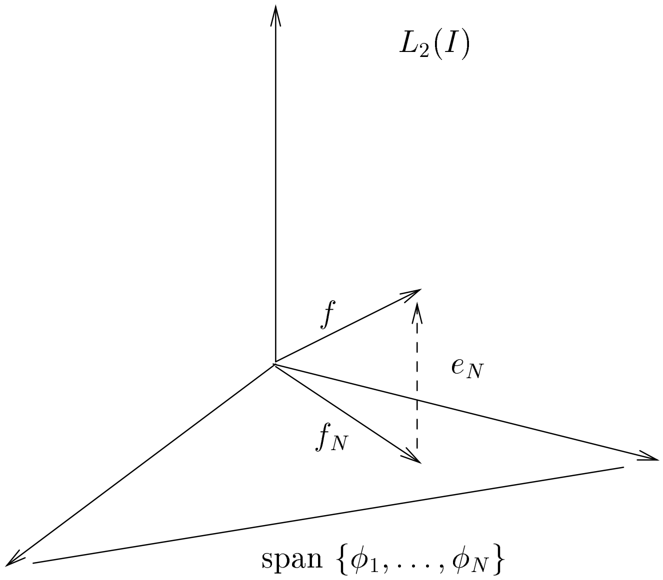
Thus

$$\langle f, f\rangle = \langle e_N + f_N, e_N + f_N\rangle$$

or

$$\|f\|^2 = \|e_N\|^2 + \|f_N\|^2. \tag{2}$$

Thus one has the Pythogorean picture:



with the error e_N orthogonal to the approximation f_N.

Corollary 1: Bessel's inequality

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|^2.$$

Proof:

$$\begin{aligned}\sum_{n=1}^{\infty} |c_n|^2 &= \lim_{N\to\infty} \sum_{n=1}^{N} |c_n|^2 = \lim_{N\to\infty} \|f_N\|^2 \\ &= \lim_{N\to\infty} (\|f\|^2 - \|e_N\|^2) \\ &\leq \|f\|^2.\end{aligned}$$

Q.E.D.

Corollary 2: L_2 version of Riemann-Lebesgue

If $f \in L_2(I)$ then the n^{th} Fourier coefficient of f w.r.t. $\{\phi_n\}$ tends to 0 as $n \to \infty$.

Proof:

n^{th} Fourier coefficient $= c_n$, and Bessel says

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|^2 < \infty;$$

hence

$$c_n \to 0 \quad \text{as} \quad n \to \infty.$$

Q.E.D.

We now bring in a concept of convergence that is matched to the idea of comparing two signals by measuring the energy of their difference signal.

Definition. A sequence $\{f_N(\cdot)\}$ converges to $f(\cdot)$ *in mean square (m.s.)* if

$$\lim_{N\to\infty} \|f - f_N\| = 0.$$

Notation: One writes $f_N \to f$ (m.s.) as $N \to \infty$, or $\underset{N\to\infty}{l.i.m.} f_N = f$ (where "$l.i.m.$" means "limit in the mean").

Thus $f_N \to f(m.s.)$ precisely when the numerical sequence $\|f - f_N\|^2$ (energy of the signal $f - f_N$) converges to 0 in the standard sense of numerical convergence.

Examples: Mean square convergence

1. Here's a simple example of how you check m.s. convergence: let

$$f_N(t) = \begin{cases} 0, & t \le 0 \\ N^{1/3}, & 0 < t < 1/N \\ 0, & t \ge 1/N \end{cases}$$

[Draw the picture!]. We claim: $f_N \to 0$ (m.s.) as $N \to \infty$. In fact:

$$\begin{aligned} \|f_N - 0\|^2 &= \int_{-\infty}^{\infty} |f_N(t) - 0|^2 dt \\ &= \int_0^{1/N} (N^{1/3})^2 dt \\ &= N^{2/3} \cdot (1/N) \\ &\to 0 \text{ as } N \to \infty. \end{aligned}$$

2. Now here's an example where you don't have m.s. convergence: let

$$f_N(t) = \begin{cases} 0, & t \le 0 \\ N^{1/2}, & 0 < t < 1/N \\ 0, & t \ge 1/N \end{cases}$$

We have: for each fixed t, $f_N(t) \to 0$ as $N \to \infty$. But

$$\begin{aligned} \|f_N - 0\|^2 &= \int_0^{1/N} (N^{1/2})^2 dt \\ &= 1, \text{ all } N! \end{aligned}$$

And it's possible to show that f_N does not converge to *anything*, in mean square [you do this by checking that $\{f_N\}$ is not a Cauchy sequence, in the norm $\|\cdot\|$].

3. Finally, here's an example of a sequence $\{f_N\}$ such that $f_N \to 0$ (m.s.), but the numerical sequence $\{f_N(t)\}$ fails to converge to anything, at any fixed t, as $N \to \infty$: first let

$$\begin{aligned} k_N &:= \text{largest integer} \leq \log_2 N \quad (N \geq 1) \\ &= \text{largest integer } k \text{ such that } 2^k \leq N; \end{aligned}$$

then let

$$t_N := \frac{N - 2^{k_N}}{2^{k_N}}, \quad N = 1, 2, \ldots;$$

and finally, on the interval $0 \leq t \leq 1$, define

$$f_N(t) := \begin{cases} 1, & t_N \leq t \leq t_N + 2^{-k_N} \\ 0, & \text{otherwise} \end{cases}$$

Exercise 1.9: Check that this example does what it was advertised to do. [From N. Wiener, *The Fourier Integral*, p. 29.]

MORAL: Pointwise convergence (i.e. at each fixed t), and m.s. convergence, are quite different things.

Nevertheless, in practice the two are often found together.

Our new concepts are related by the next result.

THEOREM

The following two statements are equivalent.

(1) For every $f \in L_2(I)$, if $\{c_n\}$ are the Fourier coefficients of f with respect to $\{\phi_n\}$, i.e. $c_n = \langle f, \phi_n \rangle$, then

$$\|f\|^2 = \sum_1^\infty |c_n|^2.$$

This says that Bessel's inequality actually becomes equality.

(2) For every $f \in L_2(I)$, the Fourier Series of f relative to $\{\phi_n\}$ converges to f in mean square, namely

$$\underset{N\to\infty}{l.i.m.} \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n = f.$$

Proof:

Assume statement (1). To prove (2), let $f \in L_2(I)$, with Fourier coefficients c_n. Then

$$\begin{aligned}
\|f - f_N\|^2 &= \|e_N\|^2 \\
&= \|f\|^2 - \|f_N\|^2 \\
&= \|f\|^2 - \sum_1^N |c_n|^2
\end{aligned}$$

and so

$$\begin{aligned}
\lim_{N\to\infty} \|f - f_N\|^2 &= \|f\|^2 - \sum_1^\infty |c_n|^2 \\
&= 0 \qquad \text{by (1).}
\end{aligned}$$

Conversely assume (2). To prove (1), we have

$$\begin{aligned}
\|f\|^2 - \sum_1^N |c_n|^2 &= \|f\|^2 - \|f_N\|^2 \\
&= \|e_N\|^2 \\
&= \|f - f_N\|^2
\end{aligned}$$

Taking the limit $N \to \infty$ gives

$$\begin{aligned}
\|f\|^2 - \sum_1^\infty |c_n|^2 &= \lim_{N\to\infty} \|f - f_N\|^2 \\
&= 0,
\end{aligned}$$

by (2) and the definition of m.s. convergence. Q.E.D.

The equation in statement (1) of the Theorem is *Parseval's formula.* Explicitly

$$\|f\|^2 = \sum_{n=1}^\infty |\langle f, \phi_n\rangle|^2$$

Parseval's formula is important in applications because, when it is valid, it allows you to calculate the 'energy' $\|f\|^2$ of f, in terms of the Fourier coefficients of f. In the standard case of Fourier trigonometric series, these coefficients are, of course, the amplitudes of the various frequency components of f. In words, Parseval's formula says that the total 'energy' of f [over one basic period] is the sum of the energies carried by the distinct sinusoidal frequency components of f. If f is periodic, a similar statement evidently holds for the 'power' of f. So, a Fourier series representation of f can be used to investigate how the power vs. frequency spectrum of f might be shaped by selective filtering. For an example, see Ex. 1.22, p. 53.

The validity of Parseval's formula (i.e. the truth of statement (1) in the Theorem) is a property (either true or false) of the particular ON system $\{\phi_n\}$ being studied. For the 'classical' systems (trigonometric functions, Legendre functions, etc.), Parseval's formula is indeed valid, this assertion being traditionally named *Parseval's Theorem.*

It's worthwhile to verify Parseval's theorem in our standard case. For technical simplicity we shall place some additional restrictions on f, but these aren't actually necessary.

Parseval's Theorem - Special Case of Fourier Trigonometric Series.

Let $I = [-\pi, \pi]$, and f real-valued on I. Assume f and f' are continuous on I, and that $f(-\pi) = f(\pi)$. Let f have Fourier trigonometric series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{1}$$

Then

$$\|f\|^2 := \int_{-\pi}^{\pi} |f(t)|^2 dt = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)\right].$$

[This shows that 'power' of f := 'energy' per unit time

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt \\ &= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)] \end{aligned}$$

Proof: By Dirichlet's theorem, in this case

$$f_N(t) \to f(t) \text{ as } N \to \infty$$

uniformly on $[-\pi, \pi]$. So (1) $\Rightarrow$

$$f(t)^2 = \frac{a_0}{2} f(t) + \sum_{n=1}^{\infty} \left[a_n f(t) \cos nt + b_n f(t) \sin nt\right],$$

the $\sum$ being a series of continuous functions, uniformly convergent on $[-\pi, \pi]$. So termwise integration is valid, giving

$$\begin{aligned} \|f\|^2 &= \int_{-\pi}^{\pi} f(t)^2 dt \\ &= \frac{a_0}{2}\int_{-\pi}^{\pi} f(t)dt + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} f(t)\cos ntdt + b_n \int_{-\pi}^{\pi} f(t)\sin ntdt\right] \\ &= \frac{a_0}{2}\left(2\pi\frac{a_0}{2}\right) + \sum_{n=1}^{\infty}\left[a_n\left(\pi a_n\right) + b_n\left(\pi b_n\right)\right] \\ &= \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]. \end{aligned}$$

Q.E.D.

Exercise 1.20: Generalize the above proof to functions merely PS on $[-\pi, \pi]$.

From the proof you may note two more handy formulas:
R.m.s. value of f on $[-\pi, \pi]$

$$\begin{aligned}
&:= \left(\text{Mean value of } |f(t)|^2 \text{ on } [-\pi, \pi]\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt\right)^{1/2} \\
&= \frac{1}{\sqrt{2\pi}}\|f\| \\
&= \left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]^{1/2}.
\end{aligned}$$

R.m.s. error of N^{th} approximation

$$:= \frac{1}{\sqrt{2\pi}}\|f - f_N\| = \frac{1}{\sqrt{2\pi}}\|e_N\|,$$

and

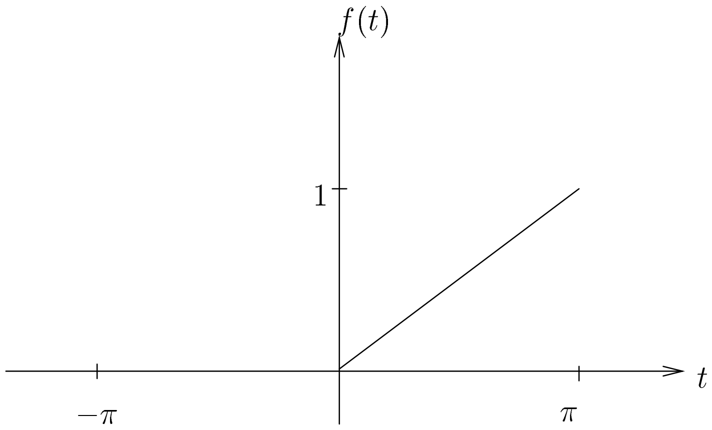
$$\begin{aligned}
\|e_N\|^2 &= \|f\|^2 - \|f_N\|^2 \\
&= \|f\|^2 - \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right] = \pi\sum_{n=N+1}^{\infty}\left(a_n^2 + b_n^2\right).
\end{aligned}$$

Tail estimation of $\|e_N\|$

Often in applications you need to know how good, in the r.m.s. error sense, your approximation to f by f_N will be. Or in other words, how large should N be taken so that r.m.s. error is within some specified tolerance? Rather than going to the trouble of computing exactly a least possible N, you may be content to have a conservative (i.e. high) estimate of N, as a rule-of-thumb. Getting such estimates is something of an art, but the following example illustrates one or two standard tricks.

Let

$$f(t) = \begin{cases} t/\pi, & 0 \le t \le \pi \\ 0, & -\pi \le t \le 0 \end{cases}$$



We have

$$f(t)=\frac{1}{4}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$

where, for $n\geq 1$,

$$a_n = \begin{cases} -2/n^2\pi^2, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$$

$$b_n = (-1)^{n+1}/n\pi$$

We know

$$\begin{aligned}\|e_N\|^2 &= \pi\sum_{n=N+1}^{\infty}\left(a_n^2+b_n^2\right)\\ &= \pi\sum_{\substack{n=N+1\\ n \text{ odd}}}^{\infty}\frac{4}{n^4\pi^4}+\pi\sum_{n=N+1}^{\infty}\frac{1}{n^2\pi^2}\\ &= \frac{4}{\pi^3}\sum_{n=\left[\frac{N+1}{2}\right]}^{\infty}\frac{1}{\left[2(n+\frac{1}{2})\right]^4}+\frac{1}{\pi}\sum_{n=N+1}^{\infty}\frac{1}{n^2} \qquad (*)\end{aligned}$$

Here $[\alpha]$ means the greatest integer $\leq \alpha$. Now recall the tail estimate derived in Example 2, Sect. 1.3: for $p>1$,

$$\sum_{n=k+1}^{\infty}\frac{1}{n^p}\leq\frac{1}{p-1}\cdot\frac{1}{k^{p-1}}\ .$$

Applying this result to our example, we get from (*),

$$\|e_N\|^2\leq\frac{4}{\pi^3\cdot 16}\cdot\frac{1}{3}\cdot\frac{1}{\left[\dfrac{N-1}{2}\right]^3}+\frac{1}{\pi}\cdot\frac{1}{1}\cdot\frac{1}{N}$$

For large N (e.g. $N \gtrsim 10$), the first term on the right is negligible compared to the second, therefore

$$\|e_N\|^2 \lesssim \frac{1}{\pi N}, \quad N \to \infty.$$

Now calculate

$$\begin{aligned} \|f\|^2 &= \int_{-\pi}^{\pi} |f(t)|^2 dt = \int_0^{\pi} (t/\pi)^2 dt \\ &= \pi/3. \end{aligned}$$

Therefore, normalized r.m.s. error

$$\begin{aligned} &:= \|e_N\|/\|f\| \\ &\lesssim \sqrt{3/N\pi^2} \\ &\leq 1/\sqrt{3N} \quad (\text{ since } \pi > 3). \end{aligned}$$

Thus normalized r.m.s. error $\leq 1\%$ if $1/\sqrt{3N} \leq 0.01$, or $N \geq N' \approx 3,300$. To see how conservative this 'tail estimate' of N may be, do

Exercise 1.21: Compute, for the foregoing example,

$$\frac{\|e_N\|}{\|f\|} = \frac{\sqrt{\|f\|^2 - \|f_N\|^2}}{\|f\|}$$

as a function of N, up to (say) $N = 4000$. Here

$$\|f_N\|^2 = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{N} \left(a_n^2 + b_n^2 \right) \right]$$

with $a_0 = \frac{1}{2}$ and a_n, b_n given on p. 52. Plot the graph of $\|e_N\|/\|f\|$ versus N. Repeat this exercise for the Fejér approximations, obtaining a counterpart formula for the Fejér (Cesàro) partial sums $\hat{f}_N$, in terms of the Fourier coefficients a_n and b_n. Warning: The normalized Fejér approximation error $\|\hat{e}_N\|/\|f\|$ is *not* given by the first formula above with f_N replaced by $\hat{f}_N$. Why not?

Exercise 1.22: Half-wave vs. full-wave rectification

For the filtered half-wave (or full-wave) rectified sine wave (Exercise 1.6, p. 12) define

$$\rho := r.m.s. \text{ ripple } := \sqrt{\frac{\text{a.c. power}}{\text{d.c. power}}}$$

$$= \sqrt{\frac{\|f - a_0/2\|^2/(2\pi/\omega_0)}{(a_0/2)^2}}$$

$$= \frac{\|f - a_0/2\|}{a_0\sqrt{\pi/2\omega_0}}$$

where f is the *filtered*, rectified output. In each of the two cases

1. Calculate an exact expression for the ripple factor when no filter is present (i.e. filter time constant $= 0$).
2. Use tail estimation to obtain conservative, but reasonable, estimates of the filter time constant required to reduce r.m.s. ripple to 5% (i.e. 0.05) and to 1% respectively.
3. Obtain accurate answers by computer and compare results.

Can you draw any conclusions about the relative virtues of full-wave vs. half-wave rectification?

HINT: Outline of procedure for full-wave case

Step 1. First suppose the filter is absent. Then

$$\begin{aligned}
\frac{a_0}{2} &= \frac{1}{T_0}\int_0^{T_0} |\sin\omega_0 t| dt, \quad T_0 = 2\pi/\omega_0 \\
&= \frac{2}{T_0}\int_0^{T_0/2} \sin\omega_0 t dt \\
&= \frac{1}{\pi}\int_0^{\pi} \sin x dx \\
&= \frac{2}{\pi}
\end{aligned}$$

$$\begin{aligned}
\|f - \frac{a_0}{2}\|^2 &= \int_0^{T_0} \left[|\sin(\omega_0 t)| - \frac{a_0}{2}\right]^2 dt \\
&= \frac{T_0}{\pi}\int_0^{\pi} (\sin x - \frac{2}{\pi})^2 dx \\
&= \frac{T_0}{2}\left(1 - \frac{8}{\pi^2}\right) \\
\rho &= \sqrt{\frac{\pi^2}{8} - 1} \simeq 0.483.
\end{aligned}$$

Now suppose the filter is present. You know from Ex. 1.6(ii), p. 12, that the filtered, rectified output has Fourier Series

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n\omega_0 t) + 2n\omega_0 T \sin(2n\omega_0 t)}{(4n^2-1)[1+(2n\omega_0 T)^2]}$$

where

$$\begin{aligned} \omega_0 &= 2\pi 60 \text{ rad sec}^{-1} \\ T &= RC = \text{ filter time constant, sec} \end{aligned}$$

Also

$$\frac{a_0}{2} = \text{ d.c. value } = \frac{2}{\pi}$$

To find an upper bound on the ripple factor, first calculate

$$\begin{aligned} \|f - \frac{a_0}{2}\|^2 &= \frac{\pi}{\omega_0}\sum_{n=1}^{\infty}\left[\frac{4}{\pi(4n^2-1)}\right]^2 \frac{1+(2n\omega_0 T)^2}{[1+(2n\omega_0 T)^2]^2} \text{ (how?)} \\ &= \frac{16}{\pi\omega_0}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\frac{1}{1+4n^2\alpha^2} \end{aligned} \tag{1}$$

where $\alpha := \omega_0 T$.

Step 2. Use tail estimation to get an upper bound on the series, in the form of some simple function of α. Start by simplifying the denominator:

$$4n^2 - 1 \geq 4(n-1)^2, \quad n \geq 1$$

and

$$1 + 4n^2\alpha^2 \geq 4n^2\alpha^2 \geq 4(n-1)^2\alpha^2, \quad n \geq 1$$

Therefore (separating the term at $n = 1$)

$$\begin{aligned} \sum_{n=1}^{\infty} &= \frac{1}{9(1+4\alpha^2)} + \sum_{n=2}^{\infty} \\ &\leq \frac{1}{9(1+4\alpha^2)} + \frac{1}{64\alpha^2}\sum_{n=2}^{\infty}\frac{1}{(n-1)^6} \\ &= \frac{1}{9(1+4\alpha^2)} + \frac{1}{64\alpha^2}\sum_{n=1}^{\infty}\frac{1}{n^6}. \end{aligned}$$

Now

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = 1 + \sum_{n=2}^{\infty}\frac{1}{n^6} \leq \frac{6}{5} \quad \text{(how?)} \tag{2}$$

So

$$\sum_{n=1}^{\infty} \leq \frac{1}{9(1+4\alpha^2)} + \frac{3}{160\alpha^2} =: \phi(\alpha), \quad \text{say}$$

and therefore

$$\|f - \frac{a_0}{2}\|^2 \leq \frac{16}{\pi\omega_0}\phi(\alpha).$$

Since

$$\rho = \frac{\|f - a_0/2\|}{a_0\sqrt{\dfrac{\pi}{2\omega_0}}}$$

you will certainly have $\rho \leq 0.05$ provided

$$\frac{16}{\pi\omega_0}\phi(\alpha) \leq \left\{a_0\sqrt{\frac{\pi}{2\omega_0}}(0.05)\right\}^2,$$

or on substituting for ω_0, a_0 and simplifying,

$$\phi(\alpha) \leq 1.25 \times 10^{-3}. \tag{3}$$

To see when (3) will be true, solve the equation $\phi(\alpha) = 1.25 \times 10^{-3}$ (as a quadratic in α^2) and check that (3) holds provided

$$\alpha \geq \ ? \tag{4}$$

and therefore $\rho \leq 0.05$ provided

$$T \geq \ ? \tag{5}$$

Verify (1) and (2), and complete (4) and (5). Repeat for the case $\rho \leq 0.01$.

Step 3. To obtain results by computer, write the right side of (1) as

$$\text{total a.c. power} = \sum_{n=1}^{\infty} (n^{th} \text{ a.c. power})$$

Replace $\sum_{n=1}^{\infty}$ by $\sum_{n=1}^{N}$ for suitably large N: e.g. for each fixed α of interest, use tail estimation to select N such that, say,

$$\sum_{N+1}^{\infty} \lesssim 10^{-6} \times \text{ (d.c. power)}$$

Compute $\sum_{n=1}^{N}$ and (to sufficient accuracy) evaluate ρ. Carry out this procedure for a range $0 \leq \alpha \leq \alpha_{\max}$, choosing $\alpha_{\max}$ large enough to satisfy the requirement $\rho \leq 0.01$; for this you could use your estimate for α in Step 2, or trial and error. Tabulate and plot ρ vs. α, and interpolate to find α for $\rho = 0.05$, $\rho = 0.01$.

Exercise 1.23: Complex exponential Fourier Series

Show that if $f \in L_2[-\pi, \pi]$, then

$$f = \underset{N\to\infty}{l.i.m.} \sum_{n=-N}^{N} c_n e^{int},$$

where c_n are the Fourier coefficients of f relative to the system $\{e^{int},\ n = 0, \pm1, \pm2, \ldots\}$. Then show

$$\|f\|^2 = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Exercise 1.24: Parseval's Theorem

Let $f \in L_2[t_0, t_1]$. Obtain the expression for $\|f\|^2$ as in Parseval's Theorem,

(a) when f is real-valued and you use the Fourier trigonometric series,

(b) when f is complex and you use the complex exponential Fourier series.

In each case obtain the expressions for r.m.s. value and r.m.s. approximation error, corresponding to those given on p. 51.

1.8 Advanced Topics: Completeness and the Riesz-Fischer Theorem

In this section we introduce more advanced topics related to Fourier series in $L_2(I)$, including completeness and the celebrated (but difficult) Riesz-Fischer theorem (1907). We state the latter first, but omit the proof as it is outside our scope.

THEOREM 1. (Riesz-Fischer)

Let $\{f_n : n = 1, 2, \ldots\}$ be a sequence of functions $f_n \in L_2(I)$ with the property that

$$\lim_{m,n\to\infty} \|f_m - f_n\| = 0 \qquad (*)$$

Then there exists a function $f \in L_2(I)$ such that

$$\lim_{n\to\infty} \|f - f_n\| = 0,$$

namely $f_n \to f$ (m.s.).

The Riesz-Fischer theorem states that every Cauchy sequence in $L_2(I)$ (the property (*)) has a mean-square limit in $L_2(I)$, namely $L_2(I)$ is "complete" in the topological sense. The following easy consequence of the Riesz-Fischer theorem is what we shall use directly.

Corollary: (Riesz-Fischer)

Let $\{\phi_n : n = 1, 2, \ldots\}$ be an ON sequence in $L_2(I)$. Let $\{c_n : n = 1, 2, \ldots\}$ be any sequence of complex numbers such that

$$\sum_{n=1}^{\infty} |c_n|^2 < \infty \qquad (**)$$

Then there exists a function $f \in L_2(I)$ such that

$$f = \underset{N\to\infty}{l.i.m.} \sum_{n=1}^{N} c_n \phi_n;$$

in other words

$$\lim_{N\to\infty} \|f - \sum_{n=1}^{N} c_n \phi_n\| = 0.$$

Proof:

For $N = 1, 2, \ldots$ let

$$f_N := \sum_{n=1}^{N} c_n \phi_n$$

Then if $M > N$,

$$\begin{aligned} \|f_M - f_N\|^2 &= \|\sum_{n=N+1}^{M} c_n \phi_n\|^2 \\ &= \sum_{n=N+1}^{M} |c_n|^2 \\ &\to 0 \text{ as } M, N \to \infty \end{aligned}$$

by (**). By the theorem, there exists $f \in L_2(I)$ such that

$$\lim_{N\to\infty} \|f - f_N\| = 0,$$

as claimed. Q.E.D.

The property (**) of a sequence of complex numbers $\{c_n\}$ is often called "square summability". The Corollary states that for *any* ON sequence in $L_2(I)$ and *any* square summable sequence in $\mathbb{C}$ there exists a mean square limit of the corresponding Fourier series. The limit function is essentially unique (i.e. up to its values on a "set of measure zero").

Next we present an elementary but fundamental property of FS partial sums which reinforces the Pythagorean picture.

Given ϕ_n ON, suppose you want to approximate f with a linear combination of the first N members of the system, $\phi_1, \ldots, \phi_N$. The next result says that the best you can do, in the sense of minimizing the norm of approximation error, is to choose the *Fourier* coefficients in your approximating sum. In other words, the Fourier series provides the 'optimal' approximation to f in the mean-square sense.

THEOREM 2: Minimizing property of the Fourier coefficients

Let $f \in L_2(I)$. Then with $\{\phi_n\}$ ON,

$$\min_{\gamma_1,\ldots,\gamma_N} \|f - \sum_{n=1}^{N} \gamma_n \phi_n\|$$

is attained for $\gamma_n = c_n := \langle f, \phi_n\rangle (n = 1, \ldots, N)$, and only for such values of γ_n .

Proof:

$$\begin{aligned}
0 &\leq \|f - \sum_1^N \gamma_n \phi_n\|^2 \\
&= \langle f - \sum_1^N \gamma_n \phi_n, f - \sum_1^N \gamma_n \phi_n\rangle \\
&= \|f\|^2 - \sum_1^N \gamma_n \langle \phi_n, f\rangle - \sum_1^N \bar{\gamma}_n \langle f, \phi_n\rangle + \sum_1^N |\gamma_n|^2 \\
&= \|f\|^2 - \sum_1^N \gamma_n \bar{c}_n - \sum_1^N \bar{\gamma}_n c_n + \sum_1^N |\gamma_n|^2 \\
&= \|f\|^2 - 2\sum_1^N \mathrm{Re}(\gamma_n \bar{c}_n) + \sum_1^N |\gamma_n|^2.
\end{aligned}$$

But

$$|\gamma_n - c_n|^2 = |\gamma_n|^2 - 2\mathrm{Re}(\gamma_n \bar{c}_n) + |c_n|^2,$$

so we get

$$\begin{aligned}
0 &\leq \|f - \sum_1^N \gamma_n \phi_n\|^2 \\
&= \|f\|^2 - \sum_1^N |c_n|^2 + \sum_1^N |\gamma_n - c_n|^2,
\end{aligned}$$

and this is obviously minimized when $\gamma_n = c_n$, for all $n \leq N$, and these minimizing values are unique. Q.E.D.

Relative to $\{\phi_n\}$, can two distinct functions in $L_2(I)$ have the same Fourier Series? The answer is (essentially) "no" provided the $\{\phi_n\}$ system has a property named 'completeness'.

Definition. $\{\phi_n\}$ is *complete* if, whenever $f \in L_2(I)$ and $\langle f, \phi_n\rangle = 0$ for all n, then necessarily $f = 0$ a.e. Otherwise, $\{\phi_n\}$ is *incomplete.*

An incomplete system can always be manufactured by taking any orthogonal system whatever (that contains at least one nonzero element) and removing one nonzero element. Thus removing ϕ_1 from $\{\phi_n\}$, with $\phi_1 \neq 0$, we have $\langle \phi_1, \phi_n \rangle = 0 (n = 2, 3, \ldots)$ so $\{\phi_n : n = 2, 3, \ldots\}$ is incomplete. On the other hand, to prove completeness of a given ON system is usually quite difficult. On the basis of Dirichlet's theorem for Fourier trigonometric or exponential series, we can sketch a partial proof, for those particular orthogonal systems, as follows.

Proposition:
The system $\{1; \cos nx, \sin nx; n = 1, 2, \ldots\}$ on $[-\pi, \pi]$ is complete.

Proof: (partial)
We know that $f \in PS[-\pi, \pi]$ implies

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (a.e.)$$

by Dirichlet's theorem, the a_n, b_n being the Fourier coefficients of f. So if it happens that

$$a_0 = a_n = b_n = 0 \quad \text{(all } n\text{)}$$

then necessarily $f(t) = 0$ a.e. on $[-\pi, \pi]$. To argue rigorously from $PS[-\pi, \pi]$ to $L_2[-\pi, \pi]$ is outside our scope: but the idea is simply that $g \in L_2$ could be approximated (in norm) by some $f \in PS$, as near as you please. For more details, see Ex. 10, p. 175. This Exercise requires Theorem 2, just proved. Q.E.D.

To make further progress, we need the famous Schwarz inequality. Recall from analytic geometry that for two vectors u, v in the plane $\mathbb{R}^2$ (with tails at the origin),

$$\langle u, v \rangle = \|u\| \, \|v\| \cos \phi,$$

ϕ being the angle between them. This yields the inequality

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|$$

Note that equality holds if $\phi = 0$ or π, namely $v = ru$ for some $r \in \mathbb{R}$. This result generalizes to $L_2(I)$, although the proof is not especially 'geometric'.

Lemma 1: Schwarz inequality

$$f, g \in L_2(I) \Rightarrow |\langle f, g \rangle| \leq \|f\| \cdot \|g\|.$$

Exercise 1.25: Prove the Schwarz inequality.

HINT

1. Real case. Suppose that a, b, c are fixed real numbers with the property that
$$a^2\lambda^2 + 2c\lambda + b^2 \geq 0$$
for all values of the real variable λ. Using the formula for the roots of a quadratic equation, show that necessarily $c^2 \leq a^2b^2$. Now let f, g be real-valued functions in $L_2(I)$. From the fact that
$$0 \leq \|\lambda f + g\|^2 = \langle \lambda f + g, \lambda f + g\rangle$$
deduce that
$$|\langle f, g\rangle| \leq \|f\|\|g\|.$$

2. Complex case. Let f, g be complex-valued functions and write $\langle f, g\rangle$ in polar form:
$$\langle f, g\rangle = e^{i\theta}|\langle f, g\rangle|, \quad \theta \text{ real.}$$
Let $\hat{f} = e^{-i\theta}f$. Note that
$$\langle \hat{f}, g\rangle = |\langle f, g\rangle|$$
is real, and $\|\hat{f}\| = \|f\|$. Use the method of "Real case" to show that $|\langle \hat{f}, g\rangle| \leq \|\hat{f}\| \cdot \|g\|$, hence etc.

The next, slightly technical lemma will be needed for the proof of our main result.

Lemma 2.
Let $\{h_N, N = 1, 2, \ldots\}$, h and $\psi \in L_2(I)$. Assume that $h_N \to h$ (m.s.) as $N \to \infty$, namely
$$h = \underset{N\to\infty}{l.i.m.}\ h_N$$
Then
$$\langle h, \psi\rangle = \lim_{N\to\infty}\langle h_N, \psi\rangle$$

Proof:
$$\begin{aligned} |\langle h, \psi\rangle - \langle h_N, \psi\rangle| &= |\langle h - h_N, \psi\rangle| \\ &\leq \|h - h_N\|\|\psi\| \quad \text{(by Schwarz)} \\ &\to 0 \text{ as } N \to \infty. \end{aligned}$$

Q.E.D.

Our final result brings together all the ideas of this Section, and can be considered a deep extension of the Theorem of Sect. 1.7.

THEOREM 3.
Let $\{\phi_n : n = 1, 2, \ldots\}$ be an ON sequence in $L_2(I)$. Then the following three statements are equivalent:

(1) $\{\phi_n\}$ is complete.

(2) If $f \in L_2(I)$, and $\{c_n\}$ are the Fourier coefficients of f with respect to $\{\phi_n\}$, i.e. $c_n = \langle f, \phi_n \rangle$, then

$$\|f\|^2 = \sum_1^\infty |c_n|^2.$$

In other words, Parseval's Theorem holds for $\{\phi_n\}$.

(3) For every $f \in L_2(I)$, the Fourier Series of f relative to $\{\phi_n\}$ converges to f in mean square, namely

$$\underset{N\to\infty}{l.i.m.} \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n = f.$$

Proof:

Note that the equivalence of statements (2) and (3) is just the Theorem of Sect. 1.7. Therefore it suffices to prove that (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). This is trivial, because

$$\begin{aligned} c_n = 0 \text{ (all } n) \quad &\Rightarrow \quad \|f\|^2 = 0 \\ &\Rightarrow \quad f = 0 \quad \text{a.e.} \end{aligned}$$

(1) $\Rightarrow$ (2). Let $f \in L_2(I)$, with Fourier coefficients $\{c_n\}$.

Bring in

$$f_N := \sum_{n=1}^{N} c_n \phi_n \qquad (N = 1, 2, \ldots)$$

Now by the Corollary to the Riesz-Fischer Theorem, since

$$\sum_{n=1}^{N} |c_n|^2 \leq \|f\|^2 < \infty$$

we have that $g := \underset{N\to\infty}{l.i.m.} f_N$ exists, as a function in $L_2(I)$.

With the aid of completeness, we shall show that $g = f$ a.e. Now

$$\begin{aligned} f - g \quad &= \quad f - \underset{N\to\infty}{l.i.m.} f_N \\ &= \quad \underset{N\to\infty}{l.i.m.} (f - f_N) \end{aligned}$$

For each fixed $k = 1, 2, \ldots$ we have, by Lemma 2,

$$\begin{aligned}
\langle f - g, \phi_k \rangle &= \lim_{N\to\infty} \langle f - f_N, \phi_k \rangle \\
&= \lim_{N\to\infty} (\langle f, \phi_k \rangle - \langle f_N, \phi_k \rangle) \\
&= \lim_{N\to\infty} (c_k - c_k) \\
&= 0.
\end{aligned}$$

So, by completeness, $f - g = 0$ a.e., and therefore $\|f - f_N\| = \|g - f_N\| \to 0$ as $N \to \infty$. From (2), p. 45, we have that

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} |c_n|^2.$$

Thus, letting $N \to \infty$ we get finally

$$0 = \|f\|^2 - \sum_{n=1}^{\infty} |c_n|^2,$$

as claimed. Q.E.D.

Chapter 2

Fourier Integral

2.1 Introduction and Heuristics

Using Fourier series, as in Ch. 1, we are able to represent a large class of functions which are either **periodic on an infinite interval**, or are defined (more or less arbitrarily) on a **finite** interval. In this chapter we extend the idea of Fourier representation to a class of **nonperiodic** functions defined on an **infinite** interval. This extension has practical advantages as well as theoretical interest – because for mathematical modelling purposes it may be much more convenient to consider a "very large" interval like $[0, T)$ simply as the infinite interval $[0, \infty)$, in order not to have to worry about T at all.

In fact we will take as our interval the whole real line $(-\infty, \infty)$. As we shall see, the Fourier **series** then gets replaced by the Fourier **integral**. First we define a suitable class of functions to work with.

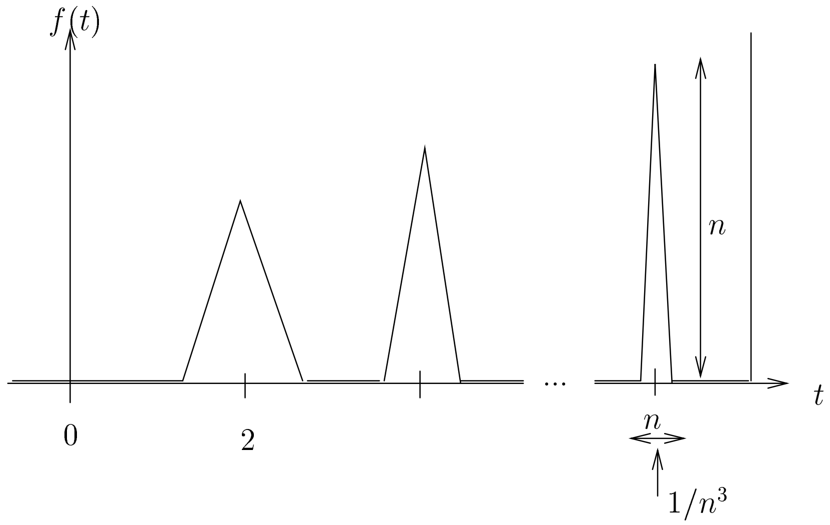
Definition.

Let $f(\cdot)$ be a (real or) complex-valued function of t, defined for $-\infty < t < \infty$. We say that $f \in PS_1(-\infty, \infty)$ if $f \in PS$ on every finite subinterval of $(-\infty, \infty)$, and

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty.$$

Examples

1. The functions
$$1, t, t^2, \ldots, (1 + |t|)^{-1}, \sin t, \cos t \notin PS_1$$
2. The functions
$$(1 + |t|)^{-\alpha} (\alpha > 1), |t|^\beta e^{-|t|} (\beta \geq 1) \in PS_1$$
3. $f \in PS_1$ need not imply that $f(t) \to 0$ as $t \to \pm\infty$; for instance let $f(t)$ be the sequence of 'tents' defined by the picture:



Then

$$\int_{-\infty}^{\infty} |f(t)|dt = \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{1}{n^3} \cdot n = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty,$$

but $f(t)$ does not $\to 0$ as $t \to \infty$: indeed, $f(\cdot)$ is actually unbounded.

Now we can introduce the

Definition.
Let $f \in PS_1(-\infty, \infty)$. The **Fourier transform** of f is the function

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt,$$

defined for all real ω, $-\infty < \omega < \infty$.

Heuristics
The standard way of motivating Fourier Transform is to give a fake derivation from Fourier Series, by letting the interval of definition of f tend to $(-\infty, \infty)$. Start from the complex exponential Fourier Series for a function f defined on $[-T, T]$:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/T}, \tag{1}$$

where

$$c_n := \frac{1}{2T} \int_{-T}^{T} f(t) e^{-in\pi t/T} dt$$

Put

$$\begin{aligned} \omega_n &:= n\pi/T, \quad n = 0, \pm 1, \pm 2, \ldots \\ &= n\omega_1. \end{aligned}$$

Next let

$$c(\omega) := \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\omega t}dt,$$

defined for $-\infty < \omega < \infty$. Then

$$c_n = c(n\omega_1), \tag{2}$$

so

$$f(t) = \sum_{n=-\infty}^{\infty} c(n\omega_1)e^{in\omega_1 t}. \tag{3}$$

Now if T is large you might expect that

$$c(\omega) \approx \frac{1}{2T}\hat{f}(\omega) \tag{4}$$

where $\hat{f}$ is the Fourier Transform defined on p. 65. Also, if $c(\omega)e^{i\omega t}$ doesn't change too fast, you could approximate

$$\begin{aligned} c(n\omega_1)e^{in\omega_1 t} &\approx \frac{1}{\omega_1}\int_{(n-\frac{1}{2})\omega_1}^{(n+\frac{1}{2})\omega_1} c(\omega)e^{i\omega t}d\omega \\ &\approx \frac{1}{\pi/T}\cdot\int_{(n-\frac{1}{2})\omega_1}^{(n+\frac{1}{2})\omega_1} \frac{1}{2T}\hat{f}(\omega)e^{i\omega t}d\omega \end{aligned} \tag{5}$$

Plugging this into (3), you get

$$f(t) \approx \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{(n-\frac{1}{2})\omega_1}^{(n+\frac{1}{2})\omega_1} \hat{f}(\omega)e^{i\omega t}d\omega$$

or simply

$$f(t) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega \tag{6}$$

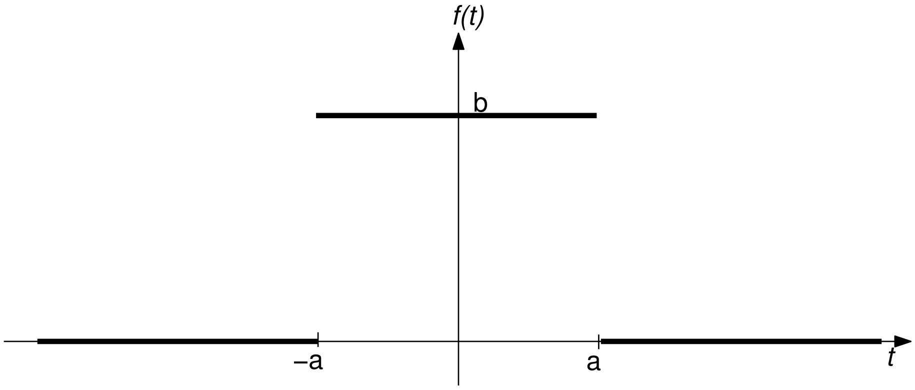
Equation (6) (with equality) is the **Fourier integral representation** of f, in terms of its Fourier transform. It's thus the "infinite interval analog" of the exponential Fourier Series representation of a function on a finite interval. Later we'll justify the Fourier integral more carefully.

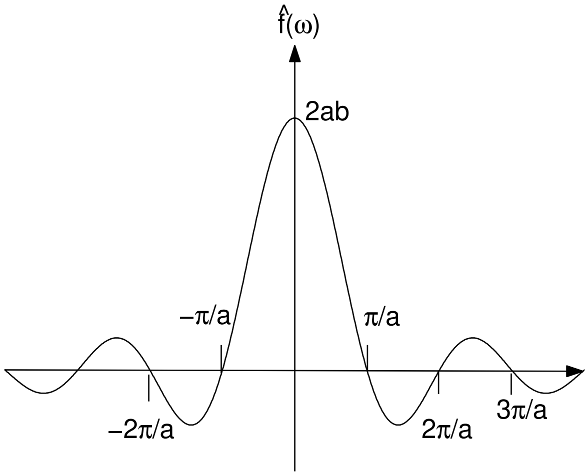
2.2 Examples of Fourier Integral Representation

Example 1. Gating function

$$f(t) = \begin{cases} b, & |t| \le a \\ 0, & |t| > a \end{cases}$$

$$\begin{aligned}
\hat{f}(\omega) &= b\int_{-a}^{a} e^{-i\omega t}dt = \frac{b}{-i\omega}(e^{-i\omega a} - e^{i\omega a}) \\
&= 2b\frac{\sin\omega a}{\omega} \\
&= 2ab\frac{\sin\omega a}{\omega a}.
\end{aligned}$$





This is a commonly occurring Fourier transform pair - worth remembering.

Now let's try to get back to f from $\hat{f}$ via the recipe [cf. (6)]:

$$f(t) \stackrel{?}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

We have

$$\begin{aligned}
RHS &= \frac{ab}{\pi}\int_{-\infty}^{\infty} \frac{\sin(a\omega)}{a\omega}e^{i\omega t}d\omega \\
&= \frac{ab}{\pi}\int_{-\infty}^{\infty} \frac{\sin(a\omega)}{a\omega}(\cos\omega t + i\sin\omega t)d\omega
\end{aligned}$$

$$
\begin{aligned}
&= \frac{ab}{\pi}\int_{-\infty}^{\infty}\frac{\sin a\omega}{a\omega}\cos\omega t d\omega \\
&= \frac{ab}{\pi}\int_{-\infty}^{\infty}\frac{1}{2a}\left\{\frac{\sin(a+t)\omega}{\omega}+\frac{\sin(a-t)\omega}{\omega}\right\}d\omega
\end{aligned}
$$

Now exploit the

$$
\begin{aligned}
\text{FACT}: \int_{-\infty}^{\infty}\frac{\sin\lambda\omega}{\omega}d\omega &= (sgn\lambda)\int_{-\infty}^{\infty}\frac{\sin x}{x}dx \\
&= (sgn\lambda)\cdot\pi,
\end{aligned}
$$

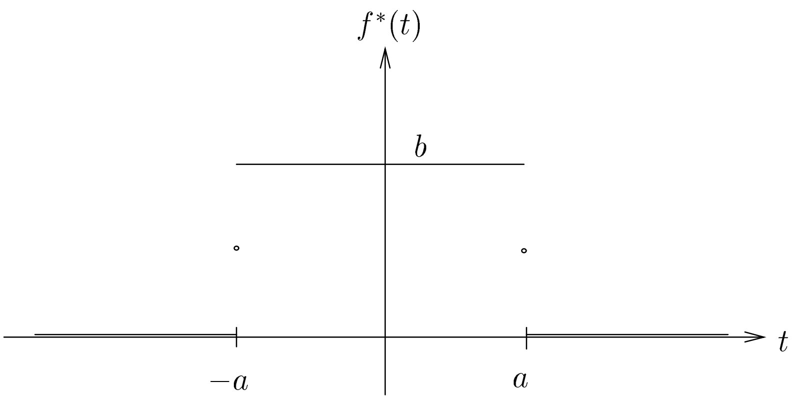
where

$$
sgn\lambda := \begin{cases} 1 & \lambda > 0 \\ 0 & \lambda = 0 \\ -1 & \lambda < 0 \end{cases}.
$$

You now get

$$
\begin{aligned}
RHS &= \frac{b}{\pi}\cdot\pi\cdot\frac{1}{2}[sgn(a+t)+sgn(a-t)] \\
&= b\cdot\begin{cases} 0, & t<-a \\ 1/2, & t=-a \\ 1, & -a<t<a \\ 1/2, & t=+a \\ 0, & t>a \end{cases} \\
&=: f^*(t), \quad \text{say.}
\end{aligned}
$$

So, $f^*(t) = f(t)$ a.e.; in fact



$$
f^*(t) = \frac{1}{2}[f(t+0)+f(t-0)],
$$

in typical Fourier style!

Continuing with this example, we have for the "energy" content of f,

$$\|f\|^2 := \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-a}^{a} b^2 dt = 2ab^2.$$

Also

$$\begin{aligned} \|\hat{f}\|^2 &:= \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = 4a^2b^2 \int_{-\infty}^{\infty} \frac{\sin^2(\omega a)}{(\omega a)^2} d\omega \\ &= 4ab^2 \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = 4\pi ab^2, \end{aligned}$$

using the FACT:

$$\int_{-\infty}^{\infty} (\sin^2 x/x^2) dx = \pi.$$

So

$$\|\hat{f}\|^2 = 2\pi \|f\|^2,$$

and this is an instance of **Parseval's Theorem for Fourier integrals.**

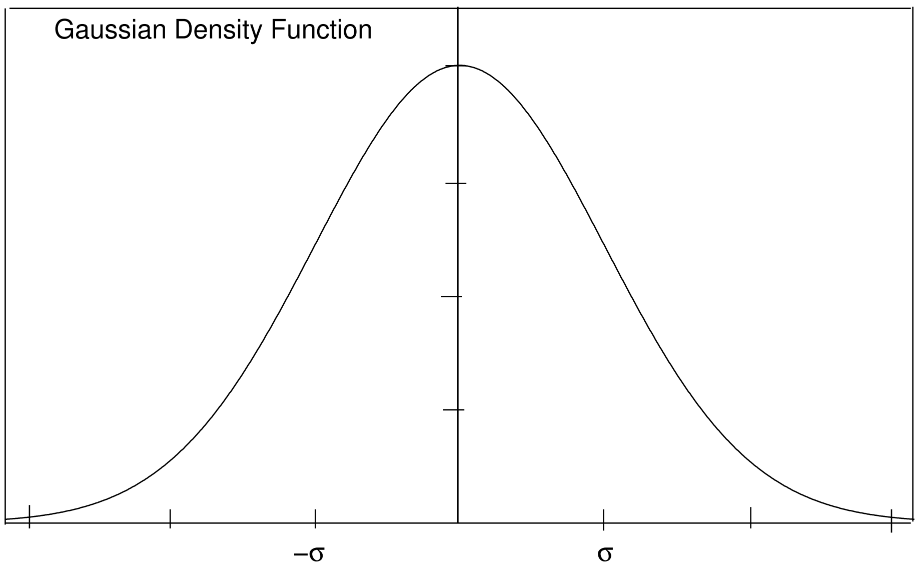
You can interpret

$$\int_{\omega_1}^{\omega_2} |\hat{f}(\omega)|^2 \frac{d\omega}{2\pi}$$

as the "energy" contributed by those frequency components of f in the range $\omega_1 \leq \omega \leq \omega_2$, and thus $\frac{|\hat{f}(\omega)|^2}{2\pi}$ as the "energy density" of f in the frequency domain. Parseval's theorem says you can compute the total "energy" by essentially the same recipe [i.e. up to the scale factor $1/2\pi$] in either the time or frequency domain.

Example 2. Gaussian distribution

$$f(t) := \frac{e^{-t^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \quad (\sigma > 0)$$



This is the "normal" or "Gaussian" function, which appears commonly in probability and statistics. We have

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} \frac{e^{-t^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dt \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t+i\omega\sigma^2)^2} e^{-\omega^2\sigma^2/2} dt \\
&= \frac{e^{-\omega^2\sigma^2/2}}{\sigma\sqrt{2\pi}} \int_{-\infty+i\omega\sigma^2}^{\infty+i\omega\sigma^2} e^{-z^2/2\sigma^2} dz \quad [\text{putting } z := t + i\omega\sigma^2] \\
&= \frac{e^{-\omega^2\sigma^2/2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2} dz \quad [\text{by an application of Cauchy's theorem}] \\
&= e^{-\omega^2\sigma^2/2} \ [\text{using } \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = \sigma\sqrt{2\pi}] \\
&= e^{-\omega^2/2(1/\sigma)^2}.
\end{aligned}
$$

Thus $\hat{f}$ is essentially the same function (of ω) as f is (of t), apart from scale changes. It is easy to verify Parseval:

$$
\begin{aligned}
\|f\|^2 &:= \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{\sigma^2 2\pi} \int_{-\infty}^{\infty} e^{-t^2/\sigma^2} dt \\
&= \frac{1}{\sigma^2 2\pi} \sqrt{2\pi}(\sigma/\sqrt{2}) = \frac{1}{2\sigma\sqrt{\pi}};
\end{aligned}
$$

and

$$
\begin{aligned}
\|\hat{f}\|^2 &:= \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \\
&= \int_{-\infty}^{\infty} e^{-\omega^2\sigma^2} d\omega = \sqrt{2\pi}\left(\frac{1}{\sigma\sqrt{2}}\right) = \sqrt{\pi}/\sigma \\
&= 2\pi\|f\|^2.
\end{aligned}
$$

The "spread" of $f(\cdot)$ in time can be measured in this example by the parameter σ (the "standard deviation", in statistical jargon). Thus the "spread" of $\hat{f}$ in frequency is $1/\sigma$.

MORAL: the smaller the spread of a signal in time, the larger its spread in frequency, the product of the two being constant.

We'll go into this "time-bandwidth tradeoff" in more detail later.

2.3 General Properties of the Fourier Transform

Before justifying the Fourier inversion rule (i.e. Fourier integral representation) rigorously we'll develop some simple properties of the Fourier transform itself. The following results are easy to work out from scratch, so don't try to memorize them. From now on, $f \in PS_1(-\infty, \infty)$.

1. Time shifting

Suppose you **delay** the function $f(\cdot)$ by a time-interval of length τ: i.e. you shift the graph of $f(\cdot)$ to the **right** along the t-axis, to get the new function

$$g(t) := f(t-\tau) \quad (\tau > 0), \quad -\infty < t < \infty$$

Then

$$\begin{aligned}
\hat{g}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} f(t-\tau)dt \\
&= e^{-i\omega\tau} \int_{-\infty}^{\infty} e^{-i\omega(t-\tau)} f(t-\tau)dt \\
&= e^{-i\omega\tau} \int_{-\infty}^{\infty} e^{-i\omega t'} f(t')dt' \\
&= e^{-i\omega\tau} \hat{f}(\omega).
\end{aligned}$$

In other words (an "advance" being a negative delay),

$$\left\{ \begin{array}{l} \text{Delay of } \tau \\ \text{Advance of } \tau \end{array} \right\} \text{ in time domain}$$

$$\equiv \left\{ \begin{array}{l} \text{Phase lag of } \omega\tau \text{ rad.} \\ \text{Phase lead of } \omega\tau \text{ rad.} \end{array} \right\} \text{ at frequency } \omega \text{, in frequency domain.}$$

2. Time scaling

Let

$$g(t) := f(\gamma t), \quad \gamma > 0.$$

If $\gamma > 1$, you *shrink* the time scale (speed things up).
If $\gamma < 1$, you *stretch* the time scale (slow things down).
Then

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(\gamma t)dt$$

$$= \gamma^{-1} \int_{-\infty}^{\infty} e^{-i(\omega/\gamma)(\gamma t)} f(\gamma t) d(\gamma t)$$

$$= \gamma^{-1} \int_{-\infty}^{\infty} e^{-i(\omega/\gamma)t'} f(t') dt'$$

$$= \gamma^{-1} \hat{f}(\omega/\gamma).$$

Namely

$$\left\{ \begin{array}{l} \text{Shrink} \\ \text{Stretch} \end{array} \right\} \text{ in time domain } \equiv \left\{ \begin{array}{l} \text{Stretch} \\ \text{Shrink} \end{array} \right\} \text{ in frequency domain.}$$

3. Combined shifting and scaling

You can combine operations of this sort in sequence. For instance, if you shift, then scale, you get e.g.

$$\begin{aligned} g_1(t) &:= f(t-\tau), \quad \hat{g}_1(\omega) = e^{-i\omega\tau} \hat{f}(\omega) \\ g_2(t) &:= g_1(\gamma t) = f(\gamma t - \tau), \\ \hat{g}_2(\omega) &= \gamma^{-1} \hat{g}_1(\omega/\gamma) = \gamma^{-1} e^{-i\omega\tau/\gamma} \hat{f}(\omega/\gamma). \end{aligned}$$

And so forth. Along this line, do

Exercise 2.1: A signal $f(t)$ starts at $t = 0$ and lasts exactly 4 minutes. You tape-record it starting at $t = 0$, stop the tape 6 minutes later, go out for coffee for 8 minutes, then wind the tape back (i.e. reverse it) at twice the recording speed. If $\hat{f}(\omega)$ is the Fourier Transform of $f(t)$, what's the Fourier Transform of the signal you will hear during wind-back? Assume that the gadgetry (and your ear) operate at perfect fidelity at all frequencies, and that no disturbing noise is present.

HINT: *Draw the picture* on the real-time axis; the required Fourier Transform is, of course, to be computed on this axis.

4. Frequency shifting (amplitude modulation)

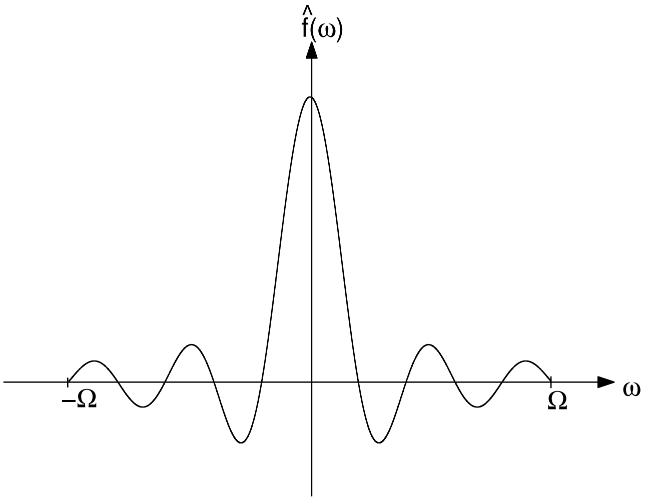
Let $f(t)$ be an arbitrary signal, and define

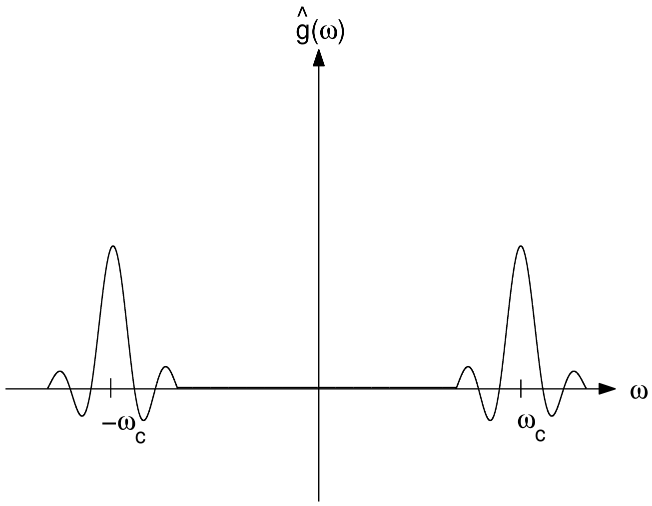
$$g(t) = f(t) \cos(\omega_c t) \quad [\omega_c = \text{“carrier frequency”}]$$

Then

$$\begin{aligned}\hat{g}(\omega) &= \int_{-\infty}^{\infty} f(t)\frac{1}{2}(e^{i\omega_c t} + e^{-i\omega_c t})e^{-i\omega t}dt \\ &= \frac{1}{2}\{\hat{f}(\omega-\omega_c) + \hat{f}(\omega+\omega_c)\}\end{aligned}$$

In communications, $\cos(\omega_c t)$ is the 'carrier' signal, with frequency $\omega_c >>$ highest frequency in the 'modulating' signal $f(\cdot)$, i.e. $|\hat{f}(\omega)| \simeq 0$ for $|\omega| > \Omega$, where $\Omega << \omega_c$. Multiplication of the carrier by f is called 'amplitude modulation'. The frequency picture is shown (roughly) below.





5. Differentiation

Assume

(i) $g := df/dt \in PS_1(-\infty,\infty)$

(ii) $\int_{-\infty}^{\infty} |f(t)|dt < \infty$

These can be shown to imply

(iii) $f(t) \to 0$ as $t \to \pm\infty$.

Then

$$\begin{aligned}\hat{g}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}\frac{df(t)}{dt}dt \\ &= \lim_{\substack{b\to+\infty \\ a\to-\infty}} \int_a^b e^{-i\omega t} f'(t)dt\end{aligned}$$

$$= \lim \left\{ e^{-i\omega t} f(t)|_a^b - \int_a^b (-i\omega e^{-i\omega t}) f(t) dt \right\}$$

$$= i\omega \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \text{ [using (iii)]}$$

$$= i\omega \hat{f}(\omega).$$

Note that the effect of differentiation is to enhance the high frequency content of the spectrum. Note also that you really have to respect assumptions (i) and (ii): e.g. $f(t) := e^{-|t|} \sin(e^{t^2})$ has a perfectly good Fourier Transform $\hat{f}(\omega)$, but $f'(t)$ does *not* have a Fourier Transform in our theory (since $\int_{-\infty}^{\infty} |f'(t)| dt$ isn't finite) as we have defined Fourier Transform only for functions in PS_1.

Exercise 2.2: Let m, n be respectively 'message' and 'noise' signals, with Fourier spectra

$$\hat{m}(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \leq 1 \\ 0, & |\omega| \geq 1 \end{cases}$$

$$\hat{n}(\omega) = \begin{cases} \nu, & |\omega| \leq 1 \\ 0, & |\omega| > 1 \end{cases}$$

Calculate

$$\frac{\text{Energy (message)}}{\text{Energy (noise)}}$$

and

$$\frac{\text{Energy } (\frac{d}{dt} \text{ message})}{\text{Energy } (\frac{d}{dt} \text{ noise})}$$

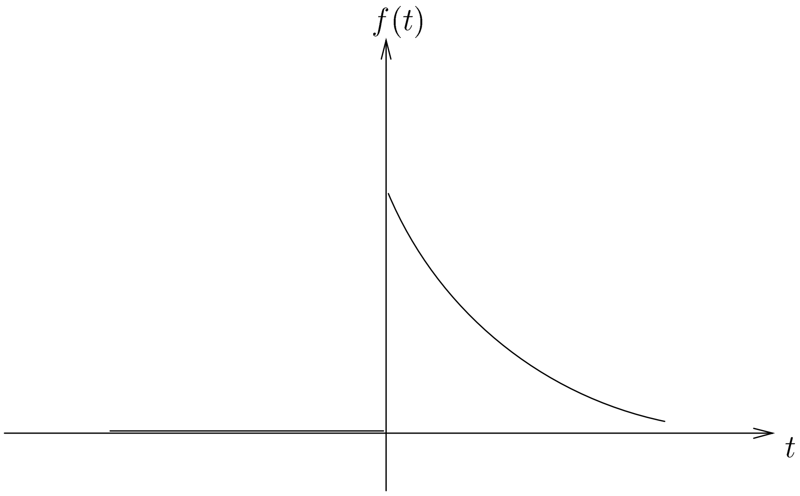
What practical conclusion might be drawn about the effect of noise on differentiation in signal processing?

6. Continuity of the Fourier Transform

Proposition. If $f \in PS_1(-\infty, \infty)$ then $\hat{f}(\omega)$ is uniformly continuous for $-\infty < \omega < \infty$.

In other words, the Fourier Transform is quite 'smooth' even when f may not be; this you would expect, from the fact that $\hat{f}$ is produced from f by an integration operation. The proof is omitted.

Example (Smoothing effect of Fourier transformation)



$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-t}, & t \geq 0 \end{cases}$$

$$\hat{f}(\omega) = \int_0^\infty e^{-i\omega t} e^{-t} dt = \frac{1}{1+i\omega}, \quad -\infty < \omega < \infty.$$

Thus $f(\cdot)$ is jumpy, but $\hat{f}(\cdot)$ is smooth, in fact

$$|\hat{f}(\omega+\alpha) - \hat{f}(\omega)| \leq |\alpha| \quad (-\infty < \omega < \infty)$$

as you can easily check.

7. Riemann-Lebesgue Lemma for the Fourier Transform

Proposition.
Let $f \in PS_1(-\infty, \infty)$. Then

$$\lim_{\omega \to \pm\infty} \hat{f}(\omega) = 0.$$

Proof:
We have

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt.$$

Let $\varepsilon > 0$ be given. Choose T so that

$$\int_{-\infty}^{-T} |f(t)| dt + \int_T^\infty |f(t)| dt < \varepsilon/2. \tag{1}$$

With T fixed, use the same argument as in the proof (p. 23 ff.) of the Riemann-Lebesgue lemma for Fourier Series, to show that there exists Ω such that

$$\left| \int_{-T}^{T} e^{-i\omega t} f(t) dt \right| < \varepsilon/2 \tag{2}$$

for all $|\omega| > \Omega$. Then (1) and (2) give, for $|\omega| > \Omega$,

$$\left|\int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt\right| \leq \int_{-\infty}^{-T} |f(t)|dt + \left|\int_{-T}^{T} f(t)e^{-i\omega t}dt\right| + \int_{T}^{\infty} |f(t)|dt < \varepsilon. \qquad Q.E.D.$$

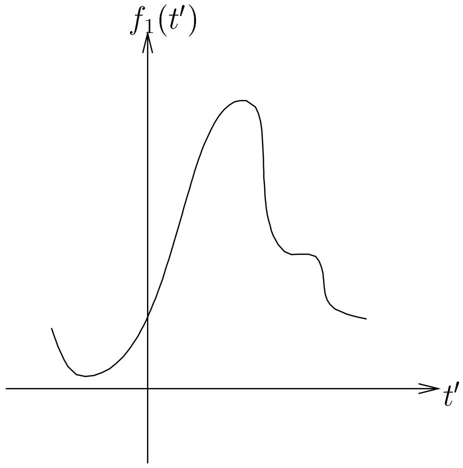
8. Convolution

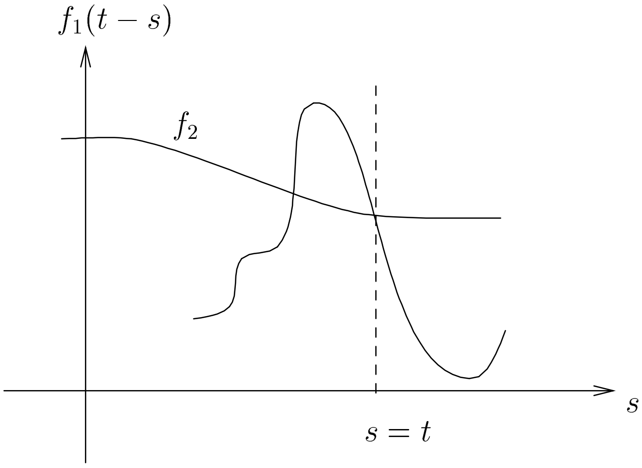
Let $f_1, f_2 \in PS_1(-\infty, \infty)$. The *convolution* of f_1 and f_2, denoted by $f_1 * f_2$, is the function

$$(f_1 * f_2)(t) := \int_{-\infty}^{\infty} f_1(t-s)f_2(s)ds, \quad -\infty < t < \infty.$$

It is a rather difficult theorem, whose proof we omit, that the convolution is well-defined, i.e. that the defining integral actually converges. [This is easy to show under the additional hypothesis that either f_1 or f_2 is bounded: try it!]

To see what the convolution means, consider the pictures:





In the second picture, consider t as a fixed point on the s-axis. You see that $(f_1 * f_2)(t)$ is simply a weighted average of $f_2(\cdot)$, with weight function a "folded-back" version of $f_1(\cdot)$. As t increases, think of the folded-f_1-graph as sliding along the s-axis to the right, while the other graph, of f_2, stays put. Thus the *function* $(f_1 * f_2)(\cdot)$ is a *moving weighted average* of $f_2(\cdot)$.

We can always reverse the roles of f_1 and f_2 in this description: indeed, by an obvious change of variable, we also have

$$(f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(s)f_2(t-s)ds.$$

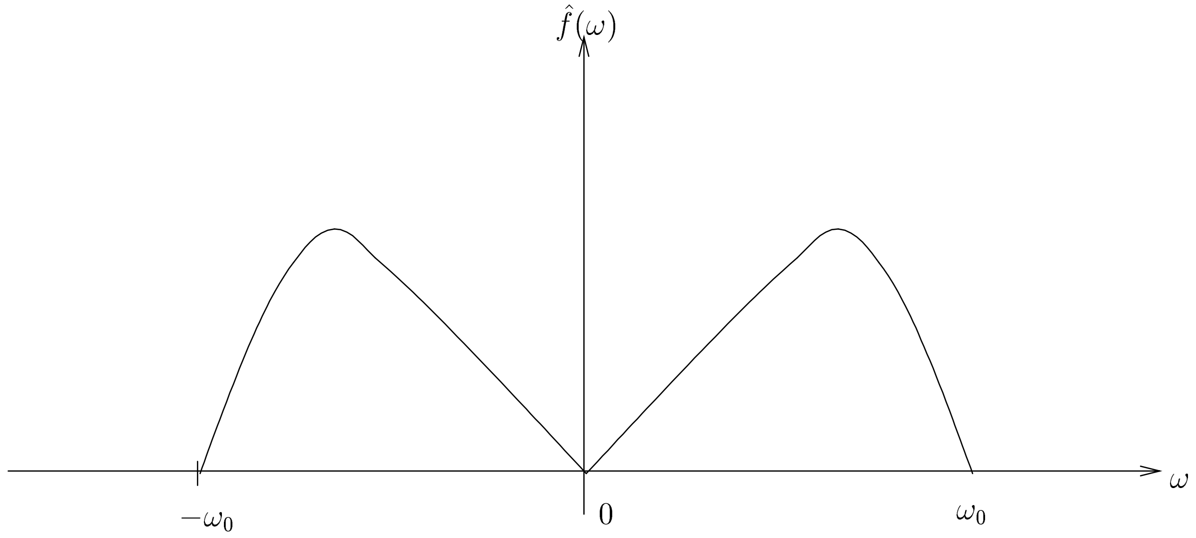
Exercise 2.3: Calculate and sketch the graph of $f_1 * f_2$, where

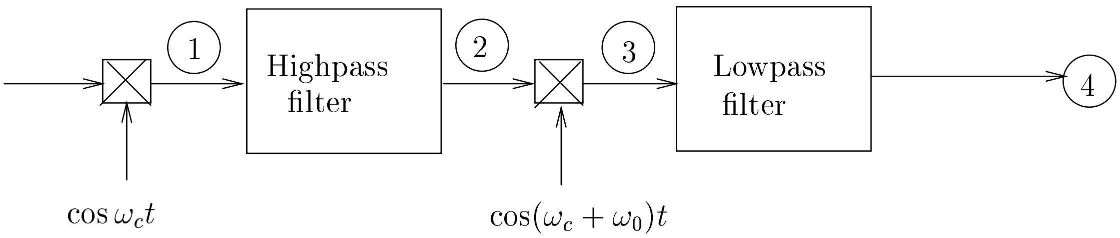
$$f_1(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$f_2(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases} \qquad (a > 0)$$

Exercise 2.4: [Frederick-Carlson]

Speech scramblers are sometimes used to foil snoopers. Analyse the simple scrambler in the figure by sketching the spectra at points (1) through (4). Assume the filters are ideal, both cutting off at $|\omega| = \omega_c$, and that $\omega_c \gg \omega_0$. How would you unscramble the scrambled output?





The boxes ⊠ denote multipliers.

To make further progress we need a slightly technical result, which justifies interchange of order of integration in a repeated integral. As our result is a special case of Fubini's theorem in Lebesgue integration, we dub it (following Loève)

Fubinito's Theorem

Let $h(t,s)$ be defined for $-\infty < t < \infty$, $-\infty < s < \infty$. Assume $h(t,\cdot) \in PS(-\infty,\infty)$, as a function of s, for each fixed t; $h(\cdot,s) \in PS(-\infty,\infty)$ as a function of t, for each fixed s; and $h(t,\cdot)$ is left continuous in s for each fixed t. All this simply guarantees that $h(\cdot,\cdot)$ is measurable. Assume in addition that *either*

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|h(t,s)|dt\right\}ds < \infty$$

or

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|h(t,s)|ds\right\}dt < \infty$$

Then both iterated integrals of $h(t,s)$ itself exist, and are equal:

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}h(t,s)dt\right\}ds = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}h(t,s)ds\right\}dt.$$

[To allay any doubts, "$f \in PS(-\infty,\infty)$" means $f(\cdot)$ is piecewise smooth, in the sense of the definition, p. 23, on every *finite* subinterval of $(-\infty,\infty)$].

Now we can go on to

9. Multiplication Rule (Fourier Transform of Convolution)

Proposition. Let $f_1, f_2 \in PS_1(-\infty,\infty)$. Let $g := f_1 * f_2$. Then

$$\hat{g}(\omega) = \hat{f}_1(\omega)\hat{f}_2(\omega), \quad -\infty < \omega < \infty,$$

i.e.

$$\hat{g} = \hat{f}_1\hat{f}_2.$$

This result is extremely useful, because it converts the 'difficult' operation of a convolution in the time domain into an 'easy' operation (ordinary multiplication of functions) in the frequency domain. Now

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\left\{\int_{-\infty}^{\infty} f_1(t-s)f_2(s)ds\right\}dt$$

Apply Fubinito to

$$h(t,s) := e^{-i\omega t} f_1(t-s)f_2(s)$$

Then

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|h(t,s)|dt\right\}ds$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|f_1(t-s)|dt\right\}|f_2(s)|ds$$

$$= \left\{ \int_{-\infty}^{\infty} |f_1(t)| dt \right\} \left\{ \int_{-\infty}^{\infty} |f_2(s)| ds \right\}$$

$$< \infty$$

since $f_1, f_2 \in PS_1(-\infty, \infty)$. Therefore

$$\begin{aligned}
\hat{g} &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-i\omega(t-s)} e^{-i\omega s} f_1(t-s) f_2(s) ds \right\} dt \\
&= \int_{-\infty}^{\infty} e^{-i\omega s} f_2(s) \left\{ \int_{-\infty}^{\infty} e^{-i\omega(t-s)} f_1(t-s) dt \right\} ds \\
&= \int_{-\infty}^{\infty} e^{-i\omega s} f_2(s) \left\{ \int_{-\infty}^{\infty} e^{-i\omega t} f_1(t) dt \right\} ds \\
&= \left\{ \int_{-\infty}^{\infty} e^{-i\omega t} f_1(t) dt \right\} \left\{ \int_{-\infty}^{\infty} e^{-i\omega s} f_2(s) ds \right\} \\
&= \hat{f}_1(\omega) \hat{f}_2(\omega) \qquad \text{Q.E.D.}
\end{aligned}$$

Exercise 2.5: Verify the foregoing result for the functions f_1, f_2 of Exercise 2.3, p. 76.

2.4 Fourier Inversion Theorem [Fourier Integral Representation]

Our aim is to justify the result

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega$$

that was obtained by heuristic methods from the "limiting" Fourier Series (pp. 65-66). A rigorous procedure can be developed much like Dirichlet's proof of the convergence of Fourier Series: in the discussion to follow, compare pp. 27-31.

So, bring in the 'truncated' Fourier integral

$$f_\beta(t) := \frac{1}{2\pi} \int_{-\beta}^{\beta} e^{i\omega t} \hat{f}(\omega) d\omega$$

defined for $\beta > 0$ and $-\infty < t < \infty$. [$f_\beta(\cdot)$ will play a role analogous to the partial sum of Fourier Series.] Substituting for $\hat{f}(\omega)$,

$$f_\beta(t) = \frac{1}{2\pi}\int_{-\beta}^{\beta} e^{i\omega t}\left\{\int_{-\infty}^{\infty} e^{-i\omega t'} f(t')dt'\right\} d\omega.$$

Now assume $f(\cdot) \in PS_1(-\infty, \infty)$. Since

$$\int_{-\beta}^{\beta}\int_{-\infty}^{\infty} |e^{i\omega t}e^{-i\omega t'} f(t')|dt'd\omega$$

$$= 2\beta\int_{-\infty}^{\infty} |f(t')|dt' < \infty,$$

Fubinito says you can write

$$\begin{aligned} f_\beta(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\beta}^{\beta} e^{i\omega(t-t')}d\omega\right\} f(t')dt' \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin\beta(t-t')}{t-t'} f(t')dt' \\ &= \int_{-\infty}^{\infty} K(t-t', \beta) f(t')dt' && \text{(say)} \\ &= \int_{-\infty}^{\infty} K(x,\beta) f(t+x)dx, && (*) \end{aligned}$$

where

$$K(x,\beta) := \frac{1}{\pi}\frac{\sin\beta x}{x}.$$

[In (*) we put $x = t' - t$ and used $K(-x,\beta) = K(x,\beta)$.]

We can now prove the important

THEOREM (Fourier Integral Representation)

Let $f \in PS_1(-\infty, \infty)$. Then

$$\lim_{\beta\to\infty} f_\beta(t) = \frac{1}{2}[f(t-0) + f(t+0)], \quad -\infty < t < \infty.$$

Proof:

Since $K(-x,\beta) = K(x,\beta)$, we have

$$f_\beta(t) = \int_0^{\infty} K(x,\beta)[f(t+x) + f(t-x)]dx.$$

Let

$$g_t(x) := \begin{cases} f(t+x)+f(t-x)-f(t+0)-f(t-0), & x>0 \\ 0, & x=0. \end{cases}$$

Then, using the fact that $\int_{-\infty}^{\infty} K(x,\beta)dx = 1$, we get

$$\begin{aligned} f_\beta(t) - \frac{1}{2}[f(t-0)+f(t+0)] &= \int_0^\infty K(x,\beta)g_t(x)dx \\ &=: F_\beta(t), \quad \text{(say)}. \end{aligned}$$

Now fix t. We have to show that $F_\beta(t) \to 0$ as $\beta \to \infty$. Let $\varepsilon > 0$ be given. Write

$$F_\beta(t) = \left(\int_0^\delta + \int_\delta^\infty\right) K(x,\beta)g_t(x)dx$$

where $\delta > 0$ will be chosen suitably, as follows.

Step 1.

Take care of $\int_0^\delta$: we have

$$\int_0^\delta K(x,\beta)g_t(x)dx = \frac{1}{\pi}\int_0^\delta \frac{\sin\beta x}{x} g_t(x)dx.$$

Bring in

$$\theta_\beta(x) := \frac{1}{\pi}\int_0^x \frac{\sin\beta u}{u}du, \quad x \geq 0,$$

and integrate by parts, to get

$$\begin{aligned} \int_0^\delta K(x,\beta)g_t(x)dx &= \int_0^\delta \theta'_\beta(x)g_t(x)dx \\ &= \theta_\beta(x)g_t(x)|_0^\delta - \int_0^\delta \theta_\beta(x)g'_t(x)dx \\ &= \theta_\beta(\delta)g_t(\delta) - \int_0^\delta \theta_\beta(x)g'_t(x)dx. \end{aligned}$$

Next observe that $\theta_\beta(x)$ is bounded in (β, x): for $\pi\theta_\beta(x) = \int_0^{\beta x} \frac{\sin\nu}{\nu}d\nu$ [by a change of variable] and the integral $\int_0^\infty \frac{\sin\nu}{\nu}d\nu$ is convergent. Hence there is a constant c_1 such that

$$|\theta_\beta(x)| \leq c_1, \text{ all } \beta > 0, x \geq 0. \tag{1}$$

Also, since $f \in PS$,

$$g_t(\delta) \to 0 \text{ as } \delta \to 0 \tag{2}$$

and

$$|g_t'(x)| < c_2 \text{ (say), all small } x \geq 0. \tag{3}$$

Therefore, by (1)-(3), we get

$$\left| \int_0^\delta K(x, \beta) g_t(x) dx \right| \leq c_1 |g_t(\delta)| + c_1 c_2 \delta \to 0 \text{ as } \delta \to 0.$$

So, finally, there exists $\delta^* > 0$ such that

$$\left| \int_0^{\delta^*} K(x, \beta) g_t(x) dx \right| < \varepsilon/2, \text{ for all } \beta > 0. \tag{4}$$

With $\delta = \delta^*$ fixed, go to

Step 2.

Take care of $\displaystyle\int_{\delta^*}^\infty$: we have for any $\delta > 0$,

$$\begin{aligned} \int_\delta^\infty K(x, \beta) g_t(x) dx &= \frac{1}{\pi} \int_\delta^\infty \frac{\sin \beta x}{x} g_t(x) dx \\ &= \int_\delta^\infty \sin \beta x \cdot G_t(x) dx \\ &- \frac{1}{\pi} [f(t+0) + f(t-0)] \int_\delta^\infty \frac{\sin \beta x}{x} dx, \end{aligned} \tag{5}$$

where

$$G_t(x) := [f(t+x) + f(t-x)]/\pi x.$$

Since

$$\begin{aligned} \int_\delta^\infty |G_t(x)| dx &\leq \frac{1}{\pi\delta} \int_\delta^\infty \{|f(t+x)| + |f(t-x)|\} dx \\ &\leq \frac{2}{\pi\delta} \int_{-\infty}^\infty |f(x)| dx < \infty, \end{aligned}$$

we see that $G_t(\cdot) \in PS_1[\delta, \infty)$ for every $\delta > 0$. Hence, by the Riemann-Lebesgue lemma (p. 23),

$$\int_\delta^\infty (\sin \beta x) G_t(x) dx \to 0 \text{ as } \beta \to \infty. \tag{6}$$

Also,

$$\int_\delta^\infty \frac{\sin \beta x}{x} dx = \int_{\beta\delta}^\infty \frac{\sin x}{x} dx \to 0 \text{ as } \beta \to \infty. \tag{7}$$

Therefore, by (5), (6) and (7) you can choose β^* so large that

$$\left| \int_{\delta^*}^{\infty} K(x,\beta) g_t(x) dx \right| < \varepsilon/2, \text{ all } \beta > \beta^*. \tag{8}$$

Finally, from (4) and (8) you get

$$\begin{aligned} |F_\beta(t)| &:= \left| \int_0^\infty K(x,\beta) g_t(x) dx \right| \\ &\le \left| \int_0^{\delta^*} \right| + \left| \int_{\delta^*}^{\infty} \right| \\ &< \varepsilon, \quad \text{for all} \quad \beta > \beta^*. \end{aligned}$$

Q.E.D.

Corollary: Uniqueness of inverse Fourier transform

Essentially distinct functions $f_1, f_2 \in PS_1(-\infty, \infty)$ have distinct Fourier transforms, i.e.

$$\hat{f}_1(\omega) = \hat{f}_2(\omega) \text{ (all } \omega) \Longrightarrow f_1(t) = f_2(t) \text{ a.e.}$$

Remark.

Notice that, in the casual version of our result, written

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega,$$

what is really meant is the so-called *principal value* of the integral, namely one should understand $\displaystyle\int_{-\infty}^{\infty}$ to mean $\displaystyle\lim_{\beta\to\infty} \int_{-\beta}^{\beta}$.

2.5 Practical Inversion of the Fourier Transform

Suppose you're given $\hat{f}(\omega)$: how do you find $f(t)$? Direct calculation of the inversion integral $(2\pi)^{-1} \displaystyle\int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega$ is seldom feasible. Instead, you usually have to use contour integration in the complex plane. Alternatively, you might find the answer in *Tables of Integral Transforms* (Bateman manuscript project), a good collection of transform pairs.

As for contour integration, here's a simple example; for much fancier stuff, see Titchmarsh (*Fourier Integral*), or books on mathematical physics like Sommerfeld, Morse and Feshbach, etc.

Example: Fourier inversion by contour integration

Given

$$\hat{f}(\omega) := \frac{e^{3i\omega}}{1+i\omega}, \qquad -\infty < \omega < \infty$$

find $f(t)$ $(-\infty < t < \infty)$. You can just consider

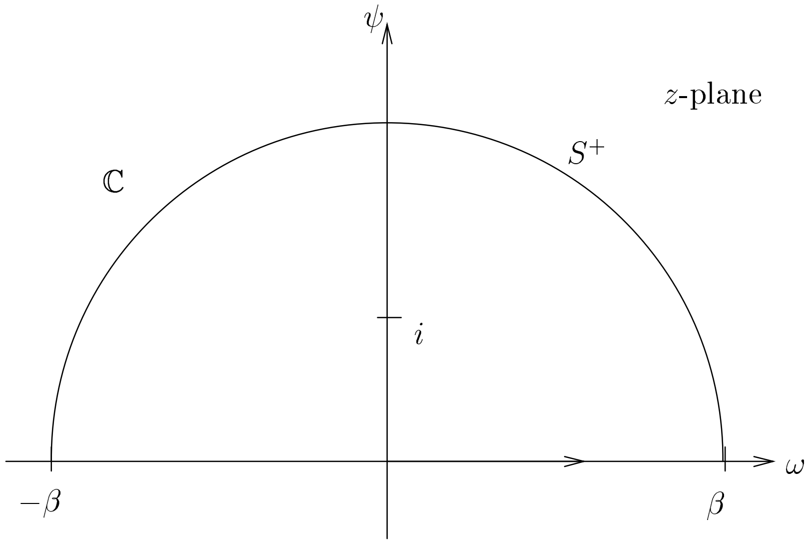
$$\hat{g}(\omega) := \frac{1}{1+i\omega} \text{ -- why?}$$

Assume (in order to get somewhere) that $\hat{g}$ really is the Fourier Transform of some function $g \in PS_1$. Then

$$g_\beta(t) = \frac{1}{2\pi i}\int_{-\beta}^{\beta} \frac{e^{i\omega t}}{\omega - i} d\omega.$$

Now bring in the complex variable $z = \omega + i\psi$.

If $t > 0$, complete the contour $(-\beta \leq \omega \leq \beta,\ \psi = 0)$ into the *upper* half z-plane, as shown, so that e^{izt} is small on the semicircle S^+.



Now for $\beta > 1$, Cauchy says

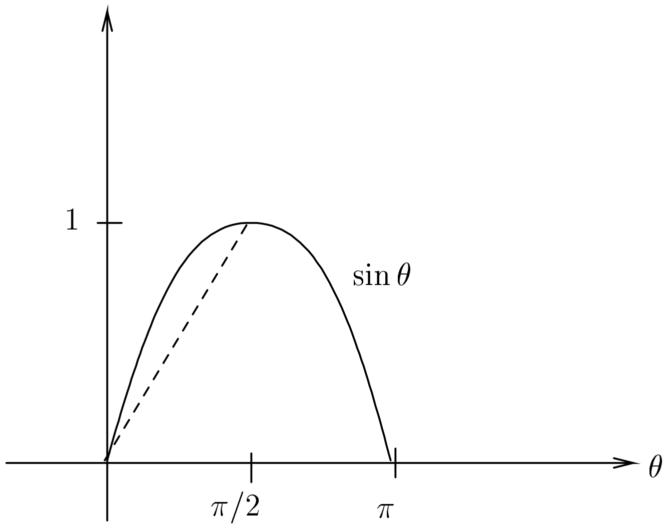
$$\begin{aligned}(2\pi i)^{-1} \oint \frac{e^{izt}}{z-i} dz &= \left\{ \begin{array}{c} \text{Residue of } \dfrac{e^{izt}}{z-i} \\ \text{at } z = i \end{array} \right\} \\ &= e^{-t}.\end{aligned}$$

Also, on S^+, $z = \beta e^{i\theta}$, $0 \leq \theta \leq \pi$, and

$$\begin{aligned}\left|\frac{e^{izt}}{z-i}\right| &= \left|\frac{\exp[it\beta(\cos\theta + i\sin\theta)]}{\beta e^{i\theta} - i}\right| \\ &= \frac{\exp(-t\beta\sin\theta)}{|\beta e^{i\theta} - i|} \\ &\leq \frac{\exp(-t\beta\sin\theta)}{\beta - 1} \text{ for } \beta > 1.\end{aligned}$$

Note that

$$\sin\theta \geq \frac{2}{\pi}\cdot\theta, \quad 0 \leq \theta \leq \pi/2,$$



and therefore on S^+, when $0 \leq \theta \leq \pi/2$,

$$\left|\frac{e^{izt}}{z-i}\right| \leq \frac{\exp(-2\beta t\theta/\pi)}{\beta - 1} \quad (\beta > 1)$$

You can now easily show that

$$\left|\int_{S^+}\frac{e^{izt}}{z-i}dz\right| \to 0 \text{ as } \beta \to \infty.$$

Exercise 2.6: Do this.

Putting everything together, you get

$$\lim_{\beta\to\infty} g_\beta(t) = e^{-t}, \quad \text{provided } t > 0.$$

To finish up, do

Exercise 2.7: Show that

$$\lim_{\beta\to\infty} g_\beta(t) = 0, \quad \text{provided } t < 0.$$

by taking a semicircle S^- in the *lower* half plane.

So now you have

$$\lim_{\beta\to\infty} g_\beta(t) = \begin{cases} e^{-t}, & t > 0 \\ 0, & t < 0 \end{cases}$$

[It's a nice exercise on complex logarithm to check directly that

$$\lim_{\beta\to\infty} g_\beta(0) = \frac{1}{2}].$$

The function $g(t)$ now defined obviously has the given $\hat{g}(\omega)$ as its Fourier Transform and all is well.

2.6 "Energy" and Parseval's Theorem

To bring in the idea of "energy" along the same lines as in the case of Fourier Series, we must introduce a new class of functions - those for which the "energy" is actually finite.

Definition
A function $f(\cdot) \in PS_2(-\infty, \infty)$ if $f \in PS(-\infty, \infty)$ [i.e. piecewise smooth on every finite subinterval of $(-\infty, \infty)$], *and also*

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty.$$

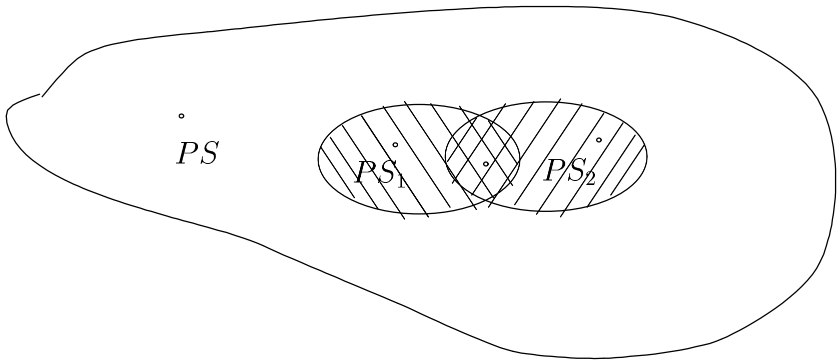
Definition
If $f \in PS_2(-\infty, \infty)$, the *norm* of f is

$$\|f\| := \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{1/2}.$$

It so happens that the function classes $PS_1(-\infty, \infty)$ and $PS_2(-\infty, \infty)$ are not the same, as illustrated by the following.

Examples (of distinct PS function classes)
Let

$$\begin{aligned}
f(t) &:= (1+|t|)^{-1}, \quad -\infty < t < \infty \\
g(t) &:= e^{-|t|}, \quad -\infty < t < \infty \\
h(t) &:= \begin{cases} n^7 - n^{16}|t-n|, & |t-n| \le n^{-9}, \quad n = 2, 3, \ldots; \\ 0, & \text{otherwise} \end{cases} \\
k(t) &:= \sin t, \quad -\infty < t < \infty.
\end{aligned}$$



Show that each of these four functions can be identified with exactly one dot $(\cdot)$ in the Venn diagram illustrated.

We shall now prove a simplified version of Parseval's Theorem for Fourier Transforms.

THEOREM (Parseval - special case)

Assume

$$f \in PS_1(-\infty, \infty), \quad \hat{f} \in PS_1(-\infty, \infty).$$

Then

$$f \in PS_2(-\infty, \infty), \quad \hat{f} \in PS_2(-\infty, \infty)$$

and

$$\|\hat{f}\|^2 = 2\pi \|f\|^2$$

Proof:

Assume f is defined at jump points so as to satisfy

$$f(t) = \frac{1}{2}[f(t-0) + f(t+0)]$$

Then by Fourier inversion

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} e^{i\omega t} \hat{f}(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega \\ &\quad (\text{since } \hat{f} \in PS_1) \end{aligned}$$

Thus

$$|f(t)|^2 = \frac{\bar{f}(t)}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega$$

$$\begin{aligned} \int_{-\infty}^{\infty} |f(t)|^2 dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(t) \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \left(\int_{-\infty}^{\infty} \overline{f(t) e^{-i\omega t}} dt \right) d\omega \text{ (by Fubinito)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \overline{\hat{f}(\omega)} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \end{aligned}$$

Namely

$$\|f\|^2 = \frac{1}{2\pi}\|\hat{f}\|^2 \qquad Q.E.D.$$

While this theorem will do for many applications, its assumptions are unnecessarily stringent, and rule out, for instance, application to the Fourier Transform pair

$$f(t) = \begin{cases} e^{-t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \qquad \hat{f}(\omega) = (1 + i\omega)^{-1} \qquad !$$

Notice from this example that $f \in PS_1$ does not imply $\hat{f} \in PS_1$. In this example we still have that $\|\hat{f}\|^2 = 2\pi\|f\|^2$, as you see by elementary integration. In fact, the general case of Parseval's theorem is much stronger, as we shall indicate shortly.

Exercise 2.8: Consider the time-gated sinusoid

$$f(t) = \begin{cases} 0, & -\infty < t \leq -T \\ \sin t, & -T \leq t \leq T \\ 0, & T \leq t < \infty \end{cases}$$

where $T = 2\pi N$, N a positive integer. Calculate the energy-frequency spectrum

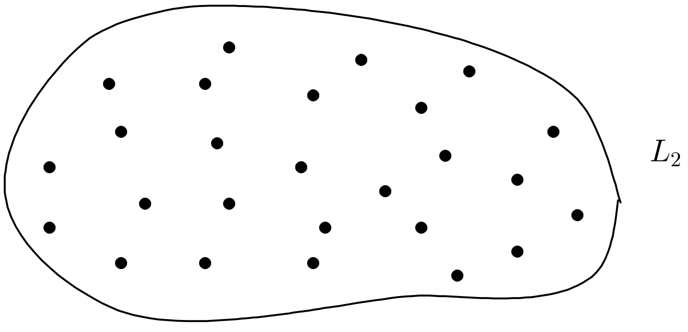
$$E_N(\omega) := |\hat{f}(\omega)|^2.$$

Show that $E_N(\pm 1) = cN^2$ for a suitable constant c. At what frequencies, if any, is $E_N(\omega) = 0$? Obtain the normalized spectrum

$$\bar{E}_N(\omega) := |\hat{f}(\omega)|^2/\|\hat{f}\|^2.$$

Compute and plot the graph of $\bar{E}_N(\omega)$ for fixed $N \gg 1$. Can you draw any conclusions about the practical problem of measuring the frequency of a sinusoidal signal? On the basis of some reasonable assumption about the response of measurement devices, estimate how long it would take to measure within 0.001% the frequency of a signal whose true frequency is about 60 Hz.

Just as with Fourier Series (review Sect. 1.7), it is convenient to extend our 'concrete' function class $PS_2(-\infty, \infty)$ to the larger class $L_2 = L_2(-\infty, \infty)$, of all functions that are Lebesgue-measurable and finitely square-integrable on $(-\infty, \infty)$. In the picture below, think of the dots as constituting PS_2 embedded in L_2.



For functions $f \in L_2$ it still is true that

$$\|f\|^2 := \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty,$$

but $\int$ is now a "Lebesgue" integral. For $f \in L_2$, $\|f\|$ is called the "L_2-norm" of f. If it happens that $f \in PS_2$, nothing is changed from before.

Recall from Sect. 1.7 that a sequence $\{f_N\}$ of functions $f_N \in L_2$ *converges in mean square*, if there exists a function $f \in L_2$ such that

$$\|f_N - f\| \to 0, \qquad N \to \infty$$

In this case one may write

$$\underset{N\to\infty}{l.i.m.}\, f_N = f, \quad \text{or} \quad f_N \to f \;\; \text{(m.s)}$$

With all this, you now have one of the world's great theorems:

THEOREM (Plancherel - 1910).
If $f \in L_2(-\infty, \infty)$, there exists a function $\hat{f} \in L_2(-\infty, \infty)$ [called the *Fourier-Plancherel transform* of f], such that

$$\hat{f}(\omega) = \underset{N\to\infty}{l.i.m.} \int_{-N}^{N} e^{-i\omega t} f(t) dt,$$

and also

$$f(t) = \underset{N\to\infty}{l.i.m.} \frac{1}{2\pi} \int_{-N}^{N} e^{i\omega t} \hat{f}(\omega) d\omega.$$

Furthermore (Parseval),

$$\|\hat{f}\|^2 = 2\pi \|f\|^2.$$

Note the beautiful symmetry, between "time" and "frequency". Unfortunately, you can't replace L_2 by PS_2 in this theorem,[1] so our proved results were not as nice as Plancherel's.

To wrap this up, if $f \in PS_1 \cap L_2$ then f has the 'ordinary' Fourier Transform

$$\hat{f} = \int_{-\infty}^{\infty} \ldots$$

as well as the 'Plancherel' Fourier Transform

$$\hat{f} = \underset{N\to\infty}{l.i.m.} \int_{-N}^{N} \ldots$$

It turns out that these amount to exactly the same function, so no inconsistency ever arises.

Verifying Plancherel's theorem by hand can be quite laborious, but here is a simple example.

[1] because $f \in PS_2$ does not imply $\hat{f} \in PS_2$: $\hat{f}$ needn't be piecewise-smooth.

Example: Illustration of Plancherel's Theorem

Let

$$f(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

We know

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt = \frac{1}{1+i\omega}.$$

We shall verify that $(f, \hat{f})$ is a Fourier transform pair in the Plancherel sense.

1. Verify

$$\hat{f} = \underset{N\to\infty}{l.i.m.} \int_{-N}^{N} f(t) e^{-i\omega t} dt$$

i.e.

$$\lim_{N\to\infty} \|\hat{f} - \int_{-N}^{N} \| = 0$$

i.e.

$$\lim \| \cdot \|^2 = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left| \hat{f}(\omega) - \int_{-N}^{N} f(t) e^{-i\omega t} dt \right|^2 d\omega = 0$$

Now $\int_{-N}^{N} f(t) e^{-i\omega t} dt = \int_{0}^{N} e^{-i\omega t} e^{-t} dt$, so

$$\begin{aligned} \int_{-\infty}^{\infty} |\cdot|^2 &= \int_{-\infty}^{\infty} \left| \frac{1}{1+i\omega} - \frac{1 - e^{-(1+i\omega)N}}{1+i\omega} \right|^2 d\omega \\ &= \int_{-\infty}^{\infty} \frac{|e^{-(1+i\omega)N}|^2}{|1+i\omega|^2} d\omega \\ &= e^{-2N} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega \to 0 \text{ as } N \to \infty, \end{aligned}$$

as required.

2. Verify

$$f = \frac{1}{2\pi} \underset{N\to\infty}{l.i.m.} \int_{-N}^{N} e^{i\omega t} \hat{f}(\omega) d\omega$$

i.e.

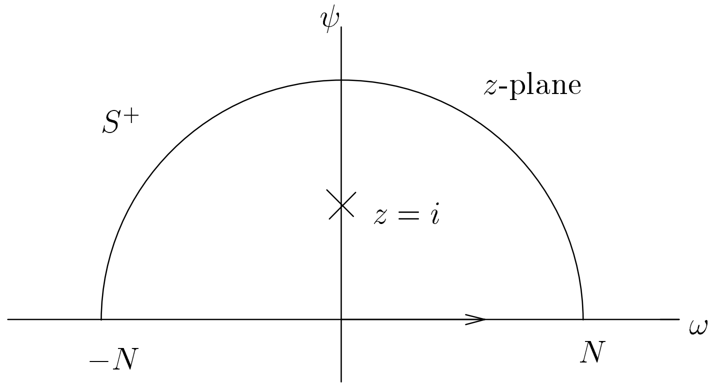
$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|f(t)-\frac{1}{2\pi}\int_{-N}^{N}\frac{e^{i\omega t}}{1+i\omega}d\omega\right|^2 dt=0.$$

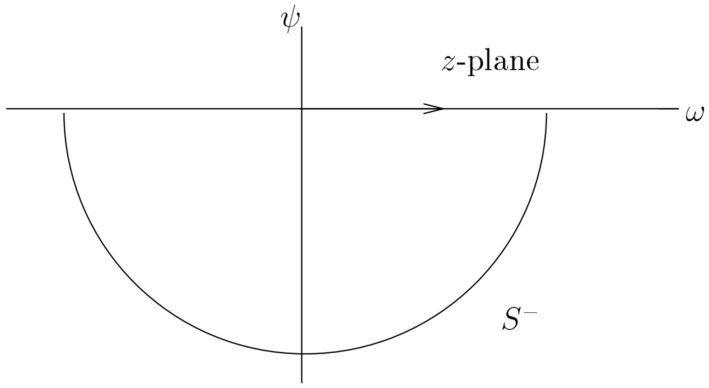
Evaluate:

$$\frac{1}{2\pi}\int_{-N}^{N}\frac{e^{i\omega t}}{1+i\omega}d\omega$$

Let $N>1$. We have:

$$\begin{aligned}\frac{1}{2\pi}\left\{\int_{-N}^{N}+\int_{S^+}\right\} &= \frac{1}{2\pi}(2\pi i)\frac{e^{-t}}{i}\\ &= e^{-t}\end{aligned}$$





and

$$\frac{1}{2\pi}\left\{\int_{-N}^{N}+\int_{S^-}\right\}=0.$$

We showed earlier:

$$t \geq 0 \Longrightarrow \left| \int_{S^+} \right| \leq 2 \int_0^{\pi/2} \frac{e^{-2Nt\theta/\pi}}{N-1} d\theta$$

$$= \frac{2}{N-1} \cdot \frac{\pi}{2Nt}(1 - e^{-Nt})$$

$$t \leq 0 \Longrightarrow \left| \int_{S^-} \right| \leq \text{ditto (replace } t \text{ by } |t|)$$

So:

$$\int_{-\infty}^{\infty} \left| f(t) - \frac{1}{2\pi} \int_{-N}^{N} \ldots \right|^2 dt$$

$$\leq \int_{-\infty}^{\infty} \left| \frac{\pi}{N-1} \left(\frac{1 - e^{-N|t|}}{N|t|} \right) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} |\ldots|^2 \frac{d(Nt)}{N}$$

$$= \frac{\pi^2}{N(N-1)^2} \cdot 2 \int_0^{\infty} \left(\frac{1 - e^{-x}}{x} \right)^2 dx$$

$$= 0(N^{-3}) \to 0 \text{ as } N \to \infty.$$

This completes the verification.

The following exercise provides a heuristic approach to Parseval's theorem which may be picturesque.

Exercise 2.9: The Fourier pair $\{\delta(t), 1\}$

Let $\delta(x)$ be the unit impulse function. Then

$$\int_{-\infty}^{\infty} e^{-i\omega t} \delta(t) dt = e^{-i\omega t}|_{t=0} = 1,$$

i.e.

$$\hat{\delta}(\omega) = 1, \quad -\infty < \omega < \infty.$$

On the basis of Fourier inversion, then, you'd get

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{\delta}(\omega)d\omega,$$

i.e.

$$\int_{-\infty}^{\infty} e^{i\omega t}d\omega = 2\pi\delta(t).$$

Similarly

$$\int_{-\infty}^{\infty} e^{-i\omega t}d\omega = 2\pi\delta(t).$$

Now give a fake proof of Parseval:

$$\begin{aligned}
\|\hat{f}\|^2 &= \|\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt\|^2 \\
&= \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt\right|^2 d\omega \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega t}f(t)e^{i\omega s}\overline{f(s)}dtdsd\omega \\
&\vdots \\
&= 2\pi\|f\|^2,
\end{aligned}$$

where the missing 3 or 4 lines are left to you.

2.7 Time-Bandwidth Tradeoff

We are now in a position to prove, what was merely suggested, by the remark on p. 70. Our discussion will at first skip details of rigor. Suppose you are given $f(\cdot)$, on $-\infty < t < \infty$. As a measure of the 'average location' of the graph of f along the t-axis, bring in the number

$$< t >:= \int_{-\infty}^{\infty} t|f(t)|^2dt/\|f\|^2.$$

Similarly, define

$$< t^2 >:= \int_{-\infty}^{\infty} t^2|f(t)|^2dt/\|f\|^2,$$

and then

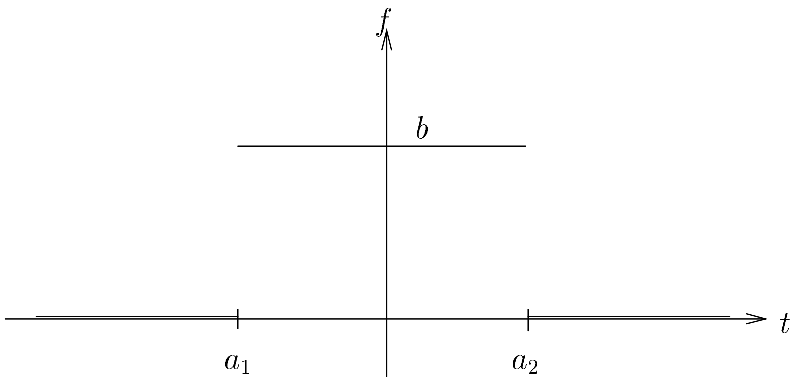
$$\text{variance of } t = \text{var } t \; := \; < t^2 > - < t >^2 .$$

$$= \; < (t- < t >)^2 > .$$

Finally, as a measure of the 'spread' of $f(\cdot)$ along the t-axis (i.e. of the 'duration' of your signal in time), let

$$T := \sqrt{\text{var } t}.$$

Example



$$f(t) \; = \; \begin{cases} b, & a_1 \le t \le a_2 \\ 0, & \text{otherwise} \end{cases}$$

$$\|f\|^2 \; = \; b^2(a_2 - a_1).$$

$$< t > \; = \; \frac{1}{2}(a_1 + a_2), \quad < t^2 >= \frac{1}{3}(a_1^2 + a_1 a_2 + a_2^2)$$

$$\text{var } t \; = \; (a_2 - a_1)^2/12, \quad T = (a_2 - a_1)/2\sqrt{3}$$

Note that T need not be the 'actual' duration of a strictly time-limited signal.

In the above definitions, we are regarding the function $|f(t)|^2/\|f\|^2$ as a sort of 'probability density' for t. Now do the same thing for ω, using $|\hat{f}(\omega)|^2/\|\hat{f}\|^2$:

$$< \omega > \; := \; \int_{-\infty}^{\infty} \omega |\hat{f}(\omega)|^2 d\omega / \|\hat{f}\|^2$$

$$< \omega^2 > \;:=\; \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega / \|\hat{f}\|^2$$

$$\text{var } \omega \;:=\; < \omega^2 > - < \omega >^2$$

and finally, as a measure of 'bandwidth',

$$B := \sqrt{\text{var } \omega}.$$

Under mild technical restrictions, we shall prove the important

TIME-BANDWIDTH UNCERTAINTY PRINCIPLE: For any signal $f(t)$, $-\infty < t < \infty$, for which the parameters $T = T(f)$, $B = B(f)$ exist, we have $TB \geq \dfrac{1}{2}$ (radians). Equivalently, putting $\omega = 2\pi\nu$, we have

$$\sqrt{\text{var } t} \cdot \sqrt{\text{var } \nu} \geq \frac{1}{4\pi}(\text{cycles}).$$

MORAL: It is impossible to design a signal which is localized arbitrarily sharply in *both* the time and frequency domains simultaneously.

For the proof we need the Schwarz inequality in the form:

$$f, g \in PS_2 \Rightarrow |\text{Re} \int_{-\infty}^{\infty} f\bar{g} dt| \leq \|f\| \cdot \|g\|.$$

Exercise 2.10: Verify this.

Now to prove the uncertainty principle, assume f is given. By a shift of t-origin if necessary (i.e. change of variable to $t' = t- < t >$), we can arrange that $< t >= 0$. Similarly we can arrange (by $\omega' = \omega- < \omega >$) that $< \omega >= 0$. (For details, see Ex. 19, p. 178.)

This means

$$T = \sqrt{< t^2 >}, \quad B = \sqrt{< \omega^2 >}.$$

So

$$\begin{aligned} \|\hat{f}\|^2 \text{var } \omega &= \|i\omega \hat{f}\|^2 \\ &= \|\widehat{\frac{df}{dt}}(\omega)\|^2 \text{ [by derivative formula p. 74]} \\ &= 2\pi \|\frac{df}{dt}\|^2 \text{ [Parseval].} \end{aligned}$$

Therefore

$$\begin{aligned}
\|f\|^2 \quad & (\text{var } t)\|\hat{f}\|^2(\text{var } \omega) \\
& = 2\pi \|tf\|^2 \|\frac{df}{dt}\|^2 \\
& \geq 2\pi \left[\text{Re} \int_{-\infty}^{\infty} tf(t)\overline{\frac{df(t)}{dt}} dt \right]^2 \quad [\text{Schwarz}].
\end{aligned}$$

Now

$$\begin{aligned}
\int_{-\infty}^{\infty} tf(t)\overline{\frac{df(t)}{dt}}dt &= tf\bar{f}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \bar{f}\frac{d}{dt}(tf)dt \\
&= 0 - \int_{-\infty}^{\infty} \bar{f}(f + t\frac{df}{dt})dt \text{ (say)} \\
&= -\int_{-\infty}^{\infty} t\bar{f}\frac{df}{dt}dt - \|f\|^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} tf\overline{\frac{df}{dt}}dt \quad &+ \quad \int_{-\infty}^{\infty} t\bar{f}\frac{df}{dt}dt \\
&= \quad 2\text{Re}\int_{-\infty}^{\infty} tf\frac{d\bar{f}}{dt}dt \\
&= \quad -\|f\|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f\|^2(\text{var } t)\|\hat{f}\|^2(\text{var } \omega) &\geq 2\pi \cdot \left[\frac{1}{2}\|f\|^2\right]^2 \\
&= 2\pi\|f\|^4/4 \\
&= \frac{1}{4} \cdot \|f\|^2 \cdot 2\pi\|f\|^2 \\
&= \frac{1}{4}\|f\|^2 \cdot \|\hat{f}\|^2 \quad [\text{Parseval}]
\end{aligned}$$

or

$$(\text{var } t)(\text{var } \omega) \geq \frac{1}{4}, \quad \text{as claimed.} \qquad \textit{Q.E.D.}$$

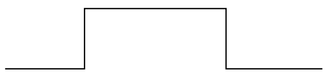
The above proof was formal, but can easily be seen to be rigorously valid if, e.g.

$$f, \quad tf, \quad \text{and} \quad \frac{df}{dt} \in PS_1 \cap PS_2$$

and

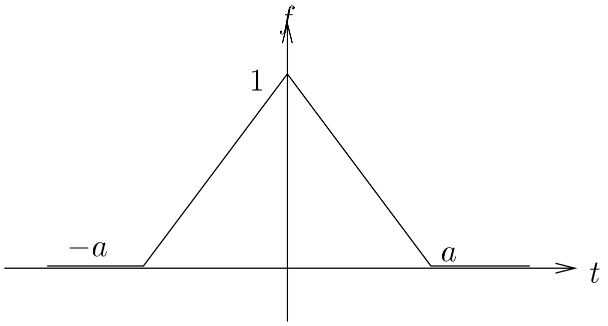
$$\sqrt{|t|}f(t) \to 0 \text{ as } t \to \pm\infty.$$

Consider our first example (p. 94).



Unfortunately, for this signal $B = +\infty$: a rectangular pulse is a little *too* sharp in time! Try this one:

Exercise 2.11: Work out var t, var ω and TB for the function f illustrated:



Do your results match intuition? Discuss. **Hint:** It will help to note that

$$\text{var } \omega = \|df/dt\|^2/\|f\|^2$$

Exercise 2.12: The optimal time-bandwidth package is Gaussian!

As another example, let

$$f(t) = (2\pi)^{-1/4}e^{-t^2/4}$$

Then

$$\hat{f}(\omega) = \sqrt{2}(2\pi)^{1/4}e^{-\omega^2}.$$

Check that

$$\text{var } t = 1, \quad \text{var } \omega = \frac{1}{4},$$

so

$$TB = \frac{1}{2}.$$

This shows that our general inequality cannot be improved, inasmuch as the Gaussian function is "at the boundary".

Exercise 2.13: In the light of $TB \geq 1/2$, revisit Exercise 2.8.

Application: Heisenberg's Uncertainty Principle

In quantum mechanics the position of a particle moving on the x-axis is described by a *wave function* $\psi(x,t)$ which satisfies Schrödinger's equation. The function

$$|\psi(x,t)|^2 / \int_{-\infty}^{\infty} |\psi(x',t)|^2 dx'$$

is interpreted as the probability density of the particle position at time t. *Fix* t, and write simply $\psi(t,x) = \psi(x)$.

Now let

$$\begin{aligned} \lambda &:= \text{ spatial wavelength} \\ k &:= 2\pi/\lambda = \text{'propagation number'} \end{aligned} \tag{1}$$

Bring in

$$\hat{\psi}(k) := \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx$$

and regard $|\hat{\psi}(k)|^2/\|\hat{\psi}\|^2$ as a probability density for k. [Thus x, k play the roles of the variables t, ω respectively in our original discussion.] By the Fourier uncertainty principle,

$$\sqrt{\text{var } x} \cdot \sqrt{\text{var } k} \geq \frac{1}{2}. \tag{2}$$

Now use de Broglie's relation (1924):

$$\lambda = h/p \tag{3}$$

where p = particle momentum, and h = Planck's constant. By (1) and (3),

$$k = \frac{2\pi}{h} p, \quad \sqrt{\text{var } k} = \frac{2\pi}{h} \sqrt{\text{var } p}. \tag{4}$$

Then by (2) and (4),

$$\sqrt{\text{var } x} \cdot \sqrt{\text{var } p} \geq h/4\pi = (1.054 \times 10^{-27})/2 = 5.27 \times 10^{-28} \text{ erg sec},$$

the Heisenberg uncertainty principle.

2.8 Nyquist-Shannon Sampling Theorem[2]

2 The famous Sampling Theorem states that a signal which is strictly band-limited, say to the frequency range

$$-\Omega < \omega < \Omega,$$

can be recovered from its values, or *samples*, spaced at time intervals of length π/Ω, i.e. taken at (at least) twice the highest frequency (Ω) present in the signal spectrum. This fact underlies numerous modern communication techniques based on digital signal encoding. For instance, the Sampling Theorem justifies the introduction of the concept of signal

[2]Primary contributors include E.T. Whittaker (1915), H. Nyquist (1928), V.A. Kotel'nikov (1933) and C. Shannon (1948).

space, along with powerful geometric methods for the development of efficient codes for the transmission of signals over noisy communication channels.

We shall first develop the result rigorously, then give a systems interpretation in terms of A/D and D/A conversion.

Definition. Let f be defined for $-\infty < t < \infty$ and have Fourier transform $\hat{f}(\omega)$. f is *band-limited* to the interval $-\Omega < \omega < \Omega$ if

$$\hat{f}(\omega) = 0, \text{ all } |\omega| \geq \Omega.$$

Let us assume first that

$$f \in PS_1(-\infty, \infty) \tag{$A.1'$}$$

with f 'standardized' at jump points (if any) according to

$$f(t) = \frac{f(t-0) + f(t+0)}{2} \tag{$A.1''$}$$

In addition we assume that

$$f \text{ is band-limited to } (-\Omega, \Omega). \tag{$A.2$}$$

By the Fourier inversion theorem, (A.1$''$) and (A.2),

$$\begin{aligned} f(t) &= \lim_{\beta \to \infty} \frac{1}{2\pi} \int_{-\beta}^{\beta} \hat{f}(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{i\omega t} d\omega, \qquad -\infty < t < \infty . \end{aligned} \tag{1}$$

By (A.1$'$), $\hat{f}(\cdot)$ is bounded; then, by use of (1), it can be shown that $f(\cdot)$ must actually be continuous and bounded on $(-\infty, \infty)$.

Now fix t, and expand $e^{i\omega t}$ in its Fourier series on $-\Omega \leq \omega \leq \Omega$:

$$e^{i\omega t} = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n(t) e^{in\pi\omega/\Omega}, \quad -\Omega < \omega < \Omega \tag{2}$$

Here

$$\begin{aligned} c_n(t) &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} e^{i\omega t} e^{-in\pi\omega/\Omega} d\omega \\ &= \frac{1}{\Omega} \int_{0}^{\Omega} \cos\left[\left(t - \frac{n\pi}{\Omega}\right)\omega\right] d\omega, \end{aligned}$$

or

$$c_n(t) = \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}. \tag{3}$$

By (1) and (2), and writing $P\sum_{-\infty}^{\infty}$ for the "principal value", as in (2), of the infinite series, we get

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\{P\sum_{n=-\infty}^{\infty} c_n(t)e^{in\pi\omega/\Omega}\}\hat{f}(\omega)d\omega \\ &= P\sum_{n=-\infty}^{\infty} c_n(t)\frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{in\pi\omega/\Omega}\hat{f}(\omega)d\omega \end{aligned}$$

[since $P\sum_{-\infty}^{\infty}$ converges boundedly]

or

$$f(t) = P\sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\Omega})\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}. \tag{4}$$

Note that (4) is obviously true at $t = k\pi/\Omega$, $k = 0, \pm 1, \ldots$

Then we have the

SAMPLING THEOREM

Subject to assumptions (A.1) and (A.2)

$$f(t) = P\sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\Omega})\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi} \tag{$*$}$$

for all $t \in (-\infty, \infty)$.

The formula (*) is sometimes called the *Nyquist-Shannon interpolation formula.*

Comments

1. A slightly sloppy but generally harmless version of (*) is:

$$f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\Omega})\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}. \tag{$\dagger$}$$

This is valid if, in addition to (A.1), (A.2) you assume, for instance, that

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{|n|}|f(\frac{n\pi}{\Omega})| < \infty. \tag{$A.3$}$$

2. Condition (A.2) holds, obviously, if

$$\hat{f}(\omega) = 0, \text{ all } |\omega| \geq \Omega',$$

for some $\Omega' < \Omega$. Let the 'best' Ω' be Ω_f:

$$\Omega_f := \inf\{\Omega : \hat{f}(\omega) = 0, \text{ all } |\omega| \geq \Omega\}.$$

Then (*) holds provided $\Omega \geq \Omega_f$.

3. Write $\Omega_f = 2\pi\nu_f$. The *Nyquist* (sampling) *rate* for f is the number

$$2\nu_f \text{ [samples/sec., say]}.$$

So (*) says that $f(\cdot)$ can be reconstructed from its samples provided sampling is carried out at a rate no lower than the Nyquist rate.

4. Warning: Here's a (false) example.

$$f(t) = \sin 2\pi t, \quad -\infty < t < \infty.$$

Nyquist rate $= 2\times 1 = 2$.
Sampling at exactly the Nyquist rate we get

$$n^{th} \text{ sample} = f(\frac{n\pi}{\Omega_f}) = f(\frac{n\pi}{2\pi}) = \sin(n\pi) = 0.$$

Then in (*),

$$\sum \ldots = 0 \neq f(t)!$$

Our theorem does not apply to this example, since $\sin 2\pi t \notin PS_1(-\infty, \infty)$. However, the theorem can be generalized to a broader class of functions, that does include $\sin 2\pi t$. In its more general form, the theorem requires sampling to be carried out at any rate *strictly higher* than the Nyquist rate.

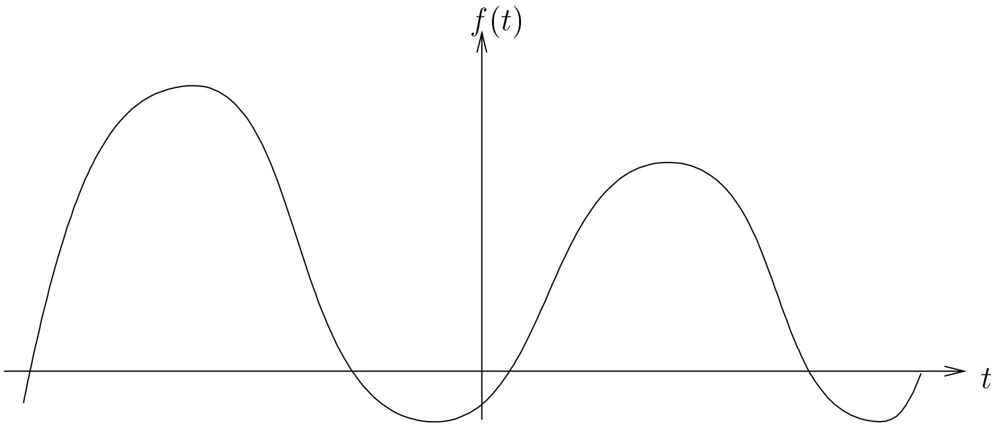
Systems derivation of the Sampling Theorem

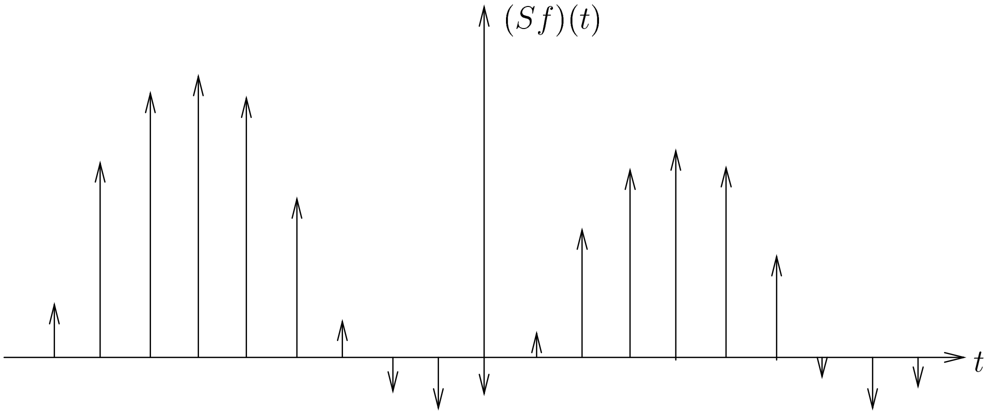
The following heuristics should serve to explain intuitively why the sampling theorem works. Formally, bring in the periodic pulse train

$$p_\tau(t) := \sum_{n=-\infty}^{\infty} \delta(t - n\tau), \quad \tau = \pi/\Omega.$$

The *sampling operator* S_τ is defined by

$$\begin{aligned}(S_\tau f)(t) &:= f(t)p_\tau(t) \\ &= \sum_{n=-\infty}^{\infty} f(n\tau)\delta(t - n\tau).\end{aligned}$$





For brevity write p, S for p_τ, S_τ. Now work out the effect of sampling in the frequency domain. For this we represent $p(t)$ by its Fourier series:

$$p(t) = \sum_{n=-\infty}^{\infty} c_n e^{in(2\pi/\tau)t}$$

where

$$\begin{aligned} c_n &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} p(t)e^{-in(2\pi/\tau)t}dt \\ &= 1/\tau; \end{aligned}$$

so

$$p(t) = (1/\tau)\sum_{n=-\infty}^{\infty} e^{in(2\pi/\tau)t}.$$

The Fourier Transform of the sampled signal can now be computed as

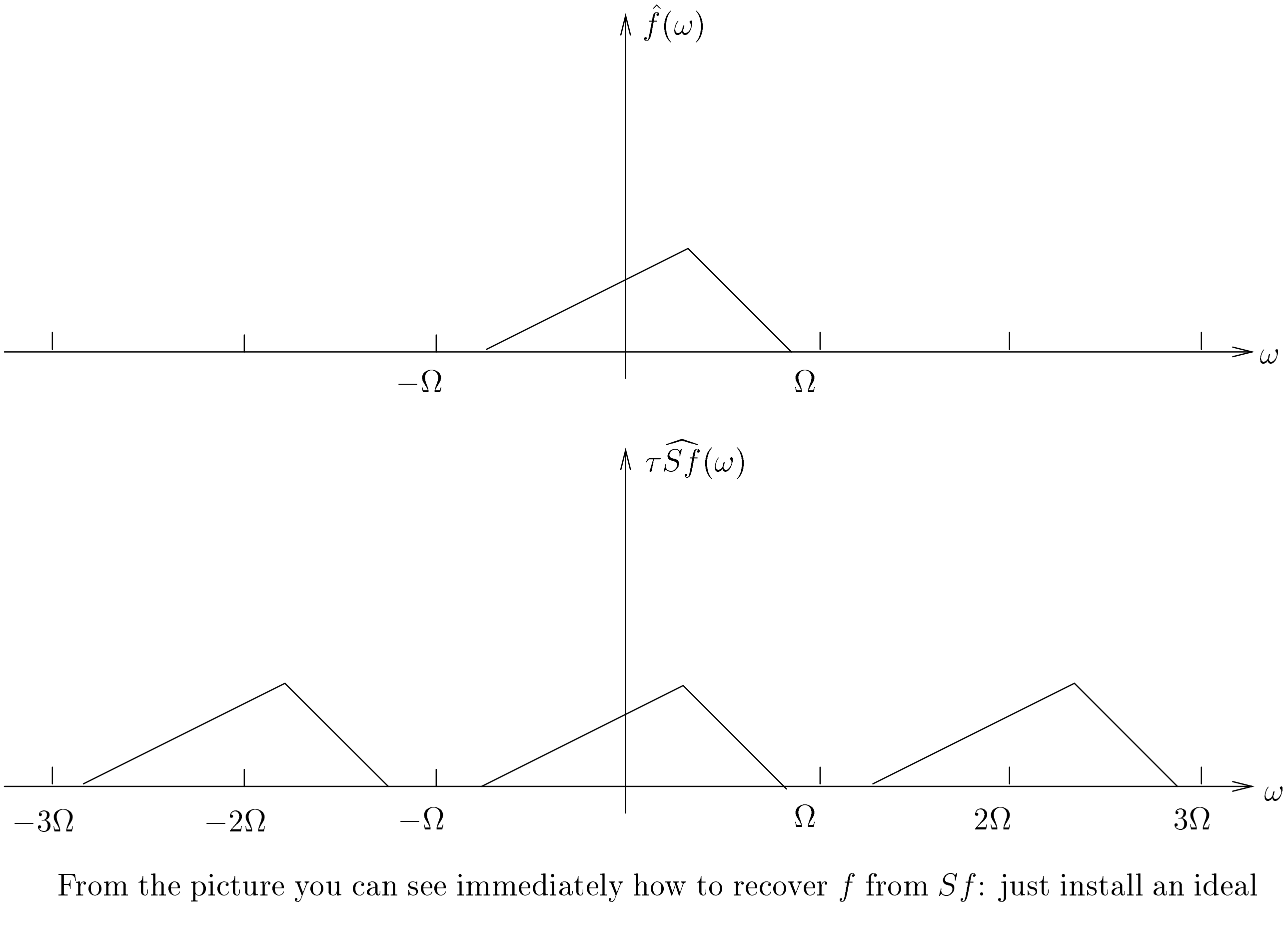
$$\widehat{Sf}(\omega) \quad = \quad \int_{-\infty}^{\infty}(Sf)(t)e^{-i\omega t}dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} f(t)p(t)e^{-i\omega t}dt \\
&= (1/\tau)\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\omega-2n\pi/\tau)t}f(t)dt.
\end{aligned}
$$

Evaluating the integral and setting $\tau = \pi/\Omega$, we get finally

$$\widehat{Sf}(\omega) = \frac{\Omega}{\pi}\sum_{n=-\infty}^{\infty}\hat{f}(\omega - 2n\Omega)$$

From this formula you can visualize the effect of sampling as simply taking the frequency spectrum of f, translating it up and down the frequency axis by integer multiples of the sampling angular frequency 2Ω, and adding up all the resulting contributions. The effect will be quite messy, in general, *except* when the various contributions never overlap, and this will be true precisely when sampling is fast enough, namely f is bandlimited to 1/2 the sampling angular frequency, $\Omega = \pi/\tau$! Here is the picture in that case:



From the picture you can see immediately how to recover f from Sf: just install an ideal low-pass filter that recovers $\hat{f}$ from $\widehat{Sf}$. Thus let

$$\hat{R}(\omega) = \begin{cases} \pi/\Omega, & |\omega| \le \Omega \\ 0, & |\omega| > \Omega \end{cases}$$

Then (provided f is bandlimited)

$$\hat{f}(\omega) = \hat{R}(\omega)\widehat{Sf}(\omega), \quad -\infty < \omega < \infty.$$

Now go back to the time domain. We have

$$\begin{aligned} R(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}(\omega)e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\cdot\frac{\pi}{\Omega}\int_{-\Omega}^{\Omega} e^{i\omega t}d\omega \\ &= \frac{\sin \Omega t}{\Omega t} \end{aligned}$$

So

$$\begin{aligned} f(t) &= (R * Sf)(t) \\ &= \int_{-\infty}^{\infty}(Sf)(t')\frac{\sin \Omega(t-t')}{\Omega(t-t')}dt' \\ &= \sum_{-\infty}^{\infty} f(n\tau)\int_{-\infty}^{\infty}\delta(t'-n\tau)\frac{\sin \Omega(t-t')}{\Omega(t-t')}dt' \\ &= \sum_{-\infty}^{\infty} f(n\tau)\frac{\sin \Omega(t-n\tau)}{\Omega(t-n\tau)} \end{aligned}$$

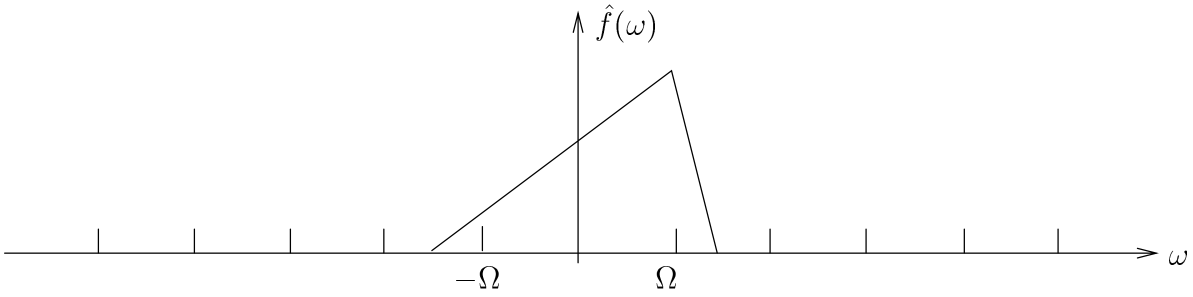
In other words

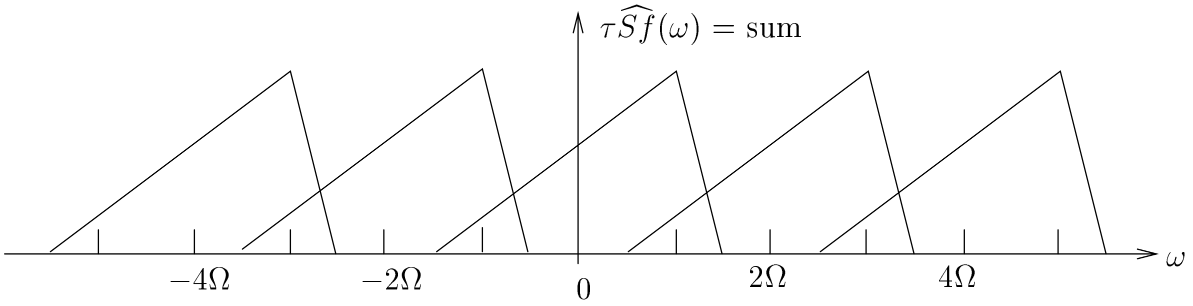
$$f(t) = \sum_{-\infty}^{\infty} f(\frac{n\pi}{\Omega})\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi},$$

the interpolation formula as derived before.

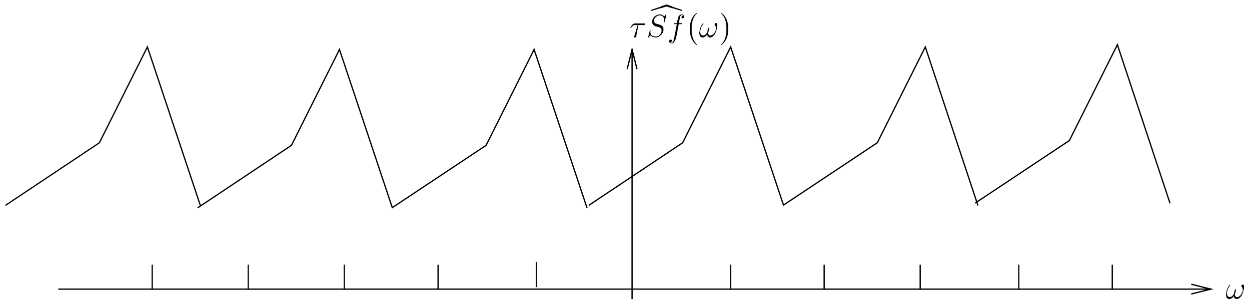
Comment 1: Aliasing

What happens if we sample too slowly? The neat picture on p. 103 now gets messed up like this:

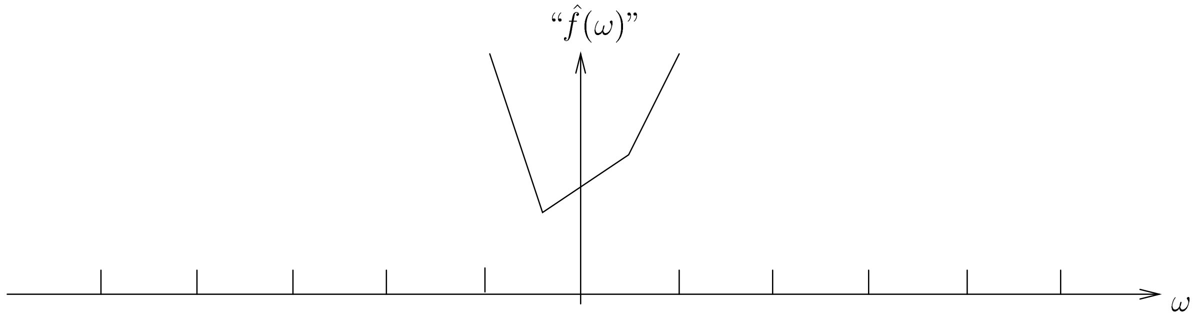




or roughly



So when you try to recover $\hat{f}$ by the low-pass filter you will get not $\hat{f}$ but this:



The output $R * Sf$ of the ideal filter will certainly not be f: because the spectrum of f is too wide to fit the sampling 'window' of width 2Ω, overlapping occurs which results in loss of information. Thus, extraneous high frequency components are introduced in the reconstructed signal. This type of error is called "aliasing" error - for reasons which should be evident from the picture.

Comment 2: Error analysis for finite sampling

Assuming $\Omega > \Omega_f$, how many samples should be taken to get a good approximation to $f(t)$? Bring in

$$f_N(t) := \sum_{n=n_t-N}^{n_t+N} f(\frac{n\pi}{\Omega})\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}$$

where

$$n_t := [\Omega t/\pi] \text{ (integer content).}$$

Let, for instance, $0 < t < \pi/\Omega$, $n_t = 0$. Then

$$e_N(t) := f(t) - f_N(t) = \sum_{-\infty}^{-(N+1)} + \sum_{N+1}^{\infty}. \tag{1}$$

Assume also that $\|f\|^2 < \infty$, and note (Ex. 2.16, p. 111)

$$\|f\|^2 = \frac{\pi}{\Omega}\sum_{n=-\infty}^{\infty}|f(\frac{n\pi}{\Omega})|^2. \tag{2}$$

Also, by Schwarz,[3]

$$|\sum_{N+1}^{\infty}|^2 \leq \sum_{N+1}^{\infty}|f(\frac{n\pi}{\Omega})|^2 \cdot \sum_{N+1}^{\infty}|\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}|^2. \tag{3}$$

But

$$\begin{aligned} \sum_{N+1}^{\infty}|\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}|^2 &\leq \frac{1}{\pi^2}\sum_{n=N+1}^{\infty}(1-n)^{-2} \text{ (since } 0 < \Omega t < \pi) \\ &= \frac{1}{\pi^2}\sum_{N}^{\infty} n^{-2} \leq (N-1)^{-1}\pi^{-2}. \end{aligned} \tag{4}$$

Combining (1) - (4), we get finally

$$|e_N(t)| \lesssim (2\Omega/\pi^3)^{1/2}\|f\|N^{-1/2} \qquad (N \gg 1)$$

and this error bound is uniform in t. The bound is quite crude, showing only that the maximum interpolation error, incurred by summing over a finite number $2N+1$ of samples, drops off rather slowly, like $1/\sqrt{N}$, as $N \to \infty$. The interesting thing about the bound is really its uniformity.

[3] In the form $|\Sigma a_n b_n|^2 \leq \Sigma|a_n|^2\Sigma|b_n|^2$.

Comment 3: Approximate signal reconstruction with delay

Notice, finally, that the recipe provided by the Sampling Theorem may be a little unrealistic in some applications, because to recover $f(t)$ you must utilize samples from the "future" (sampling instants $t_n > t$) as well as the "past" ($t_n < t$). This means that the formula (*) (p. 100) can't be evaluated in real time. If you pick N large enough (as in Comment 2 above) to make $|e_N|$ suitably small, then at time t^*, you can reconstruct $f(t^* - \frac{N\pi}{\Omega})$ to good approximation, using samples taken in the interval $t^* - \frac{(2N+1)\pi}{\Omega} \leq t \leq t^*$. But of course in so doing you incur a delay of $N\pi/\Omega$ seconds (say). In pure communications applications, such delay may not matter; but if the communication link is part of a control loop, delay could result in instability.

Comment 4: Nominal bandwidth in practice

It can be shown (Ex. 2.17) that a nonzero signal which is strictly time-limited cannot also be strictly band-limited, i.e. can never have zero frequency content outside a finite frequency interval. On the other hand, a time-limited signal could certainly have *negligibly small* frequency content outside some nominal bandwidth $|\omega| \leq \Omega$: for instance a 40-minute sound recording has negligible content beyond (say) 20 kHz. So, from a practical viewpoint, one could apply the Sampling Theorem to a time-limited signal after estimating a nominal bandwidth B_f, defined by a recipe like:

$$B_f \;:=\; \text{smallest value of } \Omega \text{ such that 99\% signal energy is trapped in } (-\Omega, \Omega):$$

$$\int_{-B_f}^{B_f} |\hat{f}(\omega)|^2 d\omega = 0.99 \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega.$$

So, if T_f (seconds) is the actual time duration, say

$$f(t) \equiv 0 \text{ if } |t| > T_f/2,$$

then you take a total of (very nearly)

$$\frac{T_f}{\pi/B_f} = 2W_f T_f$$

samples, where

$$W_f := B_f/2\pi$$

is the nominal bandwidth of f expressed in Hz. Finally, setting $N_f := W_f T_f$, you have the approximation

$$f(t) \approx \sum_{n=-N_f}^{N_f} f(\frac{n\pi}{B_f}) \frac{\sin(B_f t - n\pi)}{B_f t - n\pi}.$$

In practice, a decision as to the nominal bandwidth of a signal, for sampling purposes, may well depend on the use to which the signal will be put. For instance, a voice signal is usually adequate for telephone recognition if limited to a bandwidth of about 4 kHz. Since the Nyquist rate would then be 8 kHz, a 10 kHz sampling rate would suffice, leaving a ‘guard band’ from 4 to 5 kHz as safety margin. To prevent significant aliasing, the true voice signal (which may contain frequencies well above 5 kHz) would be prefiltered before sampling, with a lowpass filter essentially flat to 4 kHz and cutting off sharply (say to -40 db) at 5 kHz.

Example: Nyquist-Shannon sampling based on nominal bandwidth

We pick a simple, artificial signal for explicit calculation. Let

$$f(t) = e^{-|t|}, \qquad -\infty < t < \infty$$

Suppose t is measured in secs. We have

$$\hat{f}(\omega) = \frac{2}{1+\omega^2}, \qquad -\infty < \omega < \infty$$

Clearly f is not bandlimited. Now

$$\|f\|^2 = \int_{-\infty}^{\infty} e^{-2|t|} dt = 1$$

Let us determine the 99% energy bandwidth Ω_f according to

$$\frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\hat{f}(\omega)|^2 d\omega = 0.99$$

or after some calculation,

$$\frac{2}{\pi}\left(\tan^{-1}\Omega + \frac{\Omega}{1+\Omega^2}\right) = 0.99$$

from which

$$\Omega_f = 3.372251 \;\; \text{rad/sec}$$

So

$$\nu_f = \Omega_f/2\pi = 0.53671 \text{ Hz.}$$

The corresponding Nyquist sampling rate is

$$\nu_{\text{Nyq}} := 2\nu_f = 1.07342 \;\; \text{samples/sec} \tag{1}$$

Let $E(\Omega)$ denote the "sample energy" of f as given by the right side of (2), p. 106:

$$\begin{aligned} E(\Omega) &:= \frac{\pi}{\Omega}\sum_{n=-\infty}^{\infty}\left|f\left(\frac{n\pi}{\Omega}\right)\right|^2 \\ &= \frac{\pi}{\Omega}\sum_{n=-\infty}^{\infty}(e^{-|n|\pi/\Omega})^2 \\ &= (\pi/\Omega)\coth(\pi/\Omega) \end{aligned}$$

This gives

$$E(\Omega_f) = 1.274$$

Note that $E(\Omega_f)$ is the energy of the reconstructed signal obtained by passing the sampled signal through the ideal low-pass filter with bandwidth Ω_f. Since the original signal has energy 0.99 in this frequency range, we see that the reconstructed signal introduces a roughly 28% energy increase due to aliasing. Note also that

$$E(\Omega) \simeq 1 + \frac{1}{3}\left(\frac{\pi}{\Omega}\right)^2$$

for large Ω. Thus $E(\Omega) \to 1$ as $\Omega \to \infty$, as expected.

A more conservative assumed Nyquist rate of 10 samples/sec would correspond to $\Omega = 10\pi$, giving the energy for f in $[-\Omega, \Omega]$ as 0.9999 and the sample energy $E(\Omega) = 1.00333$.

Finally, a nominal time duration T_f based on 99% energy in the time domain is given by

$$\int_{-T_f/2}^{T_f/2}(e^{-|t|})^2 dt = 0.99$$

or

$$T_f = 4.6 \quad \text{sec}$$

With ν_{Nyq} given by (1), this yields a storage requirement of about 7 samples.

To appraise how well the Nyquist-Shannon interpolation formula works for more realistic signals, try the following.

Exercise 2.14: Computational experiment on sampling and interpolation

This exercise is designed to illustrate sampling for an (approximately) band-limited signal (A/D conversion) followed by signal recovery using Nyquist-Shannon interpolation (D/A conversion).

Signal "meander"

Time index integer j: $0 \le j \le T = 1024$.

$$f(j) = a\sin(\frac{j\pi}{17}) + b\sin(\frac{j\pi}{29} + \beta) + c\sin(\frac{j\pi}{53} + \gamma)$$

Preselect

$$\begin{aligned} a, b, c &= \text{random } (0,1) \\ \beta, \gamma &= \text{random } (0,2\pi) \end{aligned}$$

Signal "random walk"
Time index integer j: $0 \leq j \leq T = 1024$.
Generate $f(j)$ as the output of a 3-stage low-pass filter driven by "white noise".

$$\begin{aligned} x_0 &= 0, \quad y_0 = 0, \quad f_0 = 0. \qquad \text{For } j \geq 1, \\ x_j &= 0.96x_{j-1} + 0.04\text{random (-1,1)} \\ y_j &= 0.96y_{j-1} + 0.04x_{j-1} \\ f_j &= 0.96f_{j-1} + 0.5y_{j-1} \end{aligned}$$

Assumed signal bandwidths
Suggestions:

$$W = \frac{1}{256}, \frac{1}{128}, \ldots, \frac{1}{8}, \frac{1}{4}$$

i.e.

$$\tau = \text{'Mesh'} := \text{sampling interval } (= 1/(2W)) = 128, 64, \ldots, 4, 2$$

Nyquist-Shannon interpolation
Let $f_{NS}(j)$ $(j = 0, 1, ..., T)$ be the result from Nyquist-Shannon interpolation when the function $f(j)$ $(j = 0, 1, ..., T)$ is sampled. On setting

$$\begin{aligned} \Omega &:= \pi/\tau, \qquad N := T/\tau \\ f(j) &:= 0 \text{ for } j < 0 \text{ and } j > T \end{aligned}$$

we obtain

$$\begin{aligned} f_{NS}(j) &:= \sum_{n=-\infty}^{\infty} f(n\tau) \frac{\sin(\Omega j - n\pi)}{\Omega j - n\pi} \\ &= \sum_{n=0}^{N} f(n\tau) \frac{\sin[(j/\tau - n)\pi]}{(j/\tau - n)\pi} \end{aligned}$$

The interpolation error function is thus

$$e_{NS}(j) := f(j) - f_{NS}(j), \qquad j = 0, 1, ..., T$$

Interpolation accuracy

To measure the accuracy of interpolation, compute the normalized r.m.s. interpolation error ('deviation')

$$\text{devn} := \sqrt{\frac{\text{mean square (interpolation error)}}{\text{mean square (function } f)}}$$

Procedure (minimal requirement)

A. For each of "meander" and "random walk":

1. Generate your signal for integer j, $0 \leq j \leq T$. Plot.
2. For each of the seven suggested values of W, sample your signal at the corresponding mesh points and generate the interpolated version ($0 \leq j \leq T$) according to
   (i) Nyquist-Shannon interpolation formula
   (ii) Linear interpolation formula (joins successive sample values by a straight line).
3. For each of these fourteen runs, compute devn. Tabulate in the format

Name of signal

W	**devn (Nyquist)**	**devn (Linear)**

4. For each signal, plot the results in 3. as two graphs of devn vs. $\log_2(256{\star}W)$.

B. Conclusions should address at least the following questions.

(i) How does devn. improve with sampling rate ($= 2W$)?

(ii) Do you notice a rather dramatic improvement in devn in going from W^* to $2W^*$ for some tabulated value W^*? Discuss.

(iii) How does devn (Nyquist) compare with devn (Linear)?

(iv) What would be the theoretical bandwidth of the (continuous-time) version of "meander" if it were run for $-\infty < t < \infty$? Would you expect your table to indicate "perfect" performance for some value(s) of W? Justify your answer.

Remarks

1. You may find it handy to normalize your signal amplitude $\max|f_i|$ to unity: generate, then rescale.

2. The whole job takes less than 30 minutes to run on most personal computers.

Exercise 2.15: How to get rich

As a student of Fourier analysis, you wish to make a fortune by inventing a small plug-in unit to replace the compact disk. With current technology, how small a package could you come up with for adequate digital encoding of Beethoven's Ninth Symphony? Assume Nyquist-Shannon sampling and (say) 16 bits storage per sample.

Exercise 2.16: More on Parseval's formula

(i) Give a derivation of the following more general version of Parseval's formula:

$$\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega = 2\pi \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt,$$

where $\bar{(\cdot)}$ denotes complex conjugate.

(ii) Using the result of part (i), show that, for integer $m, n \in (-\infty, \infty)$,

$$\int_{-\infty}^{\infty} \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi} \cdot \frac{\sin(\Omega t - m\pi)}{\Omega t - m\pi} dt = \frac{\pi}{\Omega}\delta_{n,m}$$

(iii) Assuming $\|f\| < \infty$ and f given by (†), p. 100, show that

$$\|f\|^2 = \frac{\pi}{\Omega} \sum_{n=-\infty}^{\infty} |f(\frac{n\pi}{\Omega})|^2.$$

For an application of this formula see Comment 2, p. 106.

Exercise 2.17: 'Strict' time-bandwidth duality

Suppose $f \in PS$ is strictly time-limited, i.e. vanishes everywhere outside some finite subinterval of the real t-axis. Show that if f is also strictly bandlimited (namely $\hat{f}$ vanishes everywhere outside some finite subinterval of the real ω-axis) then f must be almost everywhere zero: $f(t) = 0$ a.e., $-\infty < t < \infty$. Thus a nontrivial strictly time-limited signal cannot be strictly bandlimited. For example a given performance of Beethoven's Ninth is nontrivial and is strictly time-limited (to an interval of length ∼90 minutes); hence it is not strictly bandlimited, and therefore has frequency content at arbitrarily high frequencies.[4] Of course the frequency content above 20 kHz is negligible to the human listener.

For the proof, suppose $f \in PS$ and

$$f(t) = 0 \text{ for } t < 0 \text{ and } t > T$$

Then

$$\hat{f}(\omega) = \int_0^T e^{-i\omega t} f(t)dt$$

[4] Properly speaking, our *Fourier mathematical model* of Beethoven's Ninth has arbitrarily high frequency content.

Since the range of integration is finite, $\hat{f}(\omega)$ is well-defined for all $\omega \in \mathbb{C}$. The integrand $e^{-i\omega t}f(t)$ is an everywhere analytic function of ω for each fixed t, and since $f \in PS$, $[0,T]$ can be partitioned into a finite number of subintervals on each of which $e^{-i\omega t}f(t)$ is jointly continuous in (ω, t). By a standard result in complex analysis (cf. Titchmarsh, *Theory of Functions*, Section 2.83) $\hat{f}(\omega)$ is analytic for all $\omega \in \mathbb{C}$. Thus the Taylor series of $\hat{f}$ calculated at any point ω' converges for all ω. It follows that if $\hat{f}$ and all its derivatives vanish at some point $\omega = \omega_0 \in \mathbb{C}$ then $\hat{f}$ must vanish everywhere. In particular this is true if f is strictly bandlimited. Since $f \in PS$, f can differ from its Fourier integral

$$g(t) := (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega \equiv 0$$

only at a finite set of points in $[0,T]$ (where f has jumps). That is,

$$f(t) = f(t) - g(t) = 0 \text{ a.e.}$$

as claimed.

2.9 The Real Fourier Transform

As we have seen, the complex Fourier transform and inversion integral is a powerful and convenient tool, even in applications where the underlying signals are real valued. Nevertheless, for completeness' sake we shall develop the real Fourier transform, in analogy to the real Fourier series. With the real version, appeal to intuition is perhaps more direct.

Let $f(\cdot) \in PS_1(-\infty, \infty)$, and to start with assume, as usual, that f may be complex-valued. Define the *Fourier cosine transform* of f:

$$\hat{f}_c(\omega) := \int_{-\infty}^{\infty} f(t)\cos\omega t dt$$

and the *Fourier sine transform*

$$\hat{f}_s(\omega) := \int_{-\infty}^{\infty} f(t)\sin\omega t dt$$

The standard complex Fourier transform can now be written

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} [\cos\omega t - i\sin\omega t]f(t)dt \\ &= \hat{f}_c(\omega) - i\hat{f}_s(\omega).\end{aligned}$$

Notice that $\hat{f}_c$ is an even function of ω, and $\hat{f}_s$ is odd. Therefore

$$\lim_{\beta\to\infty} \int_{-\beta}^{\beta} [\hat{f}_c(\omega)\sin\omega t - \hat{f}_s(\omega)\cos\omega t]d\omega = 0.$$

Using this fact we get, from the standard inversion formula,

$$\begin{aligned}
f(t) &= \frac{1}{2\pi} \lim_{\beta\to\infty} \int_{-\beta}^{\beta} e^{i\omega t} \hat{f}(\omega) d\omega \\
&= \frac{1}{2\pi} \lim_{\beta\to\infty} \int_{-\beta}^{\beta} (\cos\omega t + i\sin\omega t) \cdot [\hat{f}_c(\omega) - i\hat{f}_s(\omega)] d\omega \\
&= \frac{1}{2\pi} \lim_{\beta\to\infty} \int_{-\beta}^{\beta} [\hat{f}_c(\omega)\cos\omega t + \hat{f}_s(\omega)\sin\omega t] d\omega \\
&= \frac{1}{\pi} \lim_{\beta\to\infty} \int_{0}^{\beta} [\hat{f}_c(\omega)\cos\omega t + \hat{f}_s(\omega)\sin\omega t] d\omega \\
&= \frac{1}{\pi} \int_{0}^{\infty} [\hat{f}_c(\omega)\cos\omega t + \hat{f}_s(\omega)\sin\omega t] d\omega
\end{aligned}$$

The last expression is the integral analog of a standard Fourier series representation in sines and cosines.

Assume now that $f(\cdot)$ is real-valued. Then $\hat{f}_c(\cdot)$, $\hat{f}_s(\cdot)$ are real-valued as well. Therefore

$$\begin{aligned}
\mathrm{Re}\hat{f}(\omega) &= \hat{f}_c(\omega), \\
\mathrm{Im}\hat{f}(\omega) &= -\hat{f}_s(\omega)
\end{aligned}$$

Writing $\hat{f}$ in polar form

$$\hat{f}(\omega) = F(\omega)e^{-i\phi(\omega)}$$

where $F(\cdot)$ and $\phi(\cdot)$ are real-valued, we get

$$\begin{aligned}
\hat{f}_c(\omega) &= F(\omega)\cos\phi(\omega) \\
\hat{f}_s(\omega) &= F(\omega)\sin\phi(\omega)
\end{aligned}$$

Plugging these expressions into the inversion integral derived above, we get finally

$$f(t) = \frac{1}{\pi} \int_0^{\infty} F(\omega)[\cos\phi(\omega)\cos\omega t + \sin\phi(\omega)\sin\omega t] d\omega$$

or

$$f(t) = \frac{1}{\pi} \int_0^{\infty} F(\omega)\cos[\omega t - \phi(\omega)] d\omega, \text{ a.e. } -\infty < t < \infty \qquad (*)$$

A spiritual interpretation of (*) is the following. Imagine you have a celestial harp. This harp has a string corresponding to every real number $\omega \geq 0$. You also have two copies of the time-line $-\infty < t < \infty$, for this world and the next one. In this world you run $f(t)$ for $-\infty < t < \infty$, and compute $F(\omega)$ and $\phi(\omega)$. In the next world, at $t = -\infty$, you pluck the harp to set each string (ω) vibrating (in its fundamental mode only!) with amplitude $F(\omega)/\pi$ and phase $-\phi(\omega)$. Now sit back and listen to the harp. It plays $f(t)$, $-\infty < t < \infty$. For example suppose that, in this world,

$$f(t) = \begin{cases} 0, & t < 0 \\ \text{Beethoven's Ninth}, & 0 \le t \le T \\ 0, & t > T \end{cases}$$

In the next world (having started the harp at $t = -\infty$) you hear *nothing*, except Beethoven's Ninth in $0 \le t \le T$. Quite amazing.

Chapter 3

Laplace Transform

3.1 Introduction

Very often in systems analysis one is concerned with signals which 'start' at some fixed, finite time t_0 (rather than at $-\infty$) and which can be thought of as defined for $t_0 \leq t < \infty$. An example is the output of an electrical network in response to a sinusoidal voltage applied at the input from time $t = 0$, on. Knowing the initial values of the network charges and currents, we are in a position to calculate the response for $0 \leq t < \infty$. If the network is linear, and the network element values are constant in time, this calculation is readily done by use of Laplace transforms. These are tailored precisely to functions defined on $[0, \infty)$, and which need not be 'small' (in the usual sense) as $t \to \infty$. In summary, Laplace transforms are especially useful for analysis of 'transients', whereas Fourier transforms are mainly employed for signals defined on the whole t-axis. The distinction is not rigid, however; and which transform to apply may well be only a matter of convenience.

Since 'transients' in *unstable* linear systems may grow exponentially fast, we must not rule out functions like $e^t (t \geq 0)$, as we did in discussing Fourier transforms.

Definition

Let $f = f(t)$ be a complex-valued function defined for $0 \leq t < \infty$. f is of *exponential order* on $[0, \infty)$, written $f \in \mathrm{EXP}$, if there are real constants K and α such that

$$|f(t)| \leq K e^{\alpha t}, \quad 0 \leq t < \infty.$$

Thus, on $0 \leq t < \infty$, the functions

(*) $1, t, t^2, \ldots; \sin t, \cos t; e^{5t}, t^{17} e^{5t}, \ldots$ are of exponential order; but e^{t^2} is not.

Exercise 3.1: Find suitable K and α (depending on f) for each of the functions f in the list (*). Prove that no suitable K, α could exist for e^{t^2}.

3.2 Laplace Transform: Definition and Formal Properties

Write PS for the class of piecewise smooth functions on $[0,\infty)$. We shall deal mainly with functions that are both piecewise smooth and of exponential order.

Definition

Let $f \in \mathrm{PS} \cap \mathrm{EXP}$. The *Laplace transform* of f is

$$\hat{f}(s) := \int_0^\infty e^{-st} f(t)dt,$$

where s is a complex variable ($s \in \mathbb{C}$), and the integral is computed with Re s sufficiently large.

Thus if $|f(t)| \le Ke^{\alpha t}$, $(t \ge 0)$, you would take (in the integral) any s such that Re $s > \alpha$; then the integral is sure to converge. So, every function in PS$\cap$EXP is Laplace-transformable.

Examples

1. $f(t) := e^{\beta t}$, $\quad \hat{f}(s) = \frac{1}{s-\beta}$, $\quad$ Re $s >$ Re β

2. $f_n(t) := t^n$, $\quad n = 0, 1, 2, \ldots$

   $$\hat{f}_n(s) = \int_0^\infty t^n e^{-st}dt, \quad \mathrm{Re}\ s > 0$$

   $$= -\frac{1}{s} \int_0^\infty \frac{d}{dt}(e^{-st})t^n dt$$

   $$= -\frac{1}{s}(e^{-st}t^n\big|_0^\infty - n\int_0^\infty e^{-st}t^{n-1}dt)$$

   $= (n/s)\hat{f}_{n-1}(s)$.

   Since $\hat{f}_0(s) = 1/s$ we get from this,

   $$\hat{f}_n(s) = n!/s^{n+1}, \quad \mathrm{Re}\ s > 0; \quad n = 0, 1, 2, ...$$

3. $f_n(t) := t^n e^{\beta t}$, $\quad \hat{f}_n(s) = n!(s-\beta)^{-n-1}$, $\quad$ Re $s >$ Re β; $n = 0, 1, 2, ...$

4. $f(t) := \cos \omega t$

   $$\hat{f}(s) = \int_0^\infty e^{-st}\frac{1}{2}(e^{i\omega t} + e^{-i\omega t})dt$$

   $$= \frac{s}{s^2+\omega^2}, \quad \mathrm{Re}\ s > 0\,.$$

5. $f(t) := \sin \omega t\,;\quad \hat{f}(s) = \omega/(s^2+\omega^2)\,,\quad \mathrm{Re}\, s > 0$

Laplace delay operator

Suppose $f(\cdot)$ is delayed by τ:

$$g(t) \;:=\; f(t-\tau)\,,\quad \tau > 0 \,\text{and fixed.}$$

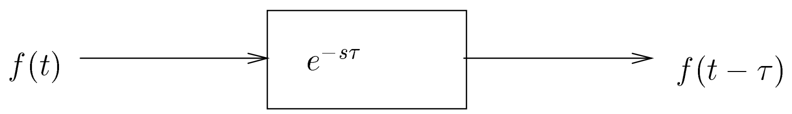
Then

$$\hat{g}(s) \;=\; \int_0^\infty e^{-st} f(t-\tau)dt$$

where we define $f(t') = 0$ for $t' < 0$. So

$$\begin{aligned} \hat{g}(s) \;&=\; e^{-s\tau} \int_0^\infty e^{-s(t-\tau)} f(t-\tau)dt \\ &=\; e^{-s\tau} \int_0^\infty e^{-st'} f(t')dt' \\ &=\; e^{-s\tau} \hat{f}(s)\,. \end{aligned}$$

In systems analysis one has the picture:



Delay element

Linearity of Laplace transform

An alternative notation for Laplace transform is $\mathcal{L} : \mathcal{L}(f) := \hat{f}$. Similarly one may write $f = \mathcal{L}^{-1}\{\hat{f}\}$.

We note that $\mathcal{L}$ is *linear*: if $f_1, f_2 \in \mathrm{PS} \cap \mathrm{EXP}$ and if $c_1, c_2 \in \mathbb{C}$, then $c_1 f_1 + c_2 f_2 \in \mathrm{PS} \cap \mathrm{EXP}$, and

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)\,.$$

Laplace transform of convolution

For $f, g \in \mathrm{PS} \cap \mathrm{EXP}$, define the *convolution*

$$(f * g)(t) := \int_0^t f(t-\tau) g(\tau) d\tau\,,\quad t \geq 0\,.$$

Proposition

If $f, g \in \mathrm{PS} \cap \mathrm{EXP}$ then $f * g \in \mathrm{PS} \cap \mathrm{EXP}$ and

$$\widehat{f * g}(s) \;=\; \hat{f}(s)\hat{g}(s)$$

Proof:

Let $|f(t)| \le Ae^{\alpha t}$, $|g(t)| \le Be^{\beta t}$, $t \ge 0$, and let $\gamma := \max(\alpha, \beta, 0)$. Writing

$$h(t) := (f * g)(t)$$

we have

$$\begin{aligned}
|h(t)| &\le \int_0^t Ae^{\alpha(t-\tau)} Be^{\beta\tau} d\tau \\
&= ABe^{\alpha t}\,\frac{1 - e^{-(\alpha-\beta)t}}{\alpha - \beta} \\
&= AB\,\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \\
&\le ABte^{\gamma t} \quad [\text{Exercise}: \quad \text{use} \quad e^x - 1 \,\le\, xe^x, \quad -\infty < x < \infty\,.] \\
&< Ce^{\delta t}
\end{aligned}$$

for any $\delta > \gamma$ and suitable C.

We have shown $h \in$ EXP. To show $h \in$ PS, note that for $t \ge 0$, $\varepsilon > 0$,

$$\begin{aligned}
h(t+\varepsilon) - h(t) &= \int_0^{t+\varepsilon} f(t+\varepsilon-\tau)g(\tau)d\tau - \int_0^t f(t-\tau)g(\tau)d\tau \\
&= \int_0^t [f(t+\varepsilon-\tau) - f(t-\tau)]g(\tau)d\tau + \int_t^{t+\varepsilon} f(t-\tau+\varepsilon)g(\tau)d\tau
\end{aligned}$$

From this it is clear that $h(t)$ is continuous for $t \ge 0$. Now *formally*, h has derivative

$$h'(t) \;=\; \int_0^t f'(t-\tau)g(\tau)d\tau \;+\; f(0)g(t)\,,$$

and this 'formula' suggests that $h'(\cdot)$ is continuous except for simple jumps at points t where either f or g undergoes a jump. A detailed analysis (omitted) verifies this, and shows that

$$\begin{aligned}
h'(t+0) \;-\; h'(t-0) &= g(0+)[f(t+0) - f(t-0)] \\
&\quad + f(0+)[g(t+0) - g(t-0)]\,.
\end{aligned}$$

So $h \in \mathrm{PS} \cap \mathrm{EXP}$. Finally, for Re s sufficiently large,

$$\begin{aligned}
\hat{h}(s) &= \int_0^\infty e^{-st}\,(\int_0^t f(t-\tau)g(\tau)d\tau)dt \\
&= \int_0^\infty g(\tau)\,(\int_\tau^\infty e^{-st} f(t-\tau)dt)d\tau \quad (\text{by Fubinito})
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-s\tau} g(\tau)\,(\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau)dt)d\tau \\
&= \int_0^\infty e^{-s\tau} g(\tau)\,(\int_0^\infty e^{-st'} f(t')dt')d\tau \\
&= (\int_0^\infty e^{-s\tau} g(\tau)d\tau)(\int_0^\infty e^{-st'} f(t')dt') \\
&= \hat{g}(s)\hat{f}(s)
\end{aligned}
$$

Q.E.D.

Laplace transform of a δ-function

Write δ_τ for the δ-function (unit impulse function) concentrated at $t = \tau > 0$. Then, formally,

$$
\begin{aligned}
\hat{\delta}_\tau(s) &= \int_0^\infty e^{-st}\delta_\tau(t)dt \\
&= \int_0^\infty e^{-st}\delta(t-\tau)dt \\
&= e^{-s\tau} \qquad (*)
\end{aligned}
$$

We see that $\hat{\delta}_\tau(s)$ is nothing other than the Laplace delay operator for delay $\tau \geq 0$. In particular, letting $\tau \to 0$, we get

$$\hat{\delta}_0(s) = 1, \text{ namely } \mathcal{L}^{-1}\{1\} = \delta(t)$$

The product rule for the transform of a convolution applies to the delay operator as well. Thus, with $\tau \geq 0$,

$$
\begin{aligned}
(\delta_\tau * f)(t) &= \int_0^t \delta_\tau(t-u)f(u)du \\
&= \int_0^t \delta(t-u-\tau)f(u)du \\
&= \int_0^t \delta(u-t+\tau)f(u)du \\
&\quad \text{since } \delta(-v) = \delta(v) \\
&= \begin{cases} f(t-\tau), & 0 \leq t-\tau \leq t \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

$$= \begin{cases} f(t-\tau), & t \geq \tau \\ 0, & \text{otherwise} \end{cases}$$

$$= f_\tau(t) \text{ i.e. } f \text{ delayed by } \tau$$

Applying the product rule on the left side, and the rule for delay on the right, we obtain

$$\hat{\delta}_\tau(s)\hat{f}(s) = e^{-s\tau}\hat{f}(s)$$

in agreement with (*).

Finally, it is well to recall that δ_τ is *not*, strictly speaking, a (real or complex-valued) *function*[1], and certainly does *not* belong to the function classes PS or EXP. However, because of its usefulness in system and signal analysis, we shall treat it separately in its own right.

Laplace transform of derivatives

Finally we shall develop some useful formulas involving derivatives of $f(\cdot)$. For this, let $f^{(m)}(t) := d^m f(t)/dt^m \in \mathrm{PS} \cap \mathrm{EXP}$. Then

$$f^{(m-1)}(t) \;=\; f^{(m-1)}(0) \,+\, \int_0^t f^{(m)}(\tau)d\tau$$

so that

$$\begin{aligned} |f^{(m-1)}(t)| \;&\leq\; A \,+\, \int_0^t Be^{\beta\tau}d\tau \qquad (\text{some}\, A, B, \beta) \\ &=\; A \,+\, B(e^{\beta t} - 1)/\beta \\ &\leq\; \begin{cases} A + Bte^{\beta t}, & \beta \geq 0 \\ A + Bt, & \beta \leq 0 \end{cases} \\ &<\; Ce^{\delta t} \quad (\text{some}\, C, \delta) \end{aligned}$$

So $f^{(m-1)} \in \mathrm{PS} \cap \mathrm{EXP}$. By iteration downward,

$$f^{(m-2)}, \ldots, f^{(1)}, \quad f \in \mathrm{PS} \cap \mathrm{EXP}\,.$$

Now let

$$\begin{aligned} \gamma_j \;&:=\; f^{(j)}(0+) \\ \hat{f}_j(s) \;&:=\; \mathcal{L}\{f^{(j)}(t)\}\,. \end{aligned}$$

Then

$$\begin{aligned} \hat{f}_j(s) \;&=\; \int_0^\infty e^{-st}\frac{d}{dt}(f^{(j-1)}(t))dt \\ &=\; e^{-st}f^{(j-1)}(t)\big|_0^\infty \,+\, s\int_0^\infty e^{-st}f^{(j-1)}(t)dt \end{aligned}$$

[1] It's called a *distribution*.

so that

$$\hat{f}_j(s) = s\hat{f}_{j-1}(s) - \gamma_{j-1}\,.$$

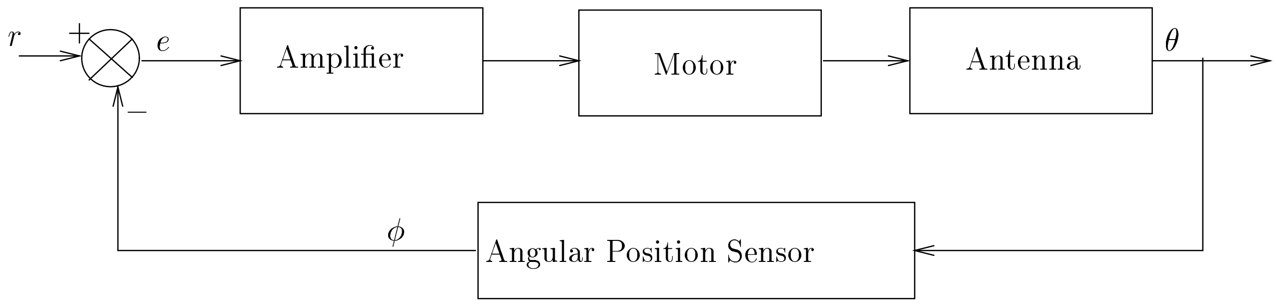
Setting $j = 1, 2, \ldots, m$ in turn, we get

$$\begin{aligned}
\hat{f}_1(s) &= s\hat{f}(s) - \gamma_0 \\
\hat{f}_2(s) &= s^2\hat{f}(s) - s\gamma_0 - \gamma_1 \\
&\vdots \\
\hat{f}_m(s) &= s^m\hat{f}(s) - s^{m-1}\gamma_0 - s^{m-2}\gamma_1 - \ldots - \gamma_{m-1}
\end{aligned}$$

The following example illustrates how the foregoing rules may be applied.

3.3 Example: Scanning Radar Antenna

Consider a radar antenna with scanning angle controlled as shown in the block diagram below.



Here

$$\begin{aligned}
r &= r(t) = \text{angular reference input [i.e. desired angle of pointing] (rad.)} \\
\theta &= \theta(t) = \text{actual angle of pointing} \\
\phi &= \phi(t) = \text{measured angle of pointing} \\
e &= r(t) - \phi(t) = \text{measured pointing error}
\end{aligned}$$

The amplifier is assumed distortionless, with gain $k > 0$; and after appropriate choice of time-scale, the forward-path dynamics are assumed to be given by

$$\frac{d^2\theta}{dt^2} + 3\,\frac{d\theta}{dt} = ke(t)\,.$$

The angle sensor is assumed to have input-output representation

$$\phi(t) = \alpha\int_0^t e^{-\alpha\tau}\theta(t-\tau)d\tau \quad (\alpha > 0)\,.$$

For definiteness suppose

$$k = 2.5, \quad \alpha = 4, \quad r(t) = \sin t.$$

Then

$$\theta''(t) + 3\theta'(t) = 2.5 \sin t - 10 \int_0^t e^{-4\tau} \theta(t-\tau) d\tau.$$

Setting $\theta(0) = \theta_0$, $\theta'(0) = \theta_1$ and taking Laplace transforms, you get by inspection,

$$s^2\hat{\theta}(s) - s\theta_0 - \theta_1 + 3[s\hat{\theta}(s) - \theta_0] = \frac{2.5}{1+s^2} - \frac{10}{s+4}\hat{\theta}(s).$$

After some arithmetic,

$$\hat{\theta}(s) \;=\; \frac{s+4}{s^3 + 7s^2 + 12s + 10} \left[(s+3)\theta_0 + \theta_1 + \frac{2.5}{s^2+1} \right]$$

Temporarily assume $\theta_0 = \theta_1 = 0$ (antenna initially at rest). Then

$$\hat{\theta}(s) \;=\; \frac{2.5(s+4)}{(s^3 + 7s^2 + 12s + 10)(s^2+1)}$$

Now

$$s^3 + 7s^2 + 12s + 10 \;=\; (s+5)(s^2 + 2s + 2)$$

Expanding $\hat{\theta}(s)$ in partial fractions you get

$$\hat{\theta}(s) \;=\; \frac{A}{s+5} + \frac{Bs + C}{s^2+1} + \frac{Ds + E}{s^2 + 2s + 2}$$

for suitable numbers A, B, C, D, E. Noting that $s^2 + 2s + 2 = (s+1)^2 + 1$ you now get, by inspection,

$$\theta(t) \;=\; Ae^{-5t} + B\cos t + C\sin t + De^{-t}(\cos t - \sin t) + Ee^{-t}\sin t\,.$$

As $t \to \infty$,

$$\theta(t) \;\simeq\; B\cos t + C\sin t \quad (\text{ steady state \ solution}).$$

It turns out (verify) that

$$B \;=\; -0.7885\,, \quad C \;=\; 0.4423$$

Thus

$$\theta(t) \;\simeq\; 0.9041\,\sin(t - 1.0596) \quad (\text{rad.})$$

in the stationary scanning mode. The actual pointing angle has amplitude 0.9041 that of the reference angle and lags the latter by about 60.7^o of phase.

If θ_0 and $\theta_1 \neq 0$, you'd get a contribution

$$\hat{\theta}_{\text{initial}}(s) \;=\; [(s+3)\theta_0 + \theta_1](s+4)/(s+5)(s^2 + 2s + 2),$$

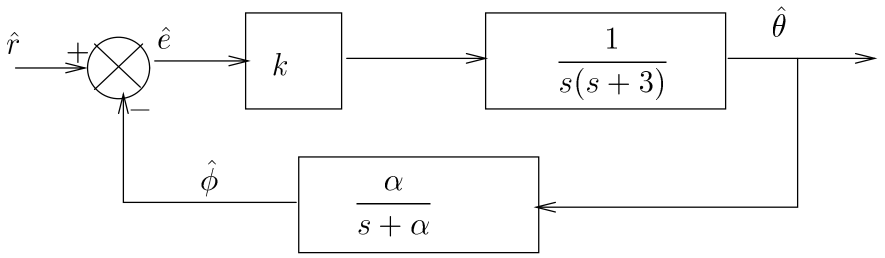
giving

$$\theta_{\text{initial}}(t) \;=\; A'e^{-5t} + B'e^{-t}\cos t + C'e^{-t}\sin t$$

("transient solution" arising from initial conditions)

$$\to 0 \text{ as } t \to \infty.$$

In commonly used *block diagram* notation, the system would be represented as shown below.



From this, by inspection (and assuming $\theta_0 = \theta_1 = \phi_0 = 0$),

$$s(s+3)\hat{\theta} \;=\; k\hat{e} \;=\; k\left(\hat{r} \;-\; \frac{\alpha\hat{\theta}}{s+\alpha}\right),$$

giving

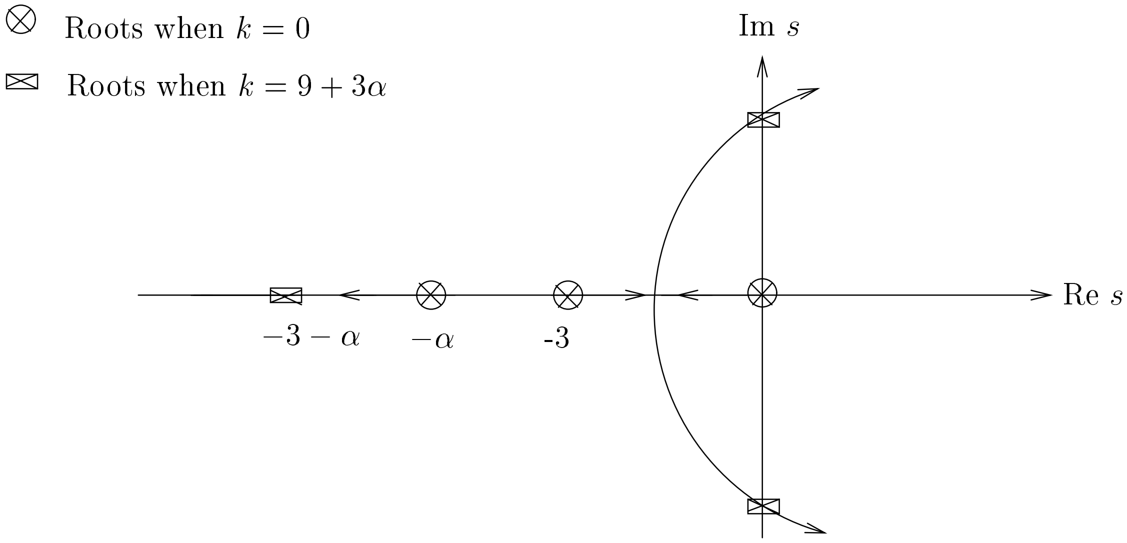
$$\frac{\hat{\theta}(s)}{\hat{r}(s)} \;=\; \frac{k(s+\alpha)}{s^3+(3+\alpha)s^2+3\alpha s+\alpha k}$$

The system will be *stable*, in the sense that the response to a bounded input $r(\cdot)$ will be bounded too, if and only if *the roots of the denominator polynomial all lie in the open left-half complex s-plane* (Re $s < 0$). It turns out[2] that this is true if and only if

$$0 < k < 9 + 3\alpha\,.$$

This means that the gain k must not be too high. Its upper limit $9 + 3\alpha$ is relatively low if the angle sensor is 'slow' ($\alpha > 0$ small), but is relatively high if the angle sensor is 'fast' ($\alpha \gg 1$). A 'root locus', showing how the roots of $s^3 + (3+\alpha)s^2 + 3\alpha s + \alpha k$ depend on k for fixed $\alpha > 0$, is sketched below (for the case $\alpha > 3$).

[2] $s^3 + a_1 s^2 + a_2 s + a_3$ (with a_1, a_2, a_3 real) has all roots in Re $s < 0$ iff $a_1 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 > 0$.



When $k = 9 + 3\alpha$, the roots are $s = \{\pm\sqrt{3\alpha}i, -3-\alpha\}$. Note that for $k > 9 + 3\alpha$, two roots have moved into the right half s-plane, and the system response 'blows up' exponentially with t.

Intuitively our result says that for stability the loop gain should not be made too high, i.e. the controller should be conservative, as long as its informational input (the measured error $r - \phi$) is a lagged version of the true error $(r - \theta)$; here the amount of lag is inversely proportional to α.

A systematic treatment of stability will be given in Ch. 4.

Exercise 3.2: Consider a simplified version of the Scanning Radar Antenna (p. 122) in which the position sensor is ideal, namely

$$\phi(t) \equiv \theta(t) .$$

(i) Calculate $\hat{\theta}(s)/\hat{r}(s)$ for general k, expressing your result as a ratio of polynomials in s [cf. above].

(ii) Calculate the roots of the denominator polynomial for the gain values:

$$k = 0,\ 1.25,\ 2.25,\ 4.5,\ 27.25 .$$

(iii) Accurately compute and plot the 'root locus' [as above]. For what range of k is the system stable?

(iv) Assume that $\theta_0 = \theta_1 = 0$, and that

$$r(t) = 1, \quad t \geq 0 .$$

Accurately compute and plot the response $\theta(t)$ for each k in (ii).

Exercise 3.3: Tracking a UFO

For the Scanning Radar Antenna suppose a UFO appears at reference angle given by

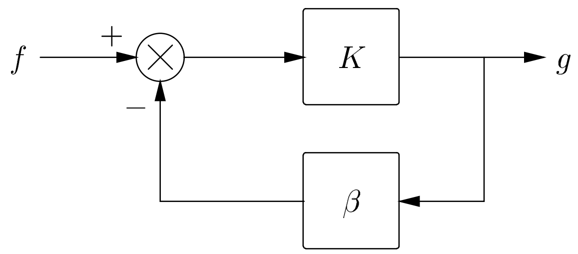
$$r(t) = 1 - t\,, \quad t \geq 0\ .$$

Assuming k and α are such that the system is stable, and that $\theta_0 = \theta_1 = 0$, obtain an expression for $\theta(t)$ when t is large (i.e. decaying transients are negligible). What does your expression reduce to when the angle sensor is perfect [i.e. $\alpha \to \infty$]? In this case, what is then the effect of varying the gain k?

HINT: From $\hat{\theta}(s)/\hat{r}(s)$ (p. 124) calculate $\hat{e}_{true}(s) = \hat{r}(s) - \hat{\theta}(s)$, and then $\lim_{t\to\infty} e_{true}(t)$.

3.4 Example: Black's Feedback Amplifier

In 1927 at the Bell Telephone Laboratories, Harold Black discovered that by use of negative feedback, high-gain amplifiers could be built that were stable and almost distortion-free. The feedback amplifier was a breakthrough in long-distance telephone communication.



A simple 'd.c.' model is shown above. A standard amplifier of gain K is modified by subtracting from the input signal f a fraction β of the output signal g. Typically K is large (say $K \sim 200$) and β is small ($\beta \sim 0.1$). We have

$$g = K(f - \beta g)$$

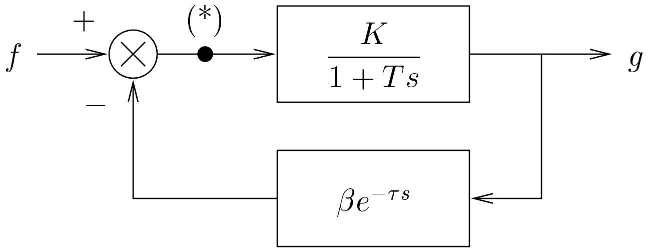
which yields the closed-loop gain

$$g/f = K_{CL} = \frac{K}{1 + \beta K}$$

where βK is the gain around the loop ('loop gain'), say ~ 20. Note that, if K is large, $K_{CL} \simeq 1/\beta$; since β can be made nearly constant in time, so can K_{CL}. Also, the relative sensitivity of the closed-loop gain K_{CL}, when the open-loop gain K undergoes a random variation dK, is diminished by approximately the loop gain factor:

$$\begin{aligned} \frac{dK_{CL}}{K_{CL}} &= \frac{1+\beta K}{K}\frac{d}{dK}\left(\frac{K}{1+\beta K}\right) dK \\ &= \frac{1}{1+\beta K}\frac{dK}{K} \end{aligned}$$

The equations suggest that, for fixed β, K should be made as large as possible. Unfortunately, K is limited by the onset of dynamic instability (the occupational hazard of feedback). To see why, let us refine the static model by assuming a frequency-dependent gain $K/(1+Ts)$ in the forward path and a small delay τ in the feedback path; our (highly idealized) dynamic model is shown below.



While the 'd.c.' behavior (at $s=0$) is unchanged, we see that at frequency ω ($s=i\omega$) the phase delay around the loop (from point (*) to (*), say) is

$$\varphi(\omega) = \tan^{-1}(\omega T) + \omega\tau + \pi \text{ rad.}, \; 0 \leq \omega < \infty$$

Instability due to 'positive feedback' will set in if the loop amplitude response

$$|\beta K e^{-i\omega\tau}/(1+i\omega T)|$$

exceeds unity when $\varphi(\omega)$ reaches 2π.

In more detail, we have

$$\hat{g}(s)/\hat{f}(s) = \hat{h}(s) = \frac{K}{1+Ts+\beta K e^{-s\tau}}$$

and look for the least value of ω for which the system characteristic function

$$p(s) := 1 + Ts + \beta K e^{-s\tau}$$

vanishes, with $s = i\omega$; at this frequency the dominant poles of $\hat{h}(s)$ first reach the unstable region Re $s \geq 0$. Now

$$p(i\omega) = 1 + Ti\omega + \beta K e^{-i\omega\tau} = 0$$

implies

$$\begin{aligned} 1 + \beta K \cos(\omega\tau) &= 0 \\ \omega T - \beta K \sin(\omega\tau) &= 0 \end{aligned}$$

Normalizing by $u := \omega\tau$, $\alpha := T/\tau$, $\gamma = \beta K$, we get

$$\begin{aligned} 1 + \gamma \cos u &= 0 \\ \alpha u - \gamma \sin u &= 0 \end{aligned}$$

which yields $\tan u = -\alpha u$. From the graphs of $y = \tan u$ and $y = -\alpha u$ for $u \geq 0$ it is clear that the least (or 'critical') root u_c satisfies $\pi/2 < u_c < \pi$ with $u_c - \frac{\pi}{2}$ small if α is large, i.e. the feedback delay is relatively small. Under that assumption, we find approximately

$$u_c \simeq \frac{\pi}{2} + \frac{2}{\alpha\pi}, \qquad \gamma_c \simeq \pi\alpha/2$$

Thus the stability range for βK is

$$0 < \beta K \lesssim (\pi/2)(T/\tau)$$

and the critical frequency of unstable oscillation is

$$\omega_c \simeq \pi/2\tau$$

Finally, assuming the stable case, we may calculate the steady-state frequency response,[3] normalized to 1 at $\omega = 0$:

$$\begin{aligned} A(u) &:= |\hat{h}(i\omega)/\hat{h}(0)| \qquad (\omega\tau = u) \\ &= \frac{1+\gamma}{[(1+\gamma\cos u)^2 + (\alpha u - \gamma \sin u)^2]^{1/2}} \end{aligned}$$

For instance at points $u = 2n\pi$,

$$\begin{aligned} A(u) &= \left[1 + \left(\frac{\alpha u}{1+\gamma}\right)^2\right]^{-\frac{1}{2}} \\ &\simeq (1+\gamma)/\alpha u, \qquad n \to \infty \end{aligned}$$

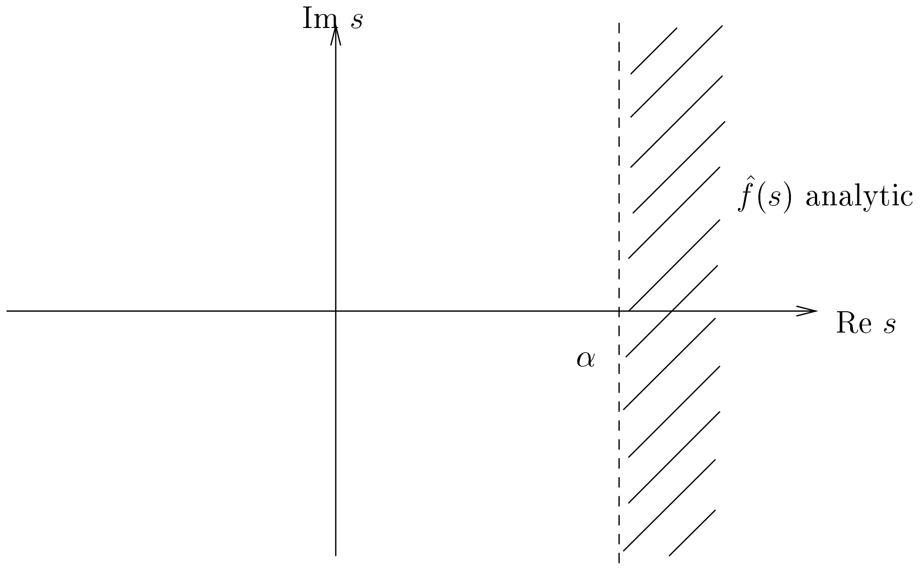
3.5 Analyticity of the Laplace Transform

To develop the inversion theory we shall need the following result, which can be obtained from complex function theory.

THEOREM

Let $f \in \mathrm{PS} \cap \mathrm{EXP}[0,\infty)$. If $|f(t)| \leq Ke^{\alpha t}$ $(t \geq 0)$, then $\hat{f}(s) := \mathcal{L}\{f\}$ is an analytic function of the complex variable s, in the open half-plane $s : \mathrm{Re}\ s > \alpha$.

[3]See Sect. 4.3, where this concept is introduced.



Remark: We could certainly have $\alpha \le 0$ in some cases: e.g. if $f(t) = te^{-5t}$ take $\alpha > -5$; if $f(t) = e^{-t^2}$ you can take α arbitrarily large and negative: $\hat{f}(s)$ is analytic in the entire s-plane. Essentially, the theorem depends on the facts that $e^{-st}f(t)$ is analytic in s for every fixed t, and the Laplace integral converges uniformly with respect to s in every closed half-plane Re $s \ge \alpha + \epsilon$ ($\epsilon > 0$).

Example

If $f(t) = t^2e^{2t}$ then $\hat{f}(s) = 2(s-2)^{-3}$ is analytic for Re $s > 2$. Note the pole on the axis of convergence, at $s = 2$.

Extension of Laplace transform by analytic continuation

With $f \in \text{PS} \cap \text{EXP}$, $\hat{f}(s)$ is initially defined for Re $s > \alpha$ for some α. But usually the domain of definition of $\hat{f}$ can be extended to the whole s-plane (apart from points s where $\hat{f}$ may have singularities), and this extension can be done in just one natural way. Thus in the example above, we *define* the Laplace transform of t^2e^{2t} to be $2(s-2)^{-3}$ at *all* points $s \ne 2$ in the complex s-plane. This definition is justified by the fact that the function

$$\phi(s) := 2(s-2)^{-3}$$

is the *unique* function which is analytic in the plane with $s = 2$ deleted, and coincides with $\hat{f}(s)$ on its original domain of definition Re $s > 2$. The extension $\phi(\cdot)$ is called the *analytic continuation* of $\hat{f}(\cdot)$ to $\mathbb{C} - \{2\}$. From now on we shall always mean by $\hat{f}(s)$ the analytic continuation [of the defining integral $\int_0^\infty e^{-st}f(t)dt$] to the whole plane [with suitable singular points, branch cuts etc. deleted].

As another example,

$$f(t) = 2\sqrt{t/\pi} \Rightarrow \hat{f}(s) = s^{-3/2},$$

defined initially for Re $s > 0$, and by analytic continuation to the whole s-plane with branch cut coinciding (say) with the nonpositive real axis.

In the light of this discussion we infer from the theorem above, the

Corollary

If $f \in \mathrm{PS} \cap \mathrm{EXP}[0, \infty)$ and $|f(t)| \leq K e^{\alpha t}$ $(t \geq 0)$ then all the singularities of $\hat{f}(s)$ (if any) occur *on or to the left of* the vertical line $\mathrm{Re}\, s = \alpha$.

3.6 Laplace Inversion Integral

We can obtain the Laplace inversion integral quickly from what we know about Fourier inversion. Let $g(\cdot) \in \mathrm{PS}_1(-\infty, \infty)$. If g has a jump at $t = t^*$ then 'standardize' (if necessary) $g(t^*)$ as

$$g(t^*) \;:=\; \frac{1}{2}\left[g(t^* - 0) \;+\; g(t^* + 0)\right]. \tag{1}$$

By the Fourier Integral Representation Theorem,

$$g(t) \;=\; \frac{1}{2\pi} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} e^{i\omega t} \int_{-\infty}^{\infty} e^{-i\omega\tau} g(\tau) d\tau d\omega\,, \quad -\infty < t < \infty\,. \tag{2}$$

Now assume

$$f(t) \;=\; 0\,, \;\; t < 0\,; \;\; f \in \mathrm{PS} \cap \mathrm{EXP}\,, \;\; t \geq 0 \tag{A.1}$$

with f standardized at jump points.

For some K and α, $|f(t)| \leq K e^{\alpha t}$, $t \geq 0$. Let

$$\sigma > \alpha \tag{A.2}$$

Then

$$|e^{-\sigma t} f(t)| \;\leq\; K e^{-(\sigma - \alpha) t}\,, \;\; t \geq 0$$

so

$$g(t) \;:=\; e^{-\sigma t} f(t) \in \mathrm{PS}_1(-\infty, \infty) \tag{3}$$

and g satisfies (1). By (1)-(3) we now have

$$e^{-\sigma t} f(t) \;=\; \frac{1}{2\pi} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} e^{i\omega t} \int_{-\infty}^{\infty} e^{-i\omega\tau} e^{-\sigma\tau} f(\tau) d\tau d\omega \tag{4}$$

Write $s := \sigma + i\omega$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-i\omega\tau} e^{-\sigma\tau} f(\tau) d\tau \;&=\; \int_{-\infty}^{\infty} e^{-s\tau} f(\tau) d\tau \\ &=\; \int_{0}^{\infty} e^{-s\tau} f(\tau) d\tau \\ &=\; \hat{f}(s). \end{aligned} \tag{5}$$

By (4) and (5),

$$f(t) \;=\; \frac{1}{2\pi} \lim_{\beta\to\infty} \int_{-\beta}^{\beta} e^{(\sigma+i\omega)t} \hat{f}(s) d\omega$$

Since $s = \sigma + i\omega$ and $-\beta \le \omega \le \beta$ in the integration, we have equivalently $ds = id\omega$, and s-plane integration on the line segment $[\sigma - i\beta, \sigma + i\beta]$. So finally

$$f(t) \;=\; \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\sigma-i\beta}^{\sigma+i\beta} e^{st} \hat{f}(s) ds\,. \tag{$*$}$$

Eq. (*) *is the representation of* $f(\cdot)$ *by its Laplace inversion integral.* The representation is valid under assumptions (A.1) and (A.2) above, and with the convention already noted about jump points.

In practice we are "given" a complex function $\phi(s)$, or arrive at it in the attempt to solve some problem, and we may not know a priori that ϕ really is the Laplace transform of some $f \in \mathrm{PS} \cap \mathrm{EXP}[0,\infty)$. But let's proceed on that *assumption.* Since we don't (yet) know $f(t)$, we don't know a suitable α, so we don't know how to pick $\sigma > \alpha$ in the inversion integral. However, analyticity now comes to the rescue. If our assumption is correct, $\phi(s)$ must be analytic for all s with Re $s >$ some α. So, look at $\phi(s)$ and determine an α such that $\phi(s)$ is analytic for all s, Re $s > \alpha$. If no such α exists, i.e. if $\phi(s)$ has singularities arbitrarily far to the right in $\mathbb{C}$, then $\phi(s)$ isn't the $\mathcal{L}$-transform of anything. However, the following can be proved. Suppose such α does exist, and that $\hat{f}(s)$ satisfies the technical condition: there are constants M, $\mu > 0$ such that for all $|s|$ sufficiently large, $|\hat{f}(s)| \le M|s|^{-\mu}$. Then if our basic assumption above is correct, the formula (*) will work with $\sigma > \alpha$.

Traditionally the 'limiting' vertical line $(\sigma - i\infty,\ \sigma + i\infty)$ is called the *Bromwich contour.*

3.7 Laplace Inversion of Strictly Proper Rational Functions

Let $\phi(s) := q(s)/p(s)$ where $q(\cdot)$ and $p(\cdot)$ are polynomials, with

$$\deg q(s) \;\le\; \deg p(s) - 1\,.$$

Such a rational function is said to be *strictly proper.*

Definition

A *zero* of $p(s)$ is a point $s = s^*$ where $p(s^*) = 0$. Let $\deg p(s) = n$, and let the zeros of $p(s)$ be $s_1, \ldots, s_n$ (multiple zeros are repeated according to multiplicity). Thus $\phi(s)$ is analytic except for poles at $s_1, \ldots, s_n$ (assume q and p have no factors in common; otherwise, cancel them). Let

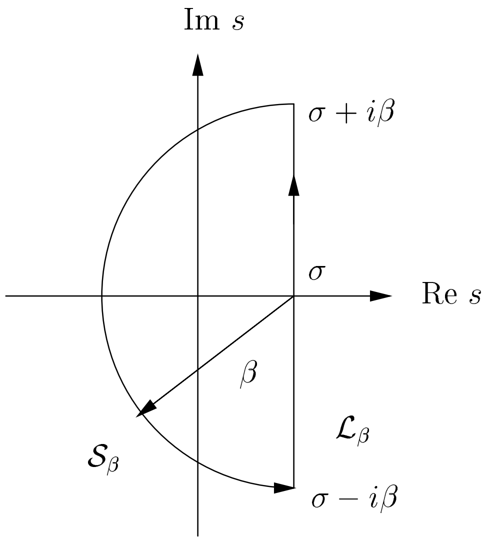
$$\sigma_0 \;=\; \max_{1\le i\le n} \text{ Re } s_i\,.$$

Then $\phi(s)$ is analytic for $\mathrm{Re}\, s \ge \sigma > \sigma_0$.

To apply the inversion theorem we shall exploit Cauchy's theorem on residues, just as we did for Fourier transforms. Let $\mathcal{L}_\beta$ be the vertical line segment $[\sigma - i\beta, \sigma + i\beta]$, and let

$\mathcal{S}_\beta$ be the semicircular contour shown, with radius β, and center $s = \sigma$ on the real axis. We shall show that for $t > 0$ and fixed,

$$\int_{\mathcal{S}_\beta} \phi(s) e^{st} ds \to 0 \text{ as } \beta \to \infty\,.$$



Without loss of generality we may assume $\sigma = 0$; otherwise, rewrite everything in terms of $s' := s - \sigma$. Now

$$\begin{aligned}\phi(s) &= \frac{q_0 + q_1 s + \ldots + q_{n-1} s^{n-1}}{p_0 + p_1 s + \ldots + p_{n-1} s^{n-1} + p_n s^n} \qquad (p_n \neq 0) \\ &= \frac{1}{s} \cdot \frac{q_0 + \ldots + q_{n-1} s^{n-1}}{p_0/s\ + p_1 + \ldots + p_n s^{n-1}}\end{aligned}$$

From this you see that for some constant $M < \infty$,

$$|\phi(s)| \leq \frac{M}{|s|}, \quad \text{all } |s| \text{ sufficiently large.}$$

Exercise 3.4: Prove this in detail.

Now, on $\mathcal{S}_\beta$, $s = \beta e^{i\theta}$, $\pi/2 \leq \theta \leq 3\pi/2$. So

$$\begin{aligned}\left|\int_{\mathcal{S}_\beta} \phi(s) e^{st} ds\right| &\leq \int_{\pi/2}^{3\pi/2} \frac{M}{|\beta e^{i\theta}|} |e^{t\beta e^{i\theta}}| |\beta i e^{i\theta}| d\theta \\ &= \int_{\pi/2}^{3\pi/2} M e^{t\beta\cos\theta} d\theta\end{aligned}$$

$$\begin{aligned} &= 2M \int_{\pi/2}^{\pi} e^{t\beta\cos\theta} d\theta \\ &\le 2M \int_{\pi/2}^{\pi} e^{-t\beta(2/\pi)(\theta-\pi/2)} d\theta \\ &= (\pi M/\beta t)(1 - e^{-t\beta}) \\ &\to 0 \text{ as } \beta \to \infty . \end{aligned}$$

By this result, and assuming (temporarily) that the inversion theorem applies to the pair $f(t)$, $\phi(s)$, with $\phi(s) = \hat{f}(s)$, we get

$$f(t) = \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\mathcal{L}_\beta + \mathcal{S}_\beta} e^{st}\phi(s) ds$$

But for all β sufficiently large, the complete contour $\mathcal{L}_\beta + \mathcal{S}_\beta$ encloses the points $s_1, \ldots, s_n$. Since $\hat{f}(s)$ is analytic except for poles at $s = s_i$, there follows by Cauchy's Residue Theorem

$$f(t) = \sum_{i=1}^{n} \{\text{Residue of } e^{st}\phi(s) \text{ at pole at } s = s_i\}.$$

If there are poles of order ≥ 2, the summation should be taken over the *distinct* poles.

Let's see what this looks like. You recall that if $g(s)$ has a pole of order k at $s = s^*$, then in a neighborhood of s^*,

$$g(s) = g_0(s) + \frac{\gamma_1}{s - s^*} + \frac{\gamma_2}{(s - s^*)^2} + \ldots + \frac{\gamma_k}{(s - s^*)^k}$$

Here $g_0(\cdot)$ is analytic in a neighborhood of s^* and the γ_i are constants with $\gamma_k \neq 0$. It is easy to verify that we get the residue γ_1 by the recipe

$$\gamma_1 = \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \{(s - s^*)^k g(s)\}|_{s=s^*}$$

Exercise 3.5: Check this in detail.

You can see now that if $\phi(s)$ has a pole of order k at $s = s^*$, then the corresponding contribution to $f(t)$ will be a linear combination of terms

$$e^{s^*t}, te^{s^*t}, \ldots, t^{k-1}e^{s^*t} .$$

Summing over all the poles, we get a total of (at most) n terms of this type.

It is now obvious that the function $f(t)$ we have calculated belongs to $\mathrm{PS} \cap \mathrm{EXP}[0, \infty)$. With a little algebra, you can verify directly that the Laplace transform $\hat{f}(s)$ of $f(t)$ really is $\phi(s)$, as we hoped.

Exercise 3.6: (For the ambitious) Carry out this verification in detail.

Example: Scanning Radar Antenna revisited

We have

$$\hat{\theta}(s) = \frac{2.5(s+4)}{(s+5)(s^2+2s+2)(s^2+1)}$$

in the case where $\theta_0 = \theta_1 = 0$.

The denominator polynomial has simple zeros at $s = -5, -1 \pm i, \pm i$. So

(i) Residue of $\hat{\theta}(s)e^{st}$ at $s = -5$ is

$$(s+5)\hat{\theta}(s)e^{st}|_{s=-5} = \frac{(2.5)(-1)}{(17)(26)} e^{-5t}.$$

(ii) (Res. at $s = +i$) + (Res. at $s = -i$)

$$\begin{aligned} &= 2\,\mathrm{Re}(\text{Res. at } s = +i) \quad \text{[Exercise]} \\ &= 2\,\mathrm{Re}\left\{\frac{(2.5)(i+4)e^{it}}{(i+5)(-1+2i+2)(2i)}\right\} \\ &= ? \end{aligned}$$

(iii) Treat the poles at $-1 \pm i$ similarly. The final result is

$$\theta(t) = ?$$

Exercise 3.7: Complete the two calculations above. Repeat for the cases where the factor $1/(s^2+1)$ in $\hat{\theta}(s)$ is replaced by $1/s$ (corresponding to a *unit step input* $r(t) = 1$) and by $1/s^2$ (corresponding to a *unit ramp input* $r(t) = t$). In the first case $1/(s^2+1)$, accurately compute and plot $\theta(t)$ for $0 \le t \le 4\pi$; for the step and ramp, plot for $0 \le t \le 6$.

Example: Rational function with delay factor

Let

$$\begin{aligned} \hat{f}(s) &= \frac{e^{-2s}}{s+3} \\ &= e^{-2s}\hat{g}(s), \text{ say.} \end{aligned}$$

Now

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}.$$

So

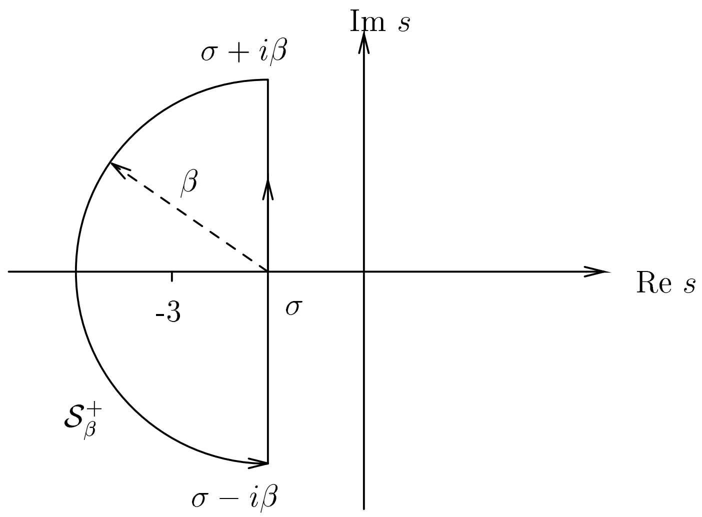
$$\begin{aligned} f(t) &= \left\{\begin{array}{ll} 0, & 0 \le t < 2 \\ g(t-2), & t \ge 2 \end{array}\right. . \\ &= \left\{\begin{array}{ll} 0, & 0 \le t < 2 \\ e^{-3(t-2)}, & t \ge 2 \end{array}\right. . \end{aligned}$$

Let's see how you'd get this from the inversion integral.

We have

$$f(t) = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\sigma-i\beta}^{\sigma+i\beta} e^{st}\frac{e^{-2s}}{s+3}\,ds\,, \ \ \sigma > -3\,.$$

For $t-2>0$, complete the contour via $\mathcal{S}_\beta^+$.



Since $\displaystyle\int_{\mathcal{S}_\beta^+}\frac{e^{s(t-2)}}{s+3}ds \to 0$ as $\beta\to\infty$, we have that

$$\lim_{\beta\to\infty} \quad \frac{1}{2\pi i}\int_{\sigma-i\beta}^{\sigma+i\beta}\frac{e^{s(t-2)}}{s+3}ds$$

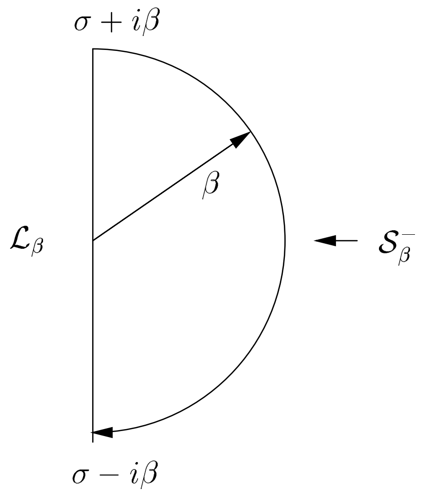
$$= \lim_{\beta\to\infty}\frac{1}{2\pi i}\oint\frac{e^{s(t-2)}}{s+3}ds$$

$$= \text{Res.}\frac{e^{s(t-2)}}{s+3} \ \text{ at } \ s=-3$$

$$= e^{-3(t-2)}\,.$$

For $t-2<0$, complete the contour via $\mathcal{S}_\beta^-$. Now for $t-2<0$,

$$\int_{\mathcal{S}_\beta^-} \to 0 \quad \text{as} \quad \beta \to \infty\,.$$



Also, $e^{s(t-2)}/(s+3)$ is analytic on and within the contour $\mathcal{L}_\beta + \mathcal{S}_\beta^-$. Hence by the same reasoning as before

$$f(t) \;=\; \lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\mathcal{L}_\beta + \mathcal{S}_\beta^-} \;=\; 0, \;\; t-2<0\,.$$

3.8 Laplace Inversion of Proper Rational Functions

As in Sect. 3.6, assume that $\phi(s) := q(s)/p(s)$ for polynomials $q(\cdot)$ and $p(\cdot)$, but now we generalize slightly to allow

$$\deg q(s) \le \deg p(s)$$

Such a rational function is said to be (merely) *proper*.

Thus the functions 1, $1/(s+1)$, $(3s-2)/(s+1)$ are proper, but of these only $1/(s+1)$ is strictly proper. The rational functions s, $(s^2-1)/(s+2)$ are not proper.

Let ϕ be proper and write

$$\phi_\infty := \lim_{s\to\infty} \phi(s)$$

Note that ϕ_∞ always exists, and that $\phi_\infty = 0$ if ϕ happens to be strictly proper. We can therefore write

$$\phi(s) = \phi_\infty + \psi(s)$$

where $\psi(s) := \phi(s) - \phi_\infty$ is strictly proper. For example

$$\frac{3s-2}{s+1} = 3 + \frac{-5}{s+1}$$

Assume temporarily that

$$\mathcal{L}^{-1}\{\phi\} =: f, \quad \mathcal{L}^{-1}\{\psi\} =: g$$

Recalling that $\mathcal{L}^{-1}\{1\} = \delta$, we have by linearity

$$f(t) = \phi_\infty \delta(t) + g(t)$$

Since ψ is strictly proper, the result of Sect. 3.6 applies, to determine g from ψ. For the example above,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{3s-2}{s+1}\right\} &= \mathcal{L}^{-1}\{3\} + \mathcal{L}^{-1}\left\{\frac{-5}{s+1}\right\} \\ &= 3\delta(t) - 5e^{-t} \end{aligned}$$

as can be seen directly by Laplace-transforming the right-hand side.

As an application, you may recall from circuit theory that transfer functions for high-pass filters may be proper but not strictly proper. For instance, a single-stage high-pass RC filter has voltage transfer function $Ts/(1+Ts)$ where $T = RC$. More details are provided in Ch. 4.

3.9 Laplace Transform of Periodic Functions

Let $f \in \mathrm{PS}[0,\infty)$, and for some $T > 0$,

$$f(t+T) = f(t)\,, \quad \text{all } t \geq 0\,.$$

Then

$$\begin{aligned} \hat{f}(s) &= \int_0^\infty e^{-st} f(t)dt \qquad (\mathrm{Re}\ s > 0) \\ &= \sum_{k=0}^\infty \int_{kT}^{(k+1)T} e^{-st} f(t)dt \\ &= \sum_{k=0}^\infty \int_0^T e^{-s(t'+kT)} f(t'+kT)dt' \\ &= \left(\sum_{k=0}^\infty e^{-ksT}\right) \int_0^T e^{-st'} f(t')dt'\,, \end{aligned}$$

or finally,

$$\hat{f}(s) = \frac{\hat{f}_T(s)}{1 - e^{-sT}}$$

where

$$\hat{f}_T(s) := \int_0^T e^{-st} f(t) dt$$

Now because $f \in$ PS and $T < \infty$, it can be shown from complex integration theory (cf. Titchmarsh, *Theory of Functions*, Sect. 2.83) that $\hat{f}_T(s)$ is analytic everywhere in the (finite) s-plane. So the only singularities of $\hat{f}(s)$ are the poles contributed by the zeros of $\phi(s) := 1 - e^{-sT}$. But $\phi(s) = 0$ iff $e^{-sT} = 1$, i.e. $sT = 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$. Also,

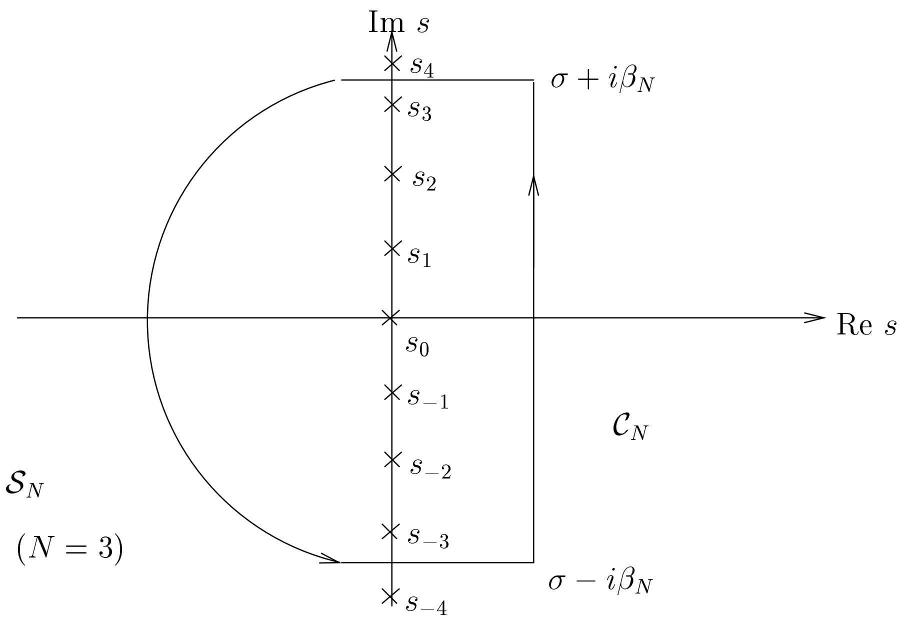
$$\phi'(s) = Te^{-sT}, \quad \phi'\left(\frac{2n\pi i}{T}\right) = T \neq 0;$$

so the zeros of ϕ are simple. Thus $\hat{f}(s)$ has, in general, simple poles at

$$s = s_n := \frac{2n\pi i}{T}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now apply the inversion integral formally, just to see what happens. Let $\sigma > 0$. Then

$$f(t) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\sigma - i\beta}^{\sigma + i\beta} e^{st} \hat{f}(s) ds$$



Let $\beta \to \infty$ along a sequence $\beta_N := \frac{(2N+1)\pi}{T}$, $N = 0, 1, \ldots$, and construct $\mathcal{S}_N$ qualitatively as shown, to sneak between s_N and s_{N+1} (s_{-N} and $s_{-(N+1)}$) as it crosses the imaginary axis. Now $\hat{f}(s)$ is analytic on and within $\mathcal{C}_N + \mathcal{S}_N$ except for simple poles at $s = s_n$, $n = 0, \pm 1, \ldots, \pm N$. Near $s = s_n$, Taylor's theorem gives

$$1 - e^{-sT} = 0 + (s - s_n)T + \text{higher terms}$$

So

$$\begin{aligned} &\text{Residue of } e^{st}\hat{f} \text{ at } (s = s_n) \\ &\qquad = \lim_{s \to s_n} \frac{(s - s_n)e^{st}\hat{f}_T(s)}{1 - e^{-sT}} \\ &\qquad = \frac{1}{T}\, e^{s_n t}\hat{f}_T(s_n)\,. \end{aligned}$$

Next, *assume* you can pick $\mathcal{S}_N$ such that

$$\lim_{N \to \infty} \int_{\mathcal{S}_N} e^{st}\hat{f}(s)ds = 0$$

Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\sigma - i\beta_N}^{\sigma + i\beta_N} e^{st}\hat{f}(s)ds \\ &= \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\mathcal{C}_N + \mathcal{S}_N} e^{st}\hat{f}(s)ds \\ &= \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{\hat{f}_T(s_n)e^{s_n t}}{T}\,. \end{aligned}$$

That is,

$$f(t) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{(2n\pi i/T)t}\,,$$

where

$$c_n := \frac{1}{T} \int_0^T e^{-in(2\pi/T)t'} f(t')dt'\,.$$

In other words, for a periodic function, evaluation of the Laplace inversion integral by residues yields precisely the complex exponential Fourier series! In particular cases, the evaluation would have to be justified by cooking up a suitable family of contours $\mathcal{S}_N$.

Sometimes you can get a more explicit answer by using the following trick.

Example: Laplace inversion by delay series

Let

$$\hat{f}(s) = \frac{1-(1+sT)e^{-sT}}{s^2(1-e^{-sT})} \tag{$*$}$$

Here

$$\hat{f}_T(s) := \frac{1-(1+sT)e^{-sT}}{s^2}$$

is analytic for all $s \in \mathbb{C}$, so $\hat{f}(s)$ is a candidate transform of some periodic function. You look for the latter by expanding

$$(1-e^{-sT})^{-1} = 1 + e^{-sT} + e^{-2sT} + \ldots,$$

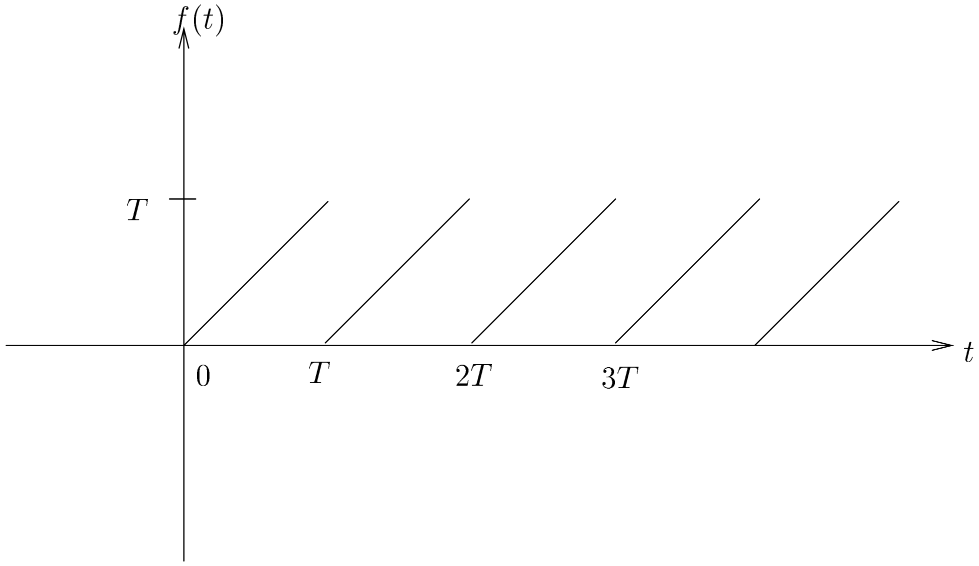
substituting in (*), and re-arranging, to get

$$\hat{f}(s) = \frac{1}{s^2} - \frac{T}{s}\,(e^{-sT} + e^{-2sT} + \ldots)\,.$$

Now note that e^{-ksT} represents a *delay*, of kT time units. So if $f = \mathcal{L}^{-1}(\hat{f})$ is to be periodic, its behavior on the basic interval $0 \le t < T$ must coincide with $\mathcal{L}^{-1}\{1/s^2\}$, namely

$$f(t) = t\,, \quad 0 \le t < T\,. \tag{$**$}$$

For $t > T$, f must be the periodic extension of (**).



It's now easy to check that indeed $\mathcal{L}(f) = \hat{f}$ as given.

Exercise 3.8: Find the inverse Laplace transform of

$$\hat{f}(s) = \frac{120\pi(1+e^{-s/120})}{(s^2 + 14400\cdot\pi^2)(1-e^{-s/60})}$$

HINT: Expand $(1-e^{-s/60})^{-1}$ in powers of $e^{-s/60}$.

3.10 Two Abelian Theorems

1. Initial Value Theorem

Let $f \in \mathrm{PS} \cap \mathrm{EXP}[0, \infty)$. Write $f(0+) = \lim f(t)(t \to 0)$. Then

$$f(0+) = \lim_{\substack{s \to \infty \\ (s\,\mathrm{real})}} s\hat{f}(s)\,.$$

Proof:

We must show

$$(\forall \varepsilon > 0)(\exists c)(\forall s) s > c \Rightarrow |s\hat{f}(s) - f(0+)| < \varepsilon$$

Now since $f \in \mathrm{EXP}$, for some K and α,

$$|f(t)| \;\leq\; Ke^{\alpha t},\;\; t \geq 0\,.$$

Also, by definition of $f(0+)$, if $\varepsilon > 0$ is given,

$$(\exists h > 0)(\forall t)\, 0 < t \leq h \Rightarrow |f(t) - f(0+)| < \varepsilon/2$$

We have

$$\begin{aligned}
s\hat{f}(s) - f(0+) &= s\int_0^\infty e^{-st} f(t)dt - f(0+) \\
&= s\int_0^\infty e^{-st}[f(t) - f(0+)]dt \\
&= s\int_0^h e^{-st}[f(t) - f(0+)]dt \,+\, s\int_h^\infty e^{-st} f(t)dt - e^{-sh} f(0+)
\end{aligned}$$

So

$$\begin{aligned}
|s\hat{f}(s) - f(0+)| &\leq s\int_0^h e^{-st}(\varepsilon/2)dt \,+\, s\int_h^\infty Ke^{-(s-\alpha)t}dt + e^{-sh}|f(0+)| \\
&= (\varepsilon/2)(1 - e^{-sh}) + \left(\frac{s}{s-\alpha}\right) Ke^{-(s-\alpha)h} + e^{-sh}|f(0+)|
\end{aligned}$$

With $h > 0$ fixed, pick c so large that

$$e^{-ch}|f(0+)| < \varepsilon/4$$

and

$$\frac{c}{c-\alpha}\, K e^{-(c-\alpha)h} < \varepsilon/4\,.$$

Then

$$|s\hat{f}(s) - f(0+)| < \varepsilon\,, \quad \text{all } s > c\,. \qquad \text{Q.E.D.}$$

Examples

1. $f(t) = \sin t$, $\hat{f}(s) = (1+s^2)^{-1}$.

$$\lim_{s\to\infty} s\hat{f}(s) = 0 = f(0+).$$

2. $f(t) = \cos t$, $\hat{f}(s) = s(1+s^2)^{-1}$.

$$\lim_{s\to\infty} s\hat{f}(s) = 1 = f(0+).$$

Why does the Initial Value Theorem work? Evidently because for large s, the function $se^{-st}(t \geq 0)$ is 'almost' a δ-function sitting a little to the right of $t = 0$.

2. Final Value Theorem

Let $f \in \mathrm{PS}[0,\infty)$ and suppose

$$f_\infty := \lim_{t\to\infty} f(t)$$

exists and is finite. Then

$$f_\infty = \lim_{\substack{s\,\to\,0\\ s\ \text{real}\\ s\,>\,0}} s\hat{f}(s)\,.$$

Proof:

Since $f \in \mathrm{PS}[0,\infty)$ and f_∞ is finite,

$$\exists K\ :\ |f(t)|\ \leq\ K\,,\ \ \text{all}\, t \geq 0\,. \tag{1}$$

Thus

$$\hat{f}(s)\ =\ \int_0^\infty e^{-st} f(t)dt\,,\ \ \operatorname{Re} s > 0\,.$$

We must show:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall s)\ s\,\text{real},\ \ 0 < s < \delta \Rightarrow |s\hat{f}(s) - f_\infty| < \varepsilon.$$

Now, since f_∞ exists, we know that

$$(\forall\varepsilon > 0)(\exists T)(\forall t) t \geq T \Rightarrow |f(t) - f_\infty| < \varepsilon/2. \tag{2}$$

Next we split up the expression

$$
\begin{aligned}
s\hat{f}(s) - f_\infty &= s\left(\int_0^T + \int_T^\infty\right) e^{-st} f(t)dt \,-\, f_\infty \\
&= s\int_0^T e^{-st} f(t)dt \,+\, s\int_T^\infty e^{-st}[f(t) - f_\infty]dt \,+\, s\int_T^\infty e^{-st} f_\infty dt - f_\infty.
\end{aligned}
$$

From this we can estimate, using (1) and (2):

$$|s\hat{f}(s) - f_\infty| \,\le\, sTK \,+\, \frac{\varepsilon}{2} e^{-sT} \,+\, |e^{-sT} - 1||f_\infty|.$$

Pick $\delta > 0$ such that

$$\delta TK \,+\, (1 - e^{-\delta T})|f_\infty| < \varepsilon/2.$$

Then, clearly $|s\hat{f}(s) - f_\infty| < \varepsilon$ for all s, $0 < s < \delta$. Q.E.D.

Examples

1. (Nonsense)

   $$\hat{f}(s) \,=\, \frac{1}{s(s-2)}\,, \quad \lim_{s\to 0} s\hat{f}(s) \,=\, -\frac{1}{2}\,.$$

   But

   $$
   \begin{aligned}
   f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1}{s-2} - \frac{1}{s}\right)\right\} \\
   &= \frac{1}{2}\,(e^{2t} - 1) \to \infty\,, \;\; t \to \infty\,!
   \end{aligned}
   $$

   Here f_∞ exists, but is infinite.

2. (Nonsense)

   $$\hat{f}(s) \,=\, \frac{1}{s^2+1}\,, \quad \lim_{s\to 0} s\hat{f}(s) \,=\, 0.$$

   But $f(t) = \sin t$, and f_∞ fails to exist!

3. Scanning Radar Antenna.
   We found

   $$\hat{\theta}(s) \,=\, \frac{k(s+\alpha)}{s^3 + (3+\alpha)s^2 + 3\alpha s + \alpha k} \cdot \hat{r}(s)$$

   The denominator polynomial has all roots s_1, s_2, s_3 stable (i.e. Re $s_i < 0$) provided $\alpha > 0$ and $0 < k < 9 + 3\alpha$. Let $r(\cdot)$ be a step reference signal applied at $t = 0$, i.e.

   $$r(t) \,=\, \theta^*\,, \;\; t \ge 0\,; \;\; \hat{r}(s) \,=\, \theta^*/s\,.$$

   Then

$$\theta(t) = \underset{\substack{\text{from simple}\\ \text{pole at } s=0}}{\text{const.}} + \underset{\substack{\text{from stable poles,}\\ \text{in Re } s<0.}}{\text{decaying exponentials}}$$

Therefore θ_∞ exists and is finite, and the final-value theorem gives

$$\theta_\infty = \lim_{s\to 0} s\hat{\theta}(s) = \frac{k\alpha}{\alpha k} \cdot \lim_{s\to 0} s \cdot \frac{\theta^*}{s} = \theta^* .$$

For a step reference signal we therefore have tracking with zero 'steady-state' error.

While this example does illustrate a correct application of the Final Value Theorem, you see that all the theorem really says in this case is: compute the constant component of time response by the usual rule for evaluating the residue of $\hat{\theta}(s)e^{st}$ at the simple pole at $s = 0$.

Why does the Final-Value Theorem work? Evidently because for small s, the function se^{-st} $(t \geq 0)$ is 'almost' a uniform 'probability density' on the half-line $0 \leq t < \infty$. Now if f_∞ exists and is finite, then $f(t) \approx f_\infty$ for all large t, that is, over 'nearly all' the t-axis. Thus its global average, as computed by

$$\lim_{s\to 0} \int_0^\infty se^{-st} f(t)dt \, ,$$

should be the constant value f_∞.

Note well, once more, that the Final-Value Theorem must be applied with some care.

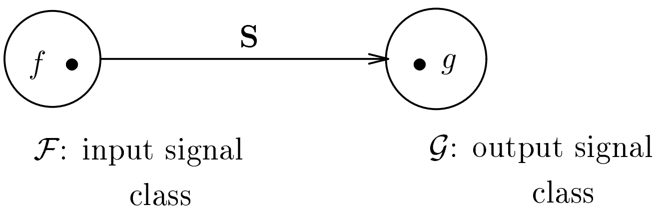
These two theorems are examples of 'abelian theorems' (after N.H. Abel, 1802-1829). Abelian theorems, in general, provide information about some 'average' of a function, starting with limited data on the function values (or limiting values). The opposite kind of theorem, giving information about some function values from data on the function 'averages', is called 'tauberian' (after A. Tauber). Tauberian theorems are harder.

Chapter 4

Systems

4.1 General Systems Properties

Broadly speaking, a 'system' is anything that is 'put together'. We'll regard a system as a device which transforms an input signal to an output signal in some specified way: in other words, a system may be represented by an operator $\mathbf{S}$, as pictured below.

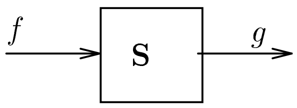


As usual, our input and output signals will be (real or) complex-valued functions of t, with $t \in (-\infty, \infty)$ or $t \in [0, \infty)$.

One writes

$$g(t) = (\mathbf{S}f)(t).$$

And sometimes one draws little pictures like:



Three general systems concepts of major importance are **causality**, **time-invariance** and **stability**. We give a brief discussion of each.

Definition: Causality

$\mathbf{S}$ is *causal* (or *nonanticipatory*) if

$$\begin{aligned}(\forall t)(\forall f_1, f_2 \in \mathcal{F}) : f_1(\tau) &= f_2(\tau) \quad (-\infty < \tau \leq t) \\ &\Rightarrow \quad (\mathbf{S}f_1)(t) = (\mathbf{S}f_2)(t)\end{aligned}$$

This means that any two inputs which agree on the 'past', up to time t (i.e. for $\tau \le t$) produce the same system output at 'the present' time t. Or, alternatively, $\mathbf{S}$ responds only to the 'past' (and 'present') of the input, and not to the 'future'. In other words, $\mathbf{S}$ is not 'clairvoyant'. Physical systems operating in real time fall into this category. On the other hand, a data processor operating on a complete, previously stored, record of data might well be modeled as a non-causal system on the 'time'-axis of the record.

Definition: Time-invariance

Write $f_\tau(t) := f(t+\tau)$. Suppose $\mathcal{F}$ is closed under time shift, i.e. $f \in \mathcal{F} \Rightarrow f_\tau \in \mathcal{F}$ for all τ. Then we say $\mathbf{S}$ is *time-invariant* (or *stationary*) if:

$$(\mathbf{S}f_\tau)(t) = (\mathbf{S}f)(t+\tau)$$

for all t, τ [which make sense] and all $f \in \mathcal{F}$.

Time-invariance just means that the action of $\mathbf{S}$ doesn't change with time. Roughly, a given input produces the same output if applied today, as the output it would produce tomorrow if it were applied tomorrow. To a good approximation many physical systems can be modeled as time-invariant over most of their operating lifetime: e.g. the component R's and C's of an electrical network. Others cannot be so modeled: e.g. inductors with hysteresis lags, or more generally anything whose behavior is modified as a result of past 'experience' or aging or changes in its environment.

To depict time-invariance in a nice way, bring in the family of *shift operators*

$$T'_\tau : \mathcal{F} \longrightarrow \mathcal{F}, \qquad -\infty < \tau < \infty$$

defined according to

$$(T'_\tau f)(t) = f(t+\tau), \qquad -\infty < t < \infty$$

for all $f \in \mathcal{F}$. Thus $T'_\tau f = f_\tau$. Similarly let T''_τ denote the shift operator on $\mathcal{G}$. The time-invariance property of $\mathbf{S}$ can now be rewritten

$$\mathbf{S}(T'_\tau f) = T''_\tau(\mathbf{S}f)$$

for all $f \in \mathcal{F}$, $\tau \in \mathbb{R}$; or simply

$$\mathbf{S} \circ T'_\tau = T''_\tau \circ \mathbf{S}, \quad \tau \in \mathbb{R}$$

where $\circ$ denotes operator composition. This states that the system operator $\mathbf{S}$ and time-shift operator T_τ ($= T'_\tau$ or T''_τ as appropriate) *commute*; and may be displayed by a so-called *commutative diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathbf{S}"] \arrow[d, "T'_\tau"'] & \mathcal{G} \arrow[d, "T''_\tau"] \\
\mathcal{F} \arrow[r, "\mathbf{S}"] & \mathcal{G}
\end{tikzcd}$$

Our third definition requires the following notation. Write

$$\|f\|_\infty = \sup\{|f(t)| : -\infty < t < \infty\}$$

That is, $\|f\|_\infty$ is the least upper bound of the values $|f(t)|$ over the domain of definition of f. If $\|f\|_\infty = +\infty$ then f is unbounded; whereas $\|f\|_\infty < \infty$ means f is bounded, and then

$$|f(t)| \leq \|f\|_\infty, \quad -\infty < t < \infty.$$

Often $\|f\|_\infty$ is called the 'sup norm' of f.

Examples: Sup norm of f

1. $f(t) = -e^{-t}$, $-\infty < t < \infty$

   $\|f\|_\infty = \infty$

   i.e. f is unbounded on $-\infty < t < \infty$.

2. $f(t) = te^{-t^2}$, $-\infty < t < \infty$

   $\|f\|_\infty = 1/\sqrt{2e}$

   Note that f hits its bound at $t = \pm 1/\sqrt{2}$.

3. $f(t) = \begin{cases} 0\ , & t < 0 \\ 1 - e^{-t}\ , & t \geq 0 \end{cases}$

   $\|f\|_\infty = 1$

   Note that in this case f never quite reaches its bound, but gets arbitrarily close for large t.

Definition: BIBO stability
S is *bounded-input, bounded-output* (BIBO) *stable* if

$$(\forall K_1)(\exists K_2)(\forall f \in \mathcal{F})\|f\|_\infty < K_1 \Rightarrow \|\mathbf{S}f\|_\infty < K_2$$

In other words $\mathbf{S}f$ is bounded whenever f is bounded, and the bound K_2 on $\mathbf{S}f$ depends only on the bound K_1 on f, not on the specific f which is bounded by K_1.

An electrical network with damping associated with each L and C, and no active elements (energy sources), is one example of a BIBO stable system. *Un*stable systems are also all too familiar: e.g. the faulty PA system to which a small, perturbing input produces a loud and unpleasant shriek; as well as (conceivably) models of national economies, world population, cancer cell division, etc. Of course, the actual physical processes involved are always *really* bounded (owing to saturation of the PA amplifier, or some catastrophe): the point is that the relevant *model* might well admit, theoretically, outputs which are unbounded even though inputs are bounded.

There is not very much you can go on to say about 'systems' at this level of generality.[1] But if you focus sharply on well-structured classes of systems, there is some hope of doing science. In this course we'll just talk about *linear* systems, where our transform methods can be used to advantage.

4.2 Linear Systems - General Theory

Let $\mathcal{F}$ and $\mathcal{G}$ be linear vector spaces over the field $\mathbb{C}$ of complex numbers. The elements of $\mathcal{F}$ and $\mathcal{G}$ will be time functions $f(\cdot)$ and $g(\cdot)$ corresponding to inputs and outputs of our system. The system itself is a map $\mathbf{S} : \mathcal{F} \to \mathcal{G}$. Then we have:

Definition: System linearity
$\mathbf{S}$ is *linear* if

$$\mathbf{S}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathbf{S} f_1 + \alpha_2 \mathbf{S} f_2$$

for all $f_1, f_2 \in \mathcal{F}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$.

Lots of physical systems, economic systems,... can be adequately represented (over reasonably small operating ranges of the variables involved) by linear models. In 'standard' (linear) network theory, for instance, the basic equations are linear integro-differential equations written down by application of Kirchhoff's laws. When you solve these equations from the time origin $\tau = -\infty$, you get a *linear input-output relation* of the form

$$(\mathbf{S}f)(t) = \int_{-\infty}^{\infty} h(t,\tau) f(\tau) d\tau, \quad -\infty < t < \infty \tag{LS}$$

where in general $h : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. The type of system representation displayed in (LS) is of rather common occurrence. The 'kernel function' $h(\cdot,\cdot)$ in the integrand is the system *weighting function* (also called *weighting pattern*, or *impulse response*).[2] The linear system model (LS) is thus specified by $h(\cdot,\cdot)$, and by some general description of $\mathcal{F}$ and $\mathcal{G}$ (e.g. $\mathcal{F} = \mathrm{PS}_1(-\infty,\infty)$, etc.) which is usually not very crucial.

For mathematical simplicity we'll assume for now that $h(\cdot,\cdot)$ is an ordinary function, free of unit impulses.

To justify the term 'impulse response' for $h(\cdot,\cdot)$ take for the input $f(\cdot)$ in (LS) a unit impulse located at time α, i.e. $f(\tau) = \delta_\alpha(\tau)$. Then

$$\begin{aligned}
(\mathbf{S}f)(t) &= \int_{-\infty}^{\infty} h(t,\tau)\delta_\alpha(\tau) d\tau \\
&= \int_{-\infty}^{\infty} h(t,\tau)\delta(\tau-\alpha) d\tau \\
&= h(t,\alpha), \quad -\infty < t < \infty.
\end{aligned}$$

[1] 99% of 'General Systems Theory', so-called, is merely bold speculation (b.s.).

[2] These are the standard terms in system theory. In books on differential equations, however, $h(\cdot,\cdot)$ is called the 'Green's function'.

Thus $h(t, \alpha)$ is the response of $\mathbf{S}$ at time t to a unit impulse input at time α. Note that, in general, there is no requirement that this response be zero for $t < \alpha$, i.e. before the impulse is applied at the input.

Without maximizing the rigor, we shall now examine what the properties of causality, time-invariance and stability imply about the structure of $h(\cdot, \cdot)$.

LINEAR SYSTEM STRUCTURE THEOREM

The linear system (LS) is

(i) causal iff $h(t, \tau) = 0$ whenever $\tau > t$;

(ii) time-invariant iff, for some function $\tilde{h}$,

$$h(t, \tau) = \tilde{h}(t - \tau), \text{ all } t, \tau.$$

(iii) BIBO stable iff there is a constant $K < \infty$ such that

$$\int_{-\infty}^{\infty} |h(t, \tau)| d\tau < K, \text{ all } t, \quad -\infty < t < \infty.$$

[In case (iii) we may consider

$$\mathcal{F} = \{f : f(\cdot) \in \mathrm{PS}(-\infty, \infty) \text{ and } f(\cdot) \text{ bounded on } (-\infty, \infty)\}]$$

Exercise 4.1: Prove each of the 'if' statements in the Linear System Structure Theorem.

Exercise 4.2: What can be said about $h(\cdot, \cdot)$ if (LS) has all three properties above?

Proof of the 'only if' statements

Variational Lemma

Suppose $k(\cdot)$ is a function such that

$$\int_I k(t)c(t)dt = 0$$

for some fixed open interval $I \subseteq \mathbb{R}$, and *all* continuous functions $c(\cdot)$ on I. Then $k(t) = 0$ a.e. on I.

Proof:

We shall prove this on the assumption that $k(\cdot)$ is continuous on I. Now if $k(\cdot)$ is not identically zero on I, then $k(t^*) > 0$ (say) for some $t^* \in I$. Since k is continuous, there is some $\delta > 0$ such that

$$(t^* - \delta, t^* + \delta) \subset I \text{ and } k(t) > 0, \quad t^* - \delta < t < t^* + \delta.$$

Pick

$$c(t) = \begin{cases} 1 - \delta^{-1}|t - t^*|, & |t - t^*| \leq \delta \\ 0, & \text{otherwise} \end{cases}$$

Then $c(\cdot)$ is continuous, but $\int_I k(t)c(t)dt > 0$, a contradiction. Q.E.D.

Now for the main proof.

(i) Suppose (LS) is causal. Then for all f, f_1, f_2, we know that

$$f_1(\tau) = f_2(\tau), \tau \leq t \Rightarrow (\mathbf{S}f_1)(t) = (\mathbf{S}f_2)(t),$$

i.e.

$$\int_t^{\infty} h(t,\tau)[f_1(\tau) - f_2(\tau)]d\tau = 0.$$

Take e.g. $f_2(\tau) = 0$, $\tau > t$; then

$$\int_t^{\infty} h(t,\tau)f_1(\tau)d\tau = 0,$$

for all $f_1(\cdot)$ on (t,∞). By the Variational Lemma,

$$h(t,\tau) = 0 \text{ a.e., } \tau > t.$$

In particular if we assume that $h(t,\cdot)$ is piecewise smooth in τ for each fixed t, then h can be defined[3] such that

$$h(t,\tau) = 0, \quad \tau > t,$$

(ii) Assume (LS) is time-invariant. Then for all $f(\cdot)$ and t, α, we have

$$(\mathbf{S}f_\alpha)(t) = (\mathbf{S}f)(t+\alpha) \qquad [f_\alpha(t) := f(t+\alpha)].$$

Thus

$$\int_{-\infty}^{\infty} h(t,\tau_1)f(\tau_1+\alpha)d\tau_1 = \int_{-\infty}^{\infty} h(t+\alpha,\tau_2)f(\tau_2)d\tau_2$$

$$\Rightarrow \int_{-\infty}^{\infty} [h(t,\tau-\alpha) - h(t+\alpha,\tau)]f(\tau)d\tau = 0$$

[putting $\tau_1 + \alpha = \tau$ in LHS integral, and $\tau_2 = \tau$ in RHS integral]. As in (i), this implies

$$h(t,\tau-\alpha) - h(t+\alpha,\tau) = 0 \quad \text{for all } t,\tau$$

$$\Rightarrow \quad h(t,-\alpha) = h(t+\alpha,0) \text{ [setting } \tau = 0] \quad \text{for all } t,\alpha$$

$$\Rightarrow \quad h(t,\tau) = h(t-\tau,0) \text{ [replacing } -\alpha \text{ by } \tau] \quad \text{for all } t,\tau$$

That is,

$$h(t,\tau) = \tilde{h}(t-\tau), \quad \text{for all } t,\tau$$

where

$$\tilde{h}(u) := h(u,0).$$

[3]This is a fussy technical point, which we gloss over.

(iii) Assume that LS is BIBO stable and let f be bounded, i.e. $|f(t)| \leq K_1$, for all $t \in (-\infty, \infty)$. We know that there exists a bound K_2, say, depending on K_1, but independent of the specific choice of f, such that the output $g(\cdot)$ satisfies

$$|g(t)| = |\int_{-\infty}^{\infty} h(t,\tau) f(\tau) d\tau| \leq K_2, \quad -\infty < t < \infty. \tag{$*$}$$

On this assumption we wish to prove the existence of K such that

$$\int_{-\infty}^{\infty} |h(t,\tau)| d\tau \leq K, \quad -\infty < t < \infty.$$

First assume $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ real-valued, fix t arbitrarily and for $-\infty < \tau < \infty$ let

$$f_t(\tau) := \begin{cases} K_1 & \text{if } h(t,\tau) > 0 \\ -K_1 & \text{if } h(t,\tau) \leq 0 \end{cases}$$

Then $|f_t(\tau)| \leq K_1$ (all τ) and

$$\begin{aligned} |g(t)| &= |\int_{-\infty}^{\infty} h(t,\tau) f_t(\tau) d\tau| \\ &= |\int_{-\infty}^{\infty} h(t,\tau) K_1 [\text{sgn } h(t,\tau)] d\tau| \\ &= K_1 \int_{-\infty}^{\infty} |h(t,\tau)| d\tau \\ &\leq K_2 \quad \text{by } (*) \end{aligned}$$

Since we can construct such $f_t(\cdot)$ at each t, we have

$$\int_{-\infty}^{\infty} |h(t,\tau)| d\tau \leq (K_2/K_1), \quad \text{for all } t,$$

and we may take $K = K_2/K_1$.

In case $h : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ takes complex values, write in polar form

$$h(t,\tau) = |h(t,\tau)| \exp(i\theta(t,\tau))$$

and put

$$f_t(\tau) = \exp(-i\theta(t,\tau))$$

Then $h(t,\tau) f_t(\tau) = |h(t,\tau)|$ and the proof is completed as before. Q.E.D.

Extension to 'distributional' impulse response functions

The foregoing results are easily extended to allow for the appearance in $h(\cdot,\cdot)$ of one or more unit impulse functions. Thus suppose

$$h(t,\tau) = h_0(t,\tau) + h_1(t)\delta(\tau - \alpha(t))$$

where $h_0(\cdot,\cdot)$, $h_1(\cdot)$ and $\alpha(\cdot)$ are ordinary well-behaved functions. Then

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} h(t,\tau)f(\tau)d\tau \\ &= \int_{-\infty}^{\infty} h_0(t,\tau)f(\tau)d\tau + h_1(t)f[\alpha(t)], \quad -\infty < t < \infty \end{aligned}$$

By suitably generalizing the previous arguments you should be able to prove that our new linear system is

(i) causal iff (for all t,τ)

$$h_0(t,\tau) = 0 \quad \text{for } \tau > t$$

and

$$\alpha(t) \le t$$

(ii) time-invariant iff, for some function $\tilde{h}_0$ and constants c_1, c_2

$$\begin{aligned} h_0(t,\tau) &= \tilde{h}_0(t-\tau), \quad \text{all } t,\tau \\ h_1(t) &= c_1, \quad \alpha(t) = t + c_2, \quad \text{all } t \end{aligned}$$

(iii) BIBO stable iff there is a constant K such that

$$\int_{-\infty}^{\infty} |h_0(t,\tau)|d\tau + |h_1(t)| < K, \quad \text{all } t$$

Notice that the conditions on the 'ordinary' part $h_0(\cdot,\cdot)$ of the impulse response function are exactly as in the previous subsection.

Exercise 4.3: Provide a detailed proof of the above generalization of the Linear System Structure Theorem.

Example: Sample-and-hold circuit

A continuous signal $f(t)$ defined for $-\infty < t < \infty$ is processed by a sample-and-hold circuit (SHC), which gives the output

$$g(t) := f(n) \quad \text{for } n \le t < n+1; \quad n = ... -1, 0, 1, ...$$

This can be written in the form

$$g(t) = \int_{-\infty}^{\infty} h(t,\tau)f(\tau)d\tau$$

where

$$h(t,\tau) = h_1(t)\delta(\tau - \alpha(t))$$

with

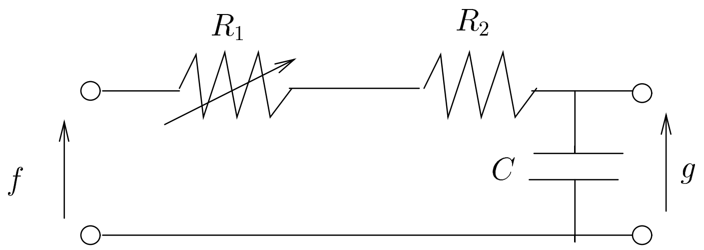
$$h_1(t) = 1, \qquad -\infty < t < \infty$$

$$\alpha(t) = n, \quad n \le t < n+1; \quad n = ... -1, 0, 1, ...$$

It follows that SHC is causal and BIBO stable, but not time-invariant.

Exercise 4.4: Explain in intuitive language why SHC is not time-invariant.

Example: Time-varying low-pass RC filter



For the circuit shown, with time-varying resistance $R_1(t) \ge 0$, consider

$$g(t) = (\mathbf{S}f)(t), \qquad -\infty < t < \infty$$

By circuit theory,

$$g'(t) + \alpha(t)g(t) = \alpha(t)f(t), \qquad -\infty < t < \infty$$

where

$$\alpha(t) = \frac{1}{[R_1(t) + R_2]C}, \qquad -\infty < t < \infty$$

Thus $0 < \alpha(t) \le 1/(R_2C)$. Assuming $g(-\infty) = 0$ we get, by causality,

$$g(t) = \int_{-\infty}^{t} h(t,\tau)f(\tau)d\tau$$

Solving the differential equation, we find

$$h(t,\tau) = \begin{cases} \alpha(\tau)\exp\left(-\int_\tau^t \alpha(u)du\right), & t \ge \tau \\ 0, & t < \tau \end{cases}$$

It is simple to calculate

$$\begin{aligned} \int_{-\infty}^{t} |h(t,\tau)|d\tau &= 1 - \exp\left(-\int_{-\infty}^{t} \alpha(u)du\right) \\ &\le 1 \qquad -\infty < t < \infty \end{aligned}$$

and so $\mathbf{S}$ is BIBO stable.

By the Linear System Structure Theorem, $\mathbf{S}$ will be time-invariant if and only if $h(t,\tau)$ is a function just of $t-\tau$, which is true just in case α is a constant, namely R_1 is constant, as expected. In this case, let us verify time-invariance directly from the definition. Take for $\mathcal{F}$ and $\mathcal{G}$ the class of signals that are bounded on $-\infty < t < \infty$; then $T'_\tau = T''_\tau = T_\tau$, say. Now with $f \in \mathcal{F}$ we have

$$\begin{aligned}
(\mathbf{S}\circ T_\tau)(f)(t) &= (\mathbf{S}(T_\tau f))(t) \\
&= \int_{-\infty}^{t} h(t,u)(T_\tau f)(u)du \\
&= \int_{-\infty}^{t} \alpha e^{-\alpha(t-u)} f(u+\tau)du \\
&= \int_{-\infty}^{t+\tau} \alpha e^{-\alpha(t+\tau-v)} f(v)dv \qquad (v = u+\tau) \\
&= (T_\tau(\mathbf{S}f))(t) \\
&= (T_\tau \circ \mathbf{S})(f)(t)
\end{aligned}$$

Since equality holds for all $t \in \mathbb{R}$, we get

$$(\mathbf{S}\circ T_\tau)(f) = (T_\tau \circ \mathbf{S})(f)$$

for all $f \in \mathcal{F}$, namely

$$\mathbf{S}\circ T_\tau = T_\tau \circ \mathbf{S}$$

for all $\tau \in \mathbb{R}$, as claimed.

4.3 Linear Time-Invariant Causal Systems

If our system $\mathbf{S}$ is linear, time-invariant and causal (LTIC) then one easily sees that the output $g(\cdot)$ for input $f(\cdot)$ is given by

$$g(t) = \int_{-\infty}^{t} h(t-\tau)f(\tau)d\tau = \int_0^\infty h(u)f(t-u)du, \quad -\infty < t < \infty$$

where the weighting function $h = h(u)$ is defined for $u \geq 0$. Assume for now that $h \in PS[0,\infty]$. What does BIBO stability mean in this case? Simply that $\mathbf{S}$ has a 'gain', in the following sense.

THEOREM: Stability of LTIC systems

Assume $\mathbf{S}$ is LTIC. Then $\mathbf{S}$ is BIBO stable iff

$$\int_0^\infty |h(u)|du < \infty. \qquad (*)$$

In that case, setting $K := \int_0^\infty |h(u)|du$, one has

$$\|g\|_\infty \leq K \cdot \|f\|_\infty \tag{†}$$

Proof. We have

$$g(t) = \int_0^\infty h(u)f(t-u)du$$

Clearly if (*) holds, and $|f(t)| \leq K_1$ $(-\infty < t < \infty)$, then

$$\begin{aligned} |g(t)| &\leq \int_0^\infty |h(u)| \cdot K_1 du \\ &= K \cdot K_1 \ (-\infty < t < \infty), \end{aligned}$$

so **S** is stable. From this,

$$\|g\|_\infty = \sup_{-\infty<t<\infty} |g(t)| \leq KK_1.$$

Taking

$$K_1 = \sup_t |f(t)| = \|f\|_\infty,$$

we obtain (†). Conversely, if (*) fails, then $\int_0^\infty |h(u)|du = +\infty$. Given $N > 0$, arbitrarily large, there exists T_N such that

$$\int_0^{T_N} |h(u)|du > N.$$

Define

$$f_N(v) = \begin{cases} 0, & v < 0 \\ \operatorname{sgn}[h(T_N - v)], & 0 \leq v \leq T_N \\ 0, & v > T_N \end{cases}$$

Then

$$\begin{aligned} g_N(T_N) &= \int_{-\infty}^\infty h(T_N - v)f_N(v)dv \\ &= \int_0^{T_N} |h(T_N - v)|dv \\ &= \int_0^{T_N} |h(u)|du \\ &> N. \end{aligned}$$

Since $\|f_N\|_\infty = 1$ for all N, our result violates the definition of BIBO stability, so **S** isn't stable. Q.E.D.

Another very useful property of stable LTIC systems is that they have a steady-state frequency response. For this, stability and time-invariance are essential. Let's now assume that $h(\cdot) \in \mathrm{PS} \cap \mathrm{EXP}[0, \infty)$, with Laplace transform $\hat{h}$. Then we have the

FREQUENCY RESPONSE THEOREM: Steady-state frequency response of stable LTIC systems

Assume the LTIC system $\mathbf{S}$ is stable and $h \in PS \cap EXP[0, \infty)$. Let

$$
\begin{aligned}
f_1(t) &= e^{i\omega t}, \quad t \geq 0 \\
f_2(t) &= \cos \omega t, t \geq 0.
\end{aligned}
$$

Then

$$(h \star f_1)(t) - \hat{h}(i\omega)e^{i\omega t} \to 0 \text{ as } t \to \infty$$

and (when h is real-valued)

$$(h \star f_2)(t) - A(\omega)\cos[\omega t + \psi(\omega)] \to 0 \text{ as } t \to \infty.$$

Here

$$
\begin{aligned}
A(\omega) &:= |\hat{h}(i\omega)| \\
\psi(\omega) &:= \arg \hat{h}(i\omega)
\end{aligned}
$$

These items are designated

$$
\begin{aligned}
\hat{h}(s) &=: \quad \text{transfer function} \\
\hat{h}(i\omega) &=: \quad \text{`complex frequency response function'} \\
A(\omega) &=: \quad \text{`amplitude' (or `gain') response function} \\
\psi(\omega) &=: \quad \text{`phase response function'.}
\end{aligned}
$$

When h is real-valued, $A(\omega)$ is an even function of ω, whereas conventionally $\psi(\omega)$ is defined so as to be odd, i.e. for $\omega < 0$, $\psi(\omega) = -\psi(-\omega)$.

Proof. Since by the previous theorem,

$$\int_0^\infty |h(\tau)|d\tau < \infty$$

it follows that the integral

$$\hat{h}(s) = \int_0^\infty e^{-st}h(t)dt$$

exists for Re $s \geq 0$. Then

$$\begin{aligned}(h * f_1)(t) - \hat{h}(i\omega)e^{i\omega t} &= \int_0^t h(\tau)e^{i\omega(t-\tau)}d\tau - \left(\int_0^\infty e^{-i\omega\tau}h(\tau)d\tau\right)e^{i\omega t} \\ &= -\int_t^\infty h(\tau)e^{i\omega(t-\tau)}d\tau.\end{aligned}$$

So

$$|\text{LHS}| \leq \int_t^\infty |h(\tau)|d\tau \to 0 \text{ as } t \to \infty.$$

The proof for f_2 is left as an exercise. Q.E.D.

In practice, a steady-state frequency response function is displayed on a logarithmic frequency axis ($\log_{10}\omega$), with $A(\omega)$ scaled in db (as $20\log_{10}A(\omega)$) and $\psi(\omega)$ scaled linearly (often in degrees instead of radians).

Exercise 4.5: Subject to the above assumptions on h, show that $\hat{h}(i\omega) \to 0$ as $|\omega| \to \infty$. In other words, "the response of an LTIC stable system (with impulse response function in $PS[0,\infty)$) is very small at very high frequencies." [**Hint:** apply Riemann-Lebesgue, as for the Fourier transform p. 75].

Example: RC low-pass filter

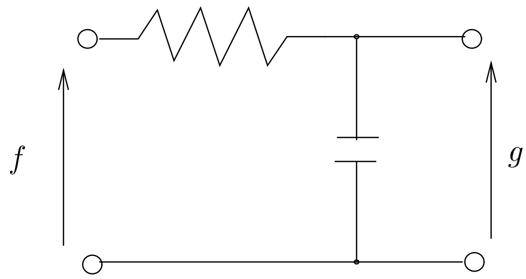
For the RC low-pass filter shown below, with RC=1, verify that

$$h(\tau) = \begin{cases} e^{-\tau}, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$$

$$\hat{h}(s) = \frac{1}{1+s}$$

$$A(\omega) = \frac{1}{\sqrt{1+\omega^2}}, \quad \psi(\omega) = -\tan^{-1}\omega, \quad -\infty < \omega < \infty.$$

Sketch the graphs of $h(\cdot)$, $A(\cdot)$ and $\varphi(\cdot)$. Explain how $h(\cdot)$ exhibits 'quasi-integrating' action at high frequencies. Discuss the circuit behavior for very low and very high frequencies.



Example: Finite-memory moving average.

For this,

$$g(t) = \int_0^t h(\tau) f(t-\tau) d\tau,$$

where we assume

$$h(\tau) \geq 0; h(\tau) = 0, \quad \tau \geq T; \text{ and}$$

$$\int_0^T h(\tau) d\tau = 1.$$

BIBO stability is obvious. As a simple special case, let

$$h(\tau) = \begin{cases} \frac{1}{T}, & 0 \leq \tau \leq T \\ 0, & \tau > T. \end{cases}$$

Then

$$g(t) = \begin{cases} \dfrac{1}{T} \displaystyle\int_0^t f(t-\tau) d\tau, & t \leq T \\ \\ \dfrac{1}{T} \displaystyle\int_0^T f(t-\tau) d\tau, & t \geq T \end{cases}$$

Now

$$\begin{aligned} \hat{h}(s) &= \int_0^T e^{-st} \cdot \frac{1}{T} dt \\ &= \frac{1}{sT}(1 - e^{-sT}). \end{aligned}$$

Notice that $\hat{h}(s)$ is analytic everywhere. The complex frequency response function is

$$\hat{h}(i\omega) = \frac{1}{i\omega T}(1 - e^{-i\omega T}), \qquad -\infty < \omega < \infty$$

So the amplitude response is

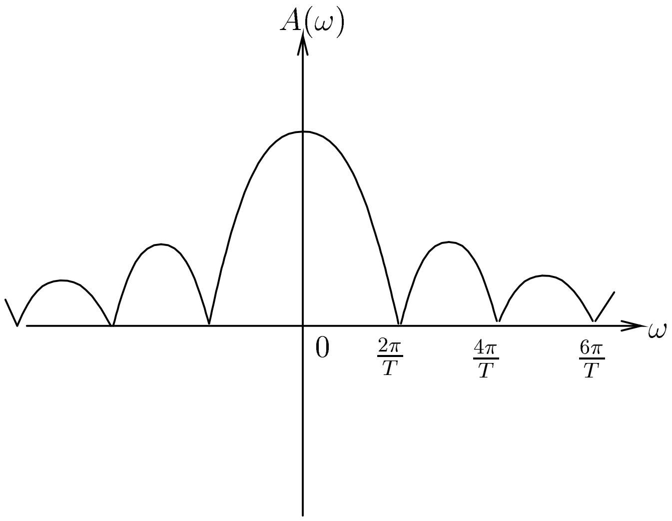
$$\begin{aligned} A(\omega) &= |\hat{h}(i\omega)| \\ &= \frac{[(1-\cos\omega T)^2 + \sin^2\omega T]^{1/2}}{|\omega T|} \\ &= [2(1-\cos\omega T)]^{1/2}/|\omega|T \\ &= \frac{|\sin(\omega T/2)|}{|\omega|T/2}, \qquad -\infty < \omega < \infty; \end{aligned}$$

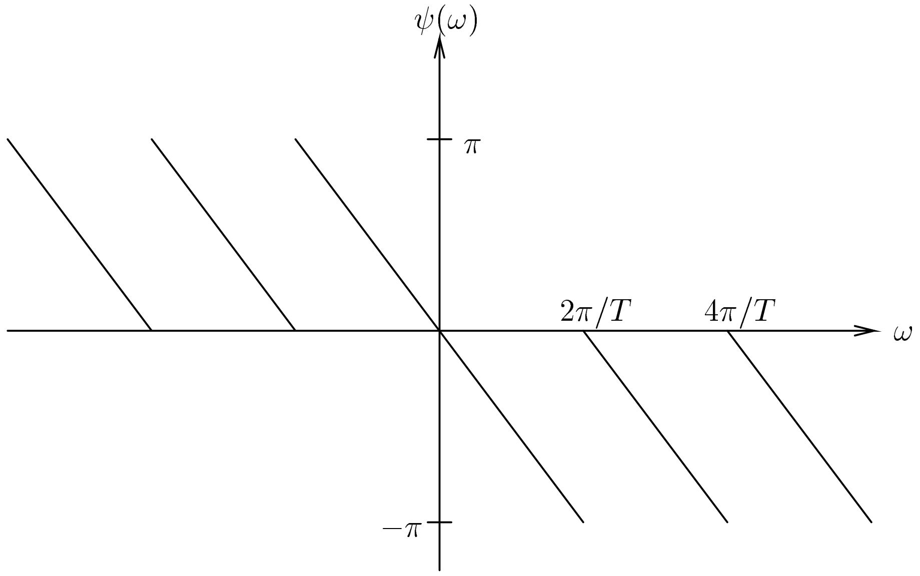
and the phase response is

$$\psi(\omega) = \arg \hat{h}(i\omega)$$

$$= \tan^{-1}\left\{\frac{-(1-\cos\omega T)}{\sin\omega T}\right\}$$

$$= \tan^{-1}(-\tan\frac{\omega T}{2}) = -\omega T/2, \qquad |\omega T| < 2\pi$$





Thus the effect of averaging (or 'smoothing') is to attenuate high frequency components, and introduce phase lag, just as you might expect.

Exercise 4.6: Explain the discontinuous phase behavior of $\psi(\omega)$ in the preceding example. **Hint:** First draw the polar plot of $\hat{h}(i\omega)$, namely the locus of the (complex) number $\hat{h}(i\omega)$ in the complex $\hat{h}$-plane as ω varies over the real interval from 0 to ∞.

Extension to 'distributional' LTIC systems

For a simple extension we'll just suppose that $h(u)$ is of the form

$$h(u) = h_0(u) + c\delta(u - \alpha),$$

defined for $u \geq 0$, with $h_0(\cdot)$ a function in PS$\cap$EXP$[0, \infty)$ as before, c an arbitrary constant, and α an arbitrary nonnegative real constant. Thus for $-\infty < t < \infty$,

$$\begin{aligned} g(t) &= \int_0^\infty h(u)f(t-u)du \\ &= \int_0^\infty h_0(u)f(t-u)dt + cf(t-\alpha) \end{aligned}$$

Notice that $\alpha \geq 0$ is required by causality.

You can easily check that the theorem on BIBO stability is essentially unchanged, the new stability condition being

$$\int_0^\infty |h_0(u)|du < \infty$$

Furthermore

$$\|g\|_\infty \leq K\|f\|_\infty$$

with

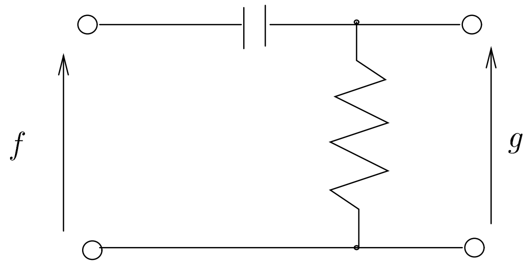
$$K = \int_0^\infty |h_0(u)|du + |c|$$

The Frequency Response Theorem is unchanged, except for the extended definition

$$\hat{h}(i\omega) = \hat{h}_0(i\omega) + ce^{-i\alpha\omega}$$

It's no longer true, however, that $\hat{h}$ vanishes at $\omega = \pm\infty$, unless of course $c = 0$.

Example: RC high-pass filter



Here

$$\frac{dg}{dt} + \beta g = \frac{df}{dt} \quad (\beta = \frac{1}{RC})$$

$$\begin{aligned}\hat{h}(s) &= \frac{s}{s+\beta} \\ &= 1 - \frac{\beta}{s+\beta} \\ h(u) &= \delta(u) - \beta e^{-\beta u}, \quad u \geq 0\end{aligned}$$

and for $-\infty < \omega < \infty$,

$$\begin{aligned}\hat{h}(i\omega) &= \frac{i\omega}{i\omega+\beta} \\ A(\omega) &= \frac{|\omega|}{(\omega^2+\beta^2)^{1/2}} \\ \varphi(\omega) &= \tan^{-1}(\beta/\omega)\end{aligned}$$

Here, by convention, $0 < \phi(\omega) < \pi/2$ for $\omega > 0$, and $\phi(\omega) = -\phi(-\omega)$ for $\omega < 0$.

Exercise 4.7: For the high-pass RC filter, sketch the polar plots of $\hat{h}(i\omega)$ (in the complex $\hat{h}$-plane, for $0 \leq \omega < \infty$), and the graphs of $h(\cdot)$, $A(\cdot)$ and $\varphi(\cdot)$. Explain how $h(\cdot)$ exhibits 'quasi-differentiating action' at low frequencies. Discuss the circuit behavior for very low and very high frequencies.

Exercise 4.8: Performance of scanning radar antenna

Referring to the scanning radar antenna of Sect. 3.3, we have the transfer function

$$\hat{h}(s) = \frac{\hat{\theta}(s)}{\hat{r}(s)} = \frac{k(s+\alpha)}{s^3 + (3+\alpha)s^2 + 3\alpha s + \alpha k}$$

Fix $\alpha = 5$. Accurately compute and plot the amplitude response function $A(\omega)$ for variable ω, $0.1 \leq \omega \leq 100$, and selected k in the stability range ($0 < k < 24$). Verify that a good design value is $k = 2.75$ (why?).

Exercise 4.9: Performance of Black's feedback amplifier

Referring to Sect. 3.4, set $\alpha = 25$. Plot $A(u)$ ($0.01 \leq u \leq 100$) for $\gamma = 0, 5, 10, 15, 20$. Notice how the amplitude response flattens out over the range $0.01 \leq u \lesssim 1$ as the loop gain γ is increased from 0 to 15. At $\gamma = 20$, however, the response exhibits a resonant peak of about 3.6 db at $u \simeq 1.15$. Repeat the experiment with $\alpha = 50$, and discuss. Note that $u = 1$ corresponds to frequency $\omega = \alpha/T$.

4.4 Ordinary Differential Equations with Constant Coefficients

Let $p(s)$, $q(s)$ be polynomials of degrees resp. n and $m \leq n$. Consider a system described by the o.d.e.

$$p\left(\frac{d}{dt}\right) g(t) = q\left(\frac{d}{dt}\right) f(t), \quad -\infty < t < \infty. \tag{1}$$

We'll assume that initial conditions are known at $t = 0$ 'just before' the input f is applied:

$$g(0) = \gamma_0, g'(0) = \gamma_1, \ldots, g^{(n-1)}(0) = \gamma_{n-1}. \tag{2}$$

Let

$$f(t) = 0, \quad t < 0, \quad \text{and} \quad f \in PS[0, \infty).$$

With these assumptions, the following is known from differential equation theory.

REPRESENTATION THEOREM: LTIC systems described by o.d.e.

For $t > 0$, $g(t)$ is given by

$$g(t) = g_0(t, \gamma) + \int_0^t h(\tau) f(t - \tau) d\tau. \tag{3}$$

Here:

(i) $g_0(t, \gamma)$ is the solution of the unforced equation

$$p\left(\frac{d}{dt}\right) g_0(t, \gamma) = 0,$$

with vector of initial values

$$\gamma := (\gamma_0, \gamma_1, \ldots, \gamma_{n-1}),$$

i.e.

$$g_0(0, \gamma) = \gamma_0, \quad g_0'(0, \gamma) = \gamma_1, \ldots, g_0^{(n-1)}(0, \gamma) = \gamma_{n-1}.$$

(ii) $h(\tau) = \mathcal{L}^{-1}\{q(s)/p(s)\}$. Since $q(s)/p(s)$ is a proper rational function, we know that h has the form

$$h(\tau) = h_0(\tau) + c\delta(\tau), \quad \tau \geq 0,$$

where $h_0 \in \mathrm{PS} \cap \mathrm{EXP}[0, \infty)$ and $c \in \mathbb{C}$.

Note: $g_0(\cdot, \cdot)$ is often called the *zero-input* solution, and $\int_0^t h(\tau) f(t - \tau) d\tau$ the *zero-state* solution.

You can find g_0 by Laplace transform as well. Suppose e.g. $n = 3$:

$$p(s) = p_0 + p_1 s + p_2 s^2 + p_3 s^3, \quad p_3 \neq 0.$$

Then

$$p_3(s^3 \hat{g}_0 - s^2\gamma_0 - s\gamma_1 - \gamma_2) \;+\; p_2(s^2 \hat{g}_0 - s\gamma_0 - \gamma_1)$$

$$+ \; p_1(s\hat{g}_0 - \gamma_0) + p_0 \hat{g}_0 = 0$$

giving

$$\hat{g}_0(s, \gamma) = \frac{p_3\gamma_0 s^2 + (p_3\gamma_1 + p_2\gamma_0)s + p_3\gamma_2 + p_2\gamma_1 + p_1\gamma_0}{p(s)}$$

There's no point memorizing a general formula - work out specific cases from scratch.

From the above formula for $\hat{g}_0(s,\gamma)$ [and its generalization to arbitrary n] you see that $\hat{g}_0(s,\gamma)$, and therefore $g_0(t,\gamma)$, *is linear with respect to the vector of initial values*

$$\gamma := (\gamma_0, \ldots, \gamma_{n-1}) \in \mathbb{C}^n.$$

If we take as the space of input forcing functions f some linear space $\mathcal{F}$, and as our space of output functions some linear space $\mathcal{G}$ (suitably large - to accommodate the actual outputs that $\mathcal{F}$ and the initial conditions generate), then you see that formula (3) really does determine a linear system operator $\mathbf{S}$,

$$\mathbf{S} : \mathbb{C}^n \oplus \mathcal{F} \to \mathcal{G}.$$

Here the input space is the 'direct sum' of the space $\mathbb{C}^n$ of initial vectors and the space $\mathcal{F}$ of forcing functions, each input being an ordered pair (γ, f).

MORAL: Linear o.d.e.'s with constant coefficients provide examples of linear, time-invariant, causal systems. A great many physical systems encountered in engineering can be successfully modeled this way.

Under what conditions is the system (so determined) BIBO stable? For this result we assume that $q(s)$ and $p(s)$ have no common factor, i.e. the poles of $\hat{h}(s)$ are precisely the zeros of $p(s)$. $p(s)$ is the *characteristic polynomial* of $\mathbf{S}$.

STABILITY THEOREM: Stability of linear o.d.e. with constant coefficients

The system $\mathbf{S}$ is BIBO stable iff all the zeros of its characteristic polynomial lie in the half-plane Re $s < 0$.

Proof:

We have that

$$\mathbf{S} : \mathbb{C}^n \oplus \mathcal{F} \to \mathcal{G}$$

where $\mathbf{S}$ has the representation

$$[\mathbf{S}(\gamma, f)](t) = g_0(t,\gamma) + \int_0^t h(u) f(t-u)\,du, \qquad t \geq 0$$

To discuss BIBO stability we define

$$\|\gamma\|_\infty := \max\{|\gamma_i|, \quad 0 \leq i \leq n-1\}$$

and

$$\|(\gamma, f)\|_\infty := \max\{\|\gamma\|_\infty, \|f\|_\infty\}$$

(If statement) Since $\hat{g}_0(s,\gamma)$ is linear in the vector γ, we have

$$\hat{g}_0(s,\gamma) = \sum_{i=0}^{n-1} \gamma_i q_i(s)/p(s)$$

where each $q_i(s)$ is a polynomial of degree $\leq n-1$ (in fact deg $q_i(s) \leq n-1-i$, as you see from the case $n=3$ displayed earlier). By hypothesis, the n zeros of $p(s)$ all lie in Re $s < 0$, so you can pick a number $\mu > 0$ such that $p(s) = 0$ implies Re $s \leq -\mu$. The result is that $\mathcal{L}^{-1}[q_i(s)/p(s)]$ is a linear combination of terms of form $t^k e^{zt}$ with Re $z \leq -\mu$, and each such term is bounded on $t \geq 0$ by, for instance, const. $e^{-\mu t/2}$ for a suitable constant. Therefore

$$|g_0(t,\gamma)| \leq \text{const.} \sum_{i=0}^{n-1} |\gamma_i| e^{-\mu t/2}, \qquad t \geq 0,$$

or

$$|g_0(t,\gamma)| \leq K_0 \;\|\gamma\|_\infty, \qquad t \geq 0$$

for a suitable constant K_0 (independent of γ).

Similarly, we know that

$$h(u) = h_0(u) + c\delta(u)$$

for some constant c, and

$$|h_0(u)| \leq \text{const. } e^{-\mu u/2}, \qquad u \geq 0$$

Therefore

$$\begin{aligned} \left|\int_0^t h(u) f(t-u) du\right| &\leq \left|c f(t) + \int_0^t h_0(u) f(t-u) du\right| \\ &\leq |c| \;\|f\|_\infty + \|f\|_\infty \text{ const.} \int_0^\infty e^{-\mu u/2} du \\ &\leq K_1 \|f\|_\infty \end{aligned}$$

for a suitable constant K_1.

Combining these results we get

$$|[\mathbf{S}(\gamma,f)](t)| \leq K_0\|\gamma\|_\infty + K_1\|f\|_\infty, \qquad t \geq 0$$

and therefore

$$\|\mathbf{S}(\gamma,f)\|_\infty \leq K\|(\gamma,f)\|_\infty$$

for a suitable constant K. This proves that $\mathbf{S}$ is BIBO stable.

(Only if statement) Suppose $\mathbf{S}$ is stable. Since $h(u) = h_0(u) + c\delta(u)$ we have, in particular, that

$$\int_0^\infty |h_0(u)| du < \infty.$$

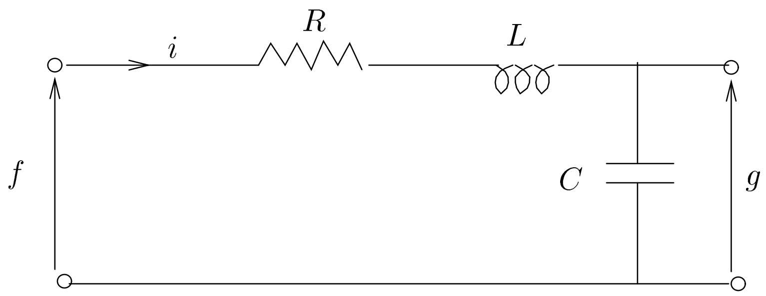
By complex function theory, $\hat{h}_0(s)$ is analytic for Re $s > 0$ [since the integral $\int_0^\infty e^{-su} h_0(u) du$ converges uniformly in the half-plane Re $s \geq 0$]. Now the *only* singularities of $\hat{h}_0(s)$ are poles. Suppose $\hat{h}_0(s)$ had a pole at $s = s^*$ with Re $s^* = 0$. Take a sequence $\{s_\nu : \nu = 1, 2, \ldots\}$ with Re $s_\nu > 0$, such that $s_\nu \to s*$ [e.g. $s_\nu = s^* + \nu^{-1}$]. Then $|\hat{h}_0(s_\nu)| \to \infty$. But for all ν,

$$\begin{aligned} |\hat{h}_0(s_\nu)| &= \left|\int_0^\infty e^{-s_\nu \tau} h_0(\tau) d\tau\right| \\ &\leq \int_0^\infty |h_0(\tau)| d\tau < \infty, \end{aligned}$$

a contradiction. So all poles of $\hat{h}_0(s)$ must lie in Re $s < 0$. Q.E.D.

Example: RLC low-pass filter

Lumped, time-invariant linear systems and circuits provide familiar applications of the foregoing theory. Consider the low-pass filter sketched below.



We have by Kirchhoff's voltage law:

$$\begin{aligned} f(t) &= Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t')dt' \\ g(t) &= \frac{1}{C}\int_{-\infty}^{t} i(t')dt' \end{aligned}$$

Thus

$$\frac{dg(t)}{dt} = \frac{1}{C}i(t), \quad \frac{d^2g(t)}{dt^2} = \frac{1}{C}\frac{di(t)}{dt}$$

from which

$$LC\frac{d^2g(t)}{dt^2} + RC\frac{dg(t)}{dt} + g(t) = f(t)$$

or

$$\frac{d^2g(t)}{dt^2} + \frac{R}{L}\frac{dg(t)}{dt} + \frac{1}{LC}g(t) = \frac{1}{LC}f(t)$$

Put $\omega_0 = 1/\sqrt{LC}$, $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$; then

$$\frac{d^2g(t)}{dt^2} + 2\omega_0\zeta\frac{dg(t)}{dt} + \omega_0^2 g(t) = \omega_0^2 f(t)$$

To apply the theorem on LTIC systems described by o.d.e., let

$$\begin{aligned} p(s) &= s^2 + 2\omega_0\zeta s + \omega_0^2 \\ q(s) &= \omega_0^2 \\ g(0) &= \gamma_0, \quad g'(0) = \gamma_1 \end{aligned}$$

To solve for g_0, we have for $\hat{g}_0$:

$$s^2\hat{g}_0 - s\gamma_0 - \gamma_1 + 2\omega_0\zeta(s\hat{g}_0 - \gamma_0) + \omega_0^2\hat{g}_0 = 0$$

or

$$\hat{g}_0(s) = \frac{s\gamma_0 + 2\omega_0\zeta\gamma_0 + \gamma_1}{p(s)}$$

The roots of $p(s)$ are given by

$$s = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_0 \qquad (*)$$

Assuming for definiteness that $R = 2$, $L = 1$, $C = 1/4$, we get

$$\zeta = \frac{1}{2}, \quad \omega_0 = 2$$

and the roots

$$s = -1 \pm i\sqrt{3}$$

So

$$\hat{g}_0(s) = \frac{(s+2)\gamma_0 + \gamma_1}{(s+1+i\sqrt{3})(s+1-i\sqrt{3})}$$

By residues we get

$$\begin{aligned} g_0(t) &= 2\mathrm{Re}[\text{ Residue } \hat{g}(s)e^{st} \text{ at } s = -1 + i\sqrt{3}] \\ &= 2\mathrm{Re}\left[\frac{(1+i\sqrt{3})\gamma_0 + \gamma_1}{2i\sqrt{3}} e^{(-1+i\sqrt{3})t}\right] \\ &= \left[\gamma_0 \cos(\sqrt{3}t) + \frac{(\gamma_0 + \gamma_1)}{\sqrt{3}} \sin(\sqrt{3}t)\right] e^{-t} \end{aligned}$$

Next we obtain

$$\begin{aligned} \hat{h}(s) &= \omega_0^2/p(s) \\ &= \frac{4}{(s+1+i\sqrt{3})(s+1-i\sqrt{3})} \end{aligned}$$

which yields similarly

$$h(\tau) = (4/\sqrt{3})e^{-\tau}\sin(\sqrt{3}\tau)$$

By the representation theorem, we conclude that for $t > 0$,

$$g(t) = g_0(t) + \int_0^t h(\tau)f(t-\tau)d\tau$$

It's clear from the stability theory that the filter is BIBO stable. Its complex frequency response function therefore exists and is

$$\hat{h}(i\omega) = \frac{4}{4 - \omega^2 + 2i\omega}$$

Exercise 4.10: For the RLC low-pass filter, fix $\omega_0 = 2$ but let ζ vary, $0 \leq \zeta < \infty$. Show that

$$\hat{h}(i\omega) = \frac{4}{4 - \omega^2 + i4\zeta\omega}, \qquad -\infty < \omega < \infty$$

Next show that $A(\omega) := |\hat{h}(i\omega)|$ has a maximum on $0 \leq \omega < \infty$, at

$$\omega_M := \begin{cases} 2\sqrt{1 - 2\zeta^2} & \text{if } 0 < \zeta \leq 1/\sqrt{2} \\ 0 & \text{if } \zeta \geq 1/\sqrt{2} \end{cases}$$

Calculate $A(\omega_M)$ as a function of ζ. For each of a set of fixed values of ζ in $0.1 \leq \zeta \leq 1.0$, display the polar plot of $\hat{h}(i\omega)$ in the complex plane, as well as the plots of $A(\omega)$ and $\psi(\omega) := \arg[\hat{h}(i\omega)]$, for $0 \leq \omega < \infty$. **Suggestion:** Pick equi-log-spaced values of ζ and ω, according to

$$\begin{aligned} \zeta &= 0.1 \star \exp(j \star \ln(10)/10), \quad j = 0, 1, ...10 \\ \omega &= 0.001 \star \exp(k \star \ln(10000)/2000), \quad k = 0, 1, ..., 2000 \end{aligned}$$

For $A(\omega)$ use a db scale, i.e. plot $20\log_{10}[A(\omega)]$, and for $\psi(\omega)$ use a linear scale; note that $0 \geq \psi(\omega) > -\pi$ for $0 \leq \omega < \infty$.

Exercise 4.11: Gain of underdamped RLC low-pass filter

Consider again the RLC low-pass filter described above. Our aim is to investigate the dependence of the 'gain' of this LTIC system as a function of the damping ratio ζ, over the underdamped range $0 \leq \zeta \leq 1$. With $L = 1$, $C = 1/4$ fixed, we have $\omega_0 = 2$ and $\zeta = R/4$; thus $0 \leq R \leq 4$. For ζ in the range indicated, you may obtain the two (complex conjugate) roots of $p(s)$ as in the formula (*) given on p. 164. Denoting the roots by $s_0, \bar{s}_0$ you then have for the transfer function

$$\hat{h}(s) = \omega_0^2/(s - s_0)(s - \bar{s}_0)$$

From this you may calculate the impulse response function $h(\tau;\zeta)$ as a function of τ ($0 \leq \tau < \infty$) with parameter ζ ($0 \leq \zeta \leq 1$). Note that $h(\tau;1)$ should be obtained as the limit of your expression for $h(\tau;\zeta)$ as $\zeta \to 1$. Next calculate the 'gain' of the system as the L_1-norm of h, namely the function

$$\text{gain}(\zeta) := \int_0^\infty |h(\tau;\zeta)| d\tau$$

For this you may find it convenient first to work out the definite integral

$$\int_0^\infty e^{-\alpha x} |\sin(x)| dx \qquad (\alpha > 0)$$

which can be done by hand. Then identify $\alpha = \zeta/(1 - \zeta^2)^{1/2}$. Finally

(i) Calculate lim gain(ζ) as $\zeta \to 0$ and as $\zeta \to 1$.

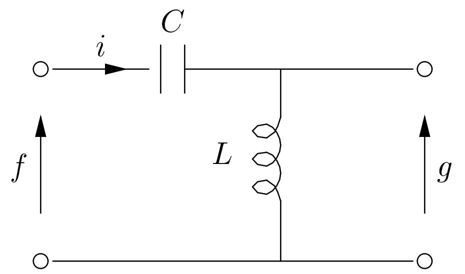
(ii) Find an approximate expression for gain(ζ) when $\zeta > 0$ is small.

(iii) Compute and plot gain(ζ) accurately for $0.01 \le \zeta \le 1$. A log-log plot is best.

(iv) Interpret and discuss your results.

Example: LC high-pass filter

Consider the high-pass filter sketched below.



By Kirchhoff's voltage law,

$$\begin{aligned} f(t) &= \frac{1}{C}\int_{-\infty}^{t} i(t')dt' + L\frac{di(t)}{dt} \\ g(t) &= L\frac{di(t)}{dt} \end{aligned}$$

which yields

$$\frac{d^2 g(t)}{dt^2} + \frac{1}{LC}g(t) = \frac{d^2 f(t)}{dt^2}$$

Let $LC = 1/4$, $g(0) = \gamma_0$, $g'(0) = \gamma_1$. To obtain the system output $g(t)$ for $t > 0$ we first calculate $\hat{g}_0(s)$ according to

$$s^2\hat{g}_0 - s\gamma_0 - \gamma_1 + 4\hat{g}_0 = 0$$

or

$$\hat{g}_0(s) = \frac{s\gamma_0 + \gamma_1}{p(s)}$$

where

$$p(s) = s^2 + 4$$

Thus, writing $\gamma = [\gamma_0, \gamma_1]$, we get

$$\begin{aligned} \hat{g}_0(s;\gamma) &= \gamma_0\frac{s}{s^2+4} + \frac{\gamma_1}{2}\frac{2}{s^2+4} \\ g_0(t;\gamma) &= \gamma_0\cos(2t) + \frac{\gamma_1}{2}\sin(2t) \end{aligned}$$

Also

$$\begin{aligned} \hat{h}(s) &= \frac{q(s)}{p(s)} = \frac{s^2}{s^2+4} \\ &= 1 - \frac{4}{s^2+4} \end{aligned}$$

so that

$$h(t) = \delta(t) - 2\sin(2t)$$

By the Representation Theorem, for $t > 0$

$$\begin{aligned} g(t) &= g_0(t,\gamma) + (h \star f)(t) \\ &= g_0(t,\gamma) + \int_0^t h(u)f(t-u)du \\ &= g_0(t,\gamma) + \int_0^t [\delta(u) - 2\sin(2u)]f(t-u)du \\ &= \gamma_0\cos(2t) + \frac{\gamma_1}{2}\sin(2t) + f(t) - 2\int_0^t \sin[2(t-u)]f(u)du \end{aligned}$$

To identify γ_0, γ_1 in terms of physical quantities, we take the appropriate limits as $t \uparrow 0$. Assuming $f(t) = 0$ $(t < 0)$ we get from the voltage equation,

$$0 = \frac{1}{C}\int_{-\infty}^0 i(t')dt' + \gamma_0$$

i.e. γ_0 is the voltage across the capacitor at $t = -0$. Similarly

$$\gamma_1 = (-1/C)i(-0).$$

Finally, as

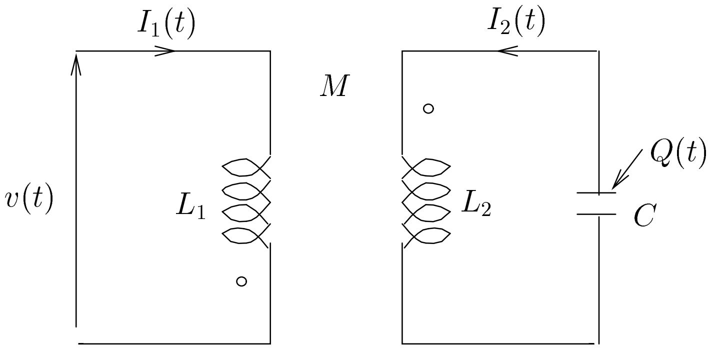
$$\int_0^\infty |h(u)|du = +\infty$$

we see that this system is not BIBO stable; its frequency response function is not defined.

Exercise 4.12: Instability of LC high-pass filter

For the foregoing example, calculate $g(t)$ $(t > 0)$ when $\gamma_0 = \gamma_1 = 0$ and $f(t) = \sin(2t)$ $(t > 0)$.

Exercise 4.13: LC-resonance



In the circuit shown the parameter values are

$$L_1 = \frac{4}{3}, \quad L_2 = 1, \quad M = 1, \quad C = \frac{1}{4}.$$

The input signal $v(t) = \cos 2t (t \geq 0)$ and is zero for $t < 0$. For $t < 0$ the currents $I_1(t)$, $I_2(t)$ and the charge $Q(t)$ are also identically zero.

Show by a Laplace transform analysis that for $t \geq 0$ the current $I_1(t)$ is sinusoidal with angular frequency 4.

How would you explain this strange (?) behavior physically?

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Appendix: Miscellaneous Exercises

1. For each of the sequences $\{f_N(t)\}$ below answer the following questions.

   (a) Does $f_N(t)$ converge (to a finite limit as $N \to \infty$) for each $t \in I$? If not, for which t does it fail to converge?

   (b) If it converges for all $t \in I$, is convergence uniform? Justify your answer.

   (i) $I = [0,1]$, i.e. $0 \le t \le 1$

   $$f_N(t) = \begin{cases} N - N^2 t, & 0 \le t \le 1/N \\ 0, & 1/N \le t \le 1 \end{cases}$$

   (ii) $I = (0,1]$, i.e. $0 < t \le 1$

   f_N as in (i)

   (iii) $I = [1/4, 1]$, i.e. $1/4 \le t \le 1$

   f_N as in (i).

2. (i) Carefully verify the statement (p.27, Notes) that

   $$\psi(u) \to 0 \text{ as } u \to 0,$$

   where

   $$\psi(u) := \begin{cases} \frac{1}{2\sin(u/2)} - \frac{1}{u}, & 0 < u \le \pi \\ 0, & u = 0 \end{cases}$$

   (ii) Sketch the graph of $\psi(u)$ for $0 \le u \le \pi$.

3. A periodic square wave $f(t)$ whose base frequency is 1 MHz has d.c. level 2 volts, peak-to-peak range 5 volts, and one of its positive half-cycles starts at $t = 2$ (time t is measured in μsec). Suppose its standard Fourier series is

   $$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)],$$

   where the a_n and b_n are assumed known. Consider another square wave $g(t)$ with base frequency 3 MHz, d.c. level 1 volt, peak-to-peak range 6 volts, and a positive half-cycle starting at $t = -\sqrt{2}$. If

$$g(t) \sim \frac{a'_0}{2} + \sum_{n=1}^{\infty} [a'_n \cos(n\omega'_0 t) + b'_n \sin(n\omega'_0 t)],$$

find the parameters a'_n, b'_n, ω'_0.

4. Let $f(\cdot)$, $g(\cdot)$ be PS functions, periodic with period 2π, on $-\infty < t < \infty$. The *convolution* of f and g, written $f * g$, is the function

$$(f * g)(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t')g(t - t')dt', \quad -\infty < t < \infty.$$

(i) Show that $f * g$ is periodic with period 2π.

(ii) Let f, g have CEFS

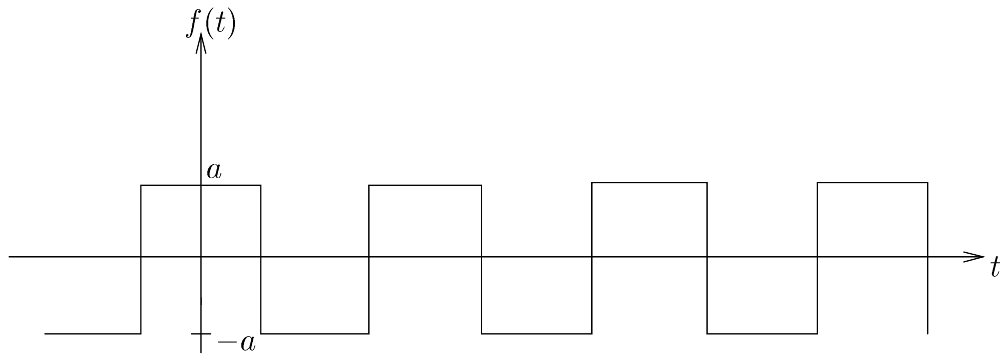
$$f(t) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{int}$$

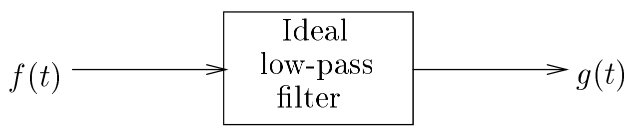
$$g(t) \sim \sum_{n=-\infty}^{\infty} \beta_n e^{int}$$

Find the CEFS of $f * g$.

(iii) Assuming $\sim$ can be replaced by $=$, find the CEFS of the product $f(t)g(t)$, in terms of the coefficients $\{\alpha_n\}$, $\{\beta_n\}$.

5.





A periodic square wave of base frequency 60 Hz is passed through an ideal low-pass filter that cuts off at frequency ν Hz. In other words

$$g(t) = \sum \begin{array}{l}\{\text{all sinusoidal components of } f \text{ having} \\ |\text{frequency}| < \nu \text{ Hz } \}.\end{array}$$

Estimate how large ν must be in order that the power of $g - f$ be no greater than 1% of the power of f.

HINT

1. Calculate FS of f, choosing phase according to convenience.
2. Calculate $\|f\|^2$ exactly, and $\|f - f_N\|^2$ as a series $\sum_{N+1}^{\infty} \dots$.
3. Show that
$$\|f - f_N\|^2 \leq \frac{\text{suitable constant}}{N}$$
4. Show that
$$\frac{\|f - f_N\|^2}{\|f\|^2} \leq 0.01 \text{ if } N \geq ?$$
5. Conclusions?

6. Let $f(t)$ be a periodic, odd function of t defined for $-\infty < t < \infty$ and having fundamental frequency ν Hz.
   (a) Write general expressions for the coefficients of the Fourier trigonometric series for f, taking into account the fact that f is odd.
   (b) State Parseval's theorem as applied to the series in part (a).
   (c) Consider a periodic square wave, of form
   $$f(t) = \begin{cases} 1, & 0 \leq t < \tau \\ -1, & \tau \leq t < 2\tau \end{cases}$$
   on the basic interval $0 \leq t \leq 2\tau$. Suppose $f(\cdot)$ has fundamental frequency 10 Hz. Applying the results of (a) and (b), estimate the minimum bandwidth of an ideal low-pass filter which would transmit $f(\cdot)$ with at most 5% loss of power.

7. The Fourier series of a certain function f is
$$f(t) = c + \sum_{n=1}^{\infty} \left(\frac{2}{n^2} \cos nt + \frac{(-1)^n}{n^3} \sin nt \right), \quad -\pi \leq t \leq \pi.$$
The number c is such that $\|f\| = 4$.

   Estimate (in a conservative but reasonable way) the number of terms of the series that should be taken to ensure a normalized r.m.s. approximation error of less than 2% (to normalize, divide r.m.s. (error) by r.m.s. (f)).

8. Let $c(\cdot)$ be a signal defined by

$$c(t) = \begin{cases} 1, & 0 \le t < \pi \\ -1, & \pi \le t < 2\pi \end{cases}$$

and by periodic extension for $t \ge 2\pi$.

(i) Compute the complex exponential Fourier series (CEFS) for $c(\cdot)$.

(ii) Define $f(\cdot)$ according to

$$f(t) = \begin{cases} t, & 0 \le t \le \pi \\ 2\pi - t, & \pi \le t \le 2\pi \end{cases}$$

and by periodic extension for $t \ge 2\pi$. From the result of part (i), compute the CEFS for $f(\cdot)$.

[Fact: $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$.]

(iii) The signal $f(\cdot)$ of part (ii) is input to a low-pass RC filter whose output $g(\cdot)$ is given by

$$T\frac{dg}{dt} + g = f.$$

Here $T > 0$ is the filter time constant. Find the CEFS for the steady-state response $g(\cdot)$ [i.e. the response for large t].

(iv) From the result of part (iii), write a formula for the average power of $g(\cdot)$ [as an infinite series].

(v) If $T = 10^{-3}$, estimate the minimum bandwidth of an ideal low-pass filter that would transmit $g(\cdot)$ with loss of average power no greater than 1% of the d.c. power of $g(\cdot)$.

9. Let $f(t) = (1-t)e^{t^2}$, $0 \le t \le 1$. Let the Fourier trigonometric series for f be, as usual,

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t). \qquad (FS)$$

For this FS answer the following:

(i) Determine ω_n.

(ii) Write down formulas for a_0, a_n and b_n [but don't evaluate the integrals].

(iii) At what points t in [0,1] does FS converge to $f(t)$?

(iv) At what points t in [0,1] does FS fail to converge to $f(t)$? What *does* FS converge to, if anything, at such points?

(v) Define what is meant by uniform convergence of a series of functions.

(vi) What can be said about uniform convergence of this FS?

(vii) Define what is meant by mean square convergence of a series of functions.

(viii) What can be said about mean square convergence of this FS?

(ix) Does this FS exhibit Gibbs' phenomenon? If so, illustrate by a sketch, and determine the approximate overshoot or undershoot of the Gibbs 'peak'.

(x) Specify the function g whose Fourier Series would be the Fourier *sine* series of f.

10. Let $\{\phi_n : n = 1, 2, \ldots\}$ be a fixed ON sequence on a finite interval I. Consider the Fourier Series of functions $f \in L_2(I)$, relative to $\{\phi_n\}$. Suppose you know that

(1) $(\forall f, \delta) f \in L_2$ and $\delta > 0 \Rightarrow (\exists g).\ g \in PS(I)$ and $\|f - g\| < \delta$.

(2) Parseval's theorem holds for any $g \in PS(I)$.

Prove that $\{\phi_n\}$ is complete, i.e.

$$(\forall f) : f \in L_2 \text{ and } (\forall n) < f, \phi_n >= 0 \Rightarrow f = 0 \text{ a.e.}$$

HINT: Note that, if

$$(\forall n) < f, \phi_n >= 0$$

and $g \in PS$ is arbitrary, then

$$\begin{aligned} \|f\| &= \|f - f_N\| \text{ (why?)} \\ &\leq \|f - g_N\| \text{ (why?)} \\ &= \|(f - g) + (g - g_N)\| \\ &\leq \|f - g\| + \|g - g_N\| \text{ (why?)} \end{aligned}$$

Now use (1) and (2) to show that, for any $\varepsilon > 0$, it must be true that $\|f\| < \varepsilon$. So?

11. Let $f(\cdot)$ be a real-valued function of t, $-\infty < t < \infty$, and assume that $f(\cdot) \in PS_1(-\infty, \infty)$. If $\hat{f}(\cdot)$ is the Fourier transform of f, write

$$\hat{f}(\omega) = F(\omega)e^{i\theta(\omega)}, \quad -\infty < \omega < \infty,$$

where $F(\omega) := |\hat{f}(\omega)|$ and $\theta(\omega) := arg\hat{f}(\omega)$. [$F$ is called the *amplitude spectrum*, and θ the *phase spectrum*, of f]. Show that

$$\overline{\hat{f}(\omega)} = \hat{f}(-\omega).$$

Then show that $F(\cdot)$ is an even function of ω, and $\theta(\cdot)$ is an odd function of ω. Restrict $\omega \neq 0$ and $|\theta| < 2n\pi$ for suitable n.

12. Let $f(\cdot)$ be the periodic square wave defined for $-\infty < t < \infty$ by

$$f(t) = \begin{cases} 1, & 0 \le t < T \\ -1, & -T \le t < 0 \end{cases}$$

(i) Calculate the complex exponential Fourier series for f. Now let $g(\cdot) \in PS_1(-\infty, \infty)$, and define

$$h(t) = f(t)g(t), \quad -\infty < t < \infty.$$

(ii) The function $h(\cdot) \in PS_1(-\infty, \infty)$. Why?

(iii) Show that $\hat{h}(\omega)$ can be represented by an infinite series involving the Fourier coefficients of f, and the Fourier transform of g.

(iv) Assume $\hat{g}(\omega) = 0$ if $|\omega| > \frac{\pi}{2T}$.
Sketch the graph of Re $\hat{h}(\omega)$, for $-\infty < \omega < \infty$.

13. (i) Let $h(t)$ be a real-valued function in $L_1(-\infty, \infty)$. Write

$$h_e(t) := \frac{1}{2}\{h(t) + h(-t)\}$$

$$h_0(t) := \frac{1}{2}\{h(t) - h(-t)\}$$

for the *even*, and *odd*, parts of h, respectively. Show that

$$\hat{h}_e(\omega) = \text{Re } \hat{h}(\omega)$$

$$i^{-1}\hat{h}_0(\omega) = \text{Im } \hat{h}(\omega).$$

(ii) A function $h(t)$ is said to be *causal* if $h(t) = 0$ for all $t < 0$. Suppose that $h(t)$ is a real-valued, causal function such that

$$\text{Re } \hat{h}(\omega) = \frac{\sin^2(\omega/2)}{(\omega/2)^2}$$

Use the result of Part (i) to find Im $\hat{h}(\omega)$.

FACT:

$$\int_0^1 (1-x)\cos(2\alpha x)dx = \frac{\sin^2 \alpha}{2\alpha^2}$$

$$\int_0^1 (1-x)\sin(2\alpha x)dx = \frac{\alpha - \sin\alpha\cos\alpha}{2\alpha^2}$$

14. Let $f(\cdot)$ and $g(\cdot)$ belong to $PS_1(-\infty, \infty)$.

(i) What is the convolution h of f and g? Illustrate its meaning by a sketch.

(ii) Derive an expression for the Fourier transform of h in terms of the transforms of f and g. [You may take for granted the validity of interchanging order of integration.]

(iii) State the Fourier integral representation theorem for f.

(iv) Find the function f given that its Fourier transform is

$$\hat{f}(\omega) = \frac{e^{3i(\omega-1)}}{1 - 2i(\omega - 1) - (\omega - 1)^2}, \quad -\infty < \omega < \infty.$$

(v) What does Plancherel's theorem have to say about the function f of Part (iv)? [You can answer this without knowing f, but only $\hat{f}$ as given!]

15. (i) Carefully state the theorem on Fourier integral representation for functions $f \in PS_1(-\infty, \infty)$.

(ii) A function $f \in PS_1(-\infty, \infty)$ is sampled periodically at a rate of ν samples per second. Derive a formula for the spectrum of the sampled signal (i.e. of the signal-modulated pulse train).

(iii) Suppose f is bandlimited to the frequency range

$$16\pi \leq |\omega| \leq 20\pi \text{ rad./sec.}$$

If f is sampled periodically at a rate of 5 samples/sec., sketch clearly the spectrum of the sampled signal f_s over the frequency range

$$-20\pi \leq \omega \leq 20\pi \text{ rad./sec.}$$

[Express frequency in Hz. if convenient].

(iv) The sampled signal f_s of part (iii) is passed through an ideal lowpass filter with passband $-5\pi \leq \omega \leq 5\pi$ rad./sec. Call the output of this filter, g. Explain how the original signal f could be reconstructed from g.

16. A signal $f(t)$ defined on $-\infty < t < \infty$, but time-limited to $|t| \leq 1$, has Fourier transform

$$\hat{f}(\omega) = \frac{2}{\omega^2}(\sin \omega - \omega \cos \omega), -\infty < \omega < \infty$$

(a) Show carefully that at least a fraction α (where $0 < \alpha < 1$ and $1 - \alpha$ is small) of the total energy of f is contained in the frequency range $|\omega| \leq \Omega$, where

$$\Omega := (16/\pi)(1 - \alpha)^{-1}\|f\|^{-2}$$

[Hint: $|\hat{f}(\omega)| \leq$ *what* simple function for large $|\omega|$?]

(b) Define a new signal

$$g(t) := \sum_{k=-\infty}^{\infty} f(t - 2k), \quad -\infty < t < \infty.$$

Find the general coefficient c_n of the complex exponential Fourier series of g.

(c) If you know that g has r.m.s. value $1/\sqrt{3}$, what value of Ω (part (a)) would ensure that you trap at least 90% of the total energy of f in the band $|\omega| \leq \Omega$?

17. Show that if a signal $f(t)(-\infty < t < \infty)$ is bandlimited to $|\omega| \leq \Omega$, then for all t,

$$|f(t)| \leq \sqrt{\frac{\Omega}{\pi}} \|f\| \quad (\|f\| \text{ is the } L_2\text{-norm})$$

Show that equality is attained for some time t if and only if, for some t_0, and some constant c with $|c| = 1$,

$$f(t) = c \frac{\|f\|}{\sqrt{\pi\Omega}} \frac{\sin \Omega(t - t_0)}{t - t_0}$$

HINT: Represent $f(t)$ in terms of $\hat{f}$ and use Schwarz.

18. A disk rotating at angular frequency 1 radian/sec. is viewed by a strobe light that flashes every $3\pi/2$ sec. What are the direction and speed of the *apparent* rotation?

HINT: Let $f(t)$ be the position of a distinguished point on the perimeter and compute the Fourier transform of f after sampling. Also, postulate that the eye-brain system processes visual inputs in the "simplest" way.

19. Let $f \in L_2(-\infty, \infty)$. In the notation for the time-bandwidth uncertainty principle (p. 93), let

$$t_0 = < t >_f, \omega_0 = < \omega >_{\hat{f}} \ .$$

Define

$$g(t) = e^{-i\omega_0(t+t_0)} f(t + t_0), \quad -\infty < t < \infty.$$

Show that

$$< t >_g = 0, < \omega >_{\hat{g}} = 0$$

and that

$$(\text{var}t)_g = (\text{var}t)_f = < t^2 >_g$$
$$(\text{var}\omega)_{\hat{g}} = (\text{var}\omega)_{\hat{f}} = < \omega^2 >_{\hat{g}} \ .$$

20. Find the signal $f(t)$, $0 \leq t < \infty$, whose Laplace transform is

$$\hat{f}(s) = s^{-2}(1 + e^{-s})^{-1}(1 - e^{-s})$$

21. For the Scanning Radar Antenna (p. 122) suppose a UFO appears at reference angle given by

$$r(t) = 1 - t, \quad t \geq 0.$$

Assuming k and α are such that the system is stable, and that $\theta_0 = \theta_1 = 0$, obtain an expression for $\theta(t)$ when t is large (i.e. decaying transients are negligible). What does your expression reduce to when the angle sensor is perfect [i.e. $\alpha \to \infty$]? In this case, what is then the effect of varying the gain k?

HINT: From $\hat{\theta}(s)/\hat{r}(s)$ (p. 124) calculate $\hat{e}_{true}(s) := \hat{r}(s) - \hat{\theta}(s)$, and then $\lim_{t\to\infty} e(t)$.

22. Find the inverse Laplace transform of

$$\hat{f}(s) = \frac{1 - e^{-s}}{s^2(1 + e^{-s})}.$$

23. Without working out $f(t)$ in detail, describe the essential qualitative features of the behavior of $f(t)$ for *large* t, given that

$$\hat{f}(s) = \frac{3s^2 - 4s + 5}{(s^4 + 4)(s^2 + 2s + 3)}.$$

24. An ideal averaging (or 'smoothing') device has input-output representation

$$(Sf)(t) = \int_{-\infty}^{\infty} h(u)f(t-u)du, \quad -\infty < t < \infty$$

where

$$h(u) = \begin{cases} 1 - |u|, & |u| \leq 1 \\ 0, & |u| \geq 1 \end{cases}$$

Is the system causal? time-invariant? BIBO stable?

25. (a) A certain time-invariant linear system has input-output representation

$$g(t) = \int_{-\infty}^{\infty} h(t, \tau) f(\tau) d\tau, \quad -\infty < t < \infty.$$

Show that there exists a function $\tilde{h}$ such that $h(t, \tau) = \tilde{h}(t - \tau)$ for all t, τ. Also, state under what conditions this system will be (i) BIBO stable, or (ii) causal.

(b) An ideal averaging device has input-output description

$$g(t) = \frac{1}{2T} \int_{t-T}^{t+T} f(u)du, \quad -\infty < t < \infty$$

Here $T > 0$ is a parameter. Find the amplitude and phase responses as functions of frequency. How large should T be taken to ensure that all frequencies above 100 Hz. are attenuated by at least 40 db.?

26. A continuous signal $f(t)$ defined for $t \geq 0$ is processed by a sample-and-hold circuit (SHC), which gives the output

$$g(t) := f(n) \text{ for } n \leq t < n + 1; \quad n = 0, 1, 2, \ldots.$$

(a) Assuming $f \in \mathrm{EXP}[0, \infty)$, find a general expression for the Laplace transform $\hat{g}(s)$, in the form of an infinite series. For what values of s is the series guaranteed to converge?

(b) Regarded as a system, is SHC linear? causal? time-invariant? stable? Justify your answers with care.

(c) For a certain *polynomial* $f(t)$, you're given that

$$\hat{g}(s) = s^{-1}(e^s - 1)^{-1}.$$

What is $f(t)$? [**HINT:** Expand everything in sight in powers of e^{-s}].

27. A linear, time-invariant filter is said to be *matched* to a specific, real-valued input $f(t)(-\infty < t < \infty)$ if its impulse response function $h(u)$ is given by

$$h(u) = Kf(-u + \tau), \quad -\infty < u < \infty,$$

where K and τ are positive constants.

(a) Calculate the Fourier transform $\hat{h}(\omega)$ of h, in terms of the Fourier transform $\hat{f}(\omega)$ of f.

(b) Show that the output $g(t)$ of the filter has Fourier transform

$$\hat{g}(\omega) = Ke^{-i\omega\tau}|\hat{f}(\omega)|^2.$$

(c) Calculate $g(\tau)$ in terms of the input energy.

(d) Show that $g(t)$ has a *maximum* at $t = \tau$. [**HINT:** Don't try to differentiate, but do use simple properties of $|\hat{f}(\omega)|^2$.]

28. A finite-memory moving average (MA) is a data-smoothing operation described by the input-output relation

$$g(t) = \frac{1}{T}\int_{t-\alpha}^{t-\alpha+T} f(u)du, \quad -\infty < t < \infty.$$

Here $T > 0$ and $\alpha \in (-\infty, \infty)$ are parameters, and the input $f(u)$ is defined for $-\infty < t < \infty$.

(i) Determine for what values of T and α, in the ranges indicated, the system MA

(a) is *not* BIBO stable

(b) is causal

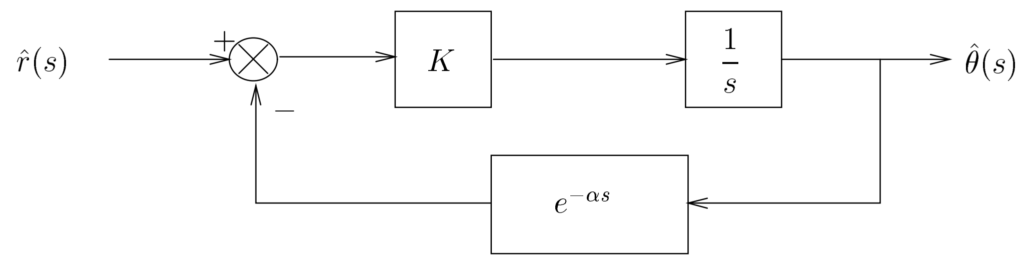
(c) is time-invariant

(ii) Let $f(u) = e^{i\omega u}$, $-\infty < u < \infty$. Assume $\alpha = T$. Find $g(t)$ in the form

$$g(t) = A(\omega)e^{i[\omega t + \phi(\omega)]}, \quad -\infty < t < \infty$$

where $A(\omega) = |g(t)|$. Sketch the graphs of $A(\omega)$ and $\phi(\omega)$ as functions of ω, $-\infty < \omega < \infty$. How do your results reflect the intuitive idea of a 'smoothing' operation?

29.



In the control system shown, the forward-path gain K and the feedback delay α are positive parameters.

(i) Find the transfer function $\hat{h}(s)$ relating $\hat{\theta}(s)$ to $\hat{r}(s)$.

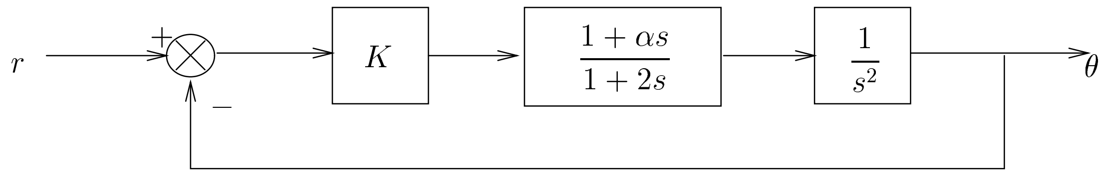
It is known that the only singularities of $\hat{h}(s)$ are poles, and that the system is stable if and only if all these poles lie in the region Re $s < 0$.

(ii) For $K = \alpha = 1$, the system is stable. Find the response $\theta(t)$ for *large* t, if $r(t) = 1 + t$ $(t \geq 0)$.

(iii) For $K = \alpha = 1$, find the amplitude and phase functions of the steady-state frequency response. Estimate the lowest frequency above which the amplitude response is at least 40 db. down from its value at zero frequency.

(iv) If $\alpha = 1$, the *critical gain* $K = K_c$ at which instability just occurs is $K_c = \pi/2$, i.e. the system is stable for $K < K_c$, unstable for $K \geq K_c$. Find a general expression for the critical gain $K_c = K_c(\alpha)$ as a function of α. In the (K, α)-plane, sketch the *stability boundary* defined by the locus $K - K_c(\alpha) = 0$, and label the regions of stability and instability.

30. Consider the tracking system with block diagram shown. Time is measured in seconds.



(a) Suppose $\alpha = 0.9$ and $K = 10$. When the input $r(t)$ is a sharp pulse, an astute observer notices that the output $\theta(t)$ includes a decaying transient component with time constant 1 second. Determine whether or not the system is BIBO stable.

(b) For $\alpha = 3$, the system is known to be BIBO stable for all positive values of K. If $r(t) = \sin \omega t$, write an expression for the amplitude $A(\omega)$ of the steady-state output response.

(c) From the result of part (b), develop a simple approximate formula for $A(\omega)$, valid when $|\omega| \gg 1$. With the aid of this formula, determine an approximate value of K such that, at frequency 10 Hz, the output amplitude has dropped to 60 db. below its value at zero frequency.

31. Let $f(t), -\infty < t < \infty$, be a periodic function, with basic interval $-1 \leq t \leq 1$, such that the coefficients of its complex exponential Fourier series are given by

$$\begin{aligned} c_0 &= 1/2 \\ c_n &= (1-(-1)^n)/(n\pi)^2, \quad n \neq 0 \end{aligned}$$

Assume that this FS converges to $f(t)$ at all t.

(i) Is f purely real-valued, or purely imaginary-valued, or neither? Justify your answer.

(ii) Is f an even function, or an odd function, or neither? Justify your answer.

(iii) Evaluate $f(1)$.

(iv) Write down an infinite series for the total power of f. From the first two terms, obtain a constant P such that

$$\text{power of } f \geq P.$$

(v) It is desired to approximate f by the Nth partial sum of its complex exponential Fourier series. Find a conservative but reasonable formula for N, such that

$$\sqrt{\frac{\text{power of approximation error}}{\text{power of } f}} \leq \alpha$$

In particular, estimate N if $\alpha = 0.01$.

(vi) Would you expect the Fourier series to exhibit Gibbs' phenomenon? (No computation is needed, but justify your answer).

32. Let $f(t)$, $-\infty < t < \infty$, be defined by

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

As usual, denote the Fourier transform of f by $\hat{f}(\omega)$ (but don't bother to calculate it!). Let $g(t)$, $-\infty < t < \infty$, have Fourier transform

$$\hat{g}(\omega) = -8e^{5i\omega}\hat{f}(2\omega)$$

(i) What is $\lim_{\omega \to -\infty} \hat{g}(\omega)$? Justify your answer.

(ii) Find $\int_{-\infty}^{\infty} g(t)dt$ (there's an easy way!).

(iii) Find the function $g(t)$, and make sure you specify it for all t, $-\infty < t < \infty$.

33. Suppose $\phi(t)$, $0 \le t < \infty$, is a real-valued function with Laplace transform $\hat{\phi}(s)$. Find (in terms of ϕ) the inverse Laplace transform of

$$\frac{1}{2}[\hat{\phi}(s - 2i + 3) + \hat{\phi}(s + 2i + 3)]$$

34. Find the inverse Laplace transform of

$$\hat{f}(s) = \frac{1 - (s+1)e^{-s}}{2s^2 \sinh(s)}$$

35. A signal $f(t)$ whose waveshape is a Gaussian function of t has a standard deviation of 100 sec.

(a) What is a reasonable estimate of its effective time duration?

(b) What is a reasonable estimate of its effective bandwidth (in Hz.)?

(c) On the basis of Shannon's Theorem, how many samples would you need for reasonably accurate signal reconstruction?

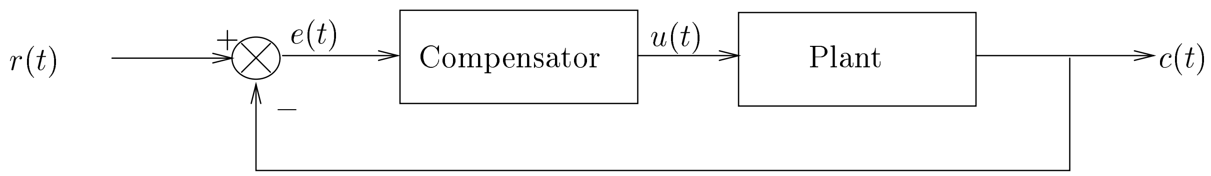
36. Let $f(t)$, $-\infty < t < \infty$, be a smooth function with Fourier transform $\hat{f}(\omega), -\infty < \omega < \infty$. Let

$$g(t) := \sum_{k=-\infty}^{\infty} f(t - 2k), \quad -\infty < t < \infty.$$

Show that

$$g(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{f}(n\pi) e^{in\pi t}.$$

37.



In the control system shown, the plant is described by the differential equation

$$\frac{d^2}{dt^2} c(t) = ku(t),$$

while the compensator is described by

$$\frac{du(t)}{dt} + bu(t) = \frac{de(t)}{dt} + ae(t)$$

Here a, b and k are constants.

(i) On the assumption that the variables

$$c, \quad dc/dt, \quad u, \quad e$$

are initially zero, find the Laplace transform of e in terms of that of r. Express your answer in the form

$$\hat{e}(s) = H(s)\hat{r}(s)$$

for a suitable function $H(s)$.

(ii) For what values of a, b and k is the system stable?

(iii) Let $a = 1/2$, $b = 3$, $k = 4$. Evaluate $\lim_{t\to\infty} e(t)$ when
(1) $r(t) = t \quad t \geq 0$
(2) $r(t) = t^2 \quad t \geq 0$

38. Let $f(t)$ $(-\infty < t < \infty)$ be a periodic symmetric rectangular wave taking the values ± 1, and having period T.

(i) Obtain the complex exponential Fourier series (FS) for f. (You need not work out the coefficients explicitly, provided you give the correct integral formula for them.)

(ii) At what points t in $(-\infty, \infty)$ does this FS converge? To what value does it converge at such a point?

(iii) In what subintervals of $(-\infty, \infty)$ does this FS converge uniformly?

(iv) What is meant by the statement that this FS converges to f in mean square?

(v) Sketch the graph of the partial sum of this FS for a large but finite number of terms. Indicate the approximate location of the points of maximum overshoot and undershoot, and the value of the FS partial sum at such points.

39. Let

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t < 0 \text{ and } t > 1 \end{cases}$$

$$g(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

(i) Obtain the convolution $(f * g)(t)$. (You need not work out the result explicitly, provided you give the correct integral formula or formulas for it.)

(ii) On the basis of your understanding of what the convolution operation means, sketch the graph of $(f * g)(t)$ approximately.

40. Let $f(t)$, $-\infty < t < \infty$, be a smooth, complex-valued function such that

$$\int_{-\infty}^{\infty} |f(t)|^k dt < \infty$$

for $k = 1$ and $k = 2$. Suppose the Fourier transform $\hat{f}(\omega)$ of f vanishes for all ω with $|\omega| \geq \pi$.

(i) Show that, for $|\omega| < \pi$,

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-in\omega}$$

(ii) From the result in (i), derive a formula for the energy of f in terms of the numbers $f(n)$.

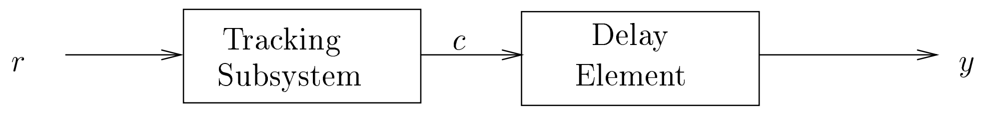
41. You are given a linear, time-invariant, causal filter with impulse response function

$$h(u) = ue^{-u}, \quad u \geq 0$$

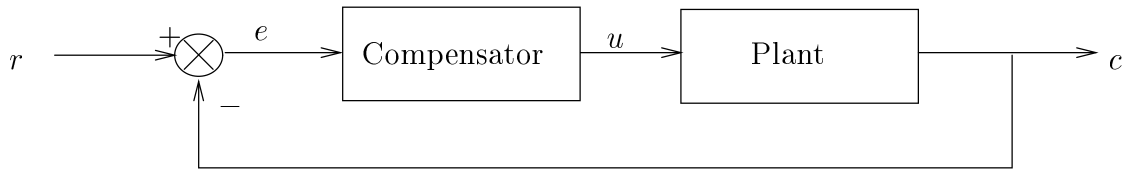
Suppose the input signal is a symmetric periodic square wave with period 4 sec., running for $0 \leq t < \infty$.

Carefully estimate a nominal bandwidth (W Hz.) for the steady-state output signal, such that only a fraction 10^{-8} of the total input power is carried (by the output) at frequencies higher than W.

42.



In the overall system displayed above, the delay element introduces a pure delay of 1 sec. The tracking subsystem is specified below.



Plant: $\dfrac{d^2c}{dt^2} - \dfrac{dc}{dt} = ku$

Compensator: $\dfrac{du}{dt} + bu = \dfrac{de}{dt} + ae$

Here a, b and k are constants. Initially c, dc/dt, u and e are zero.

(i) Find the transfer function from r to y, expressing your answer in the form

$$\hat{y}(s) = G(s)\hat{r}(s)$$

(ii) For what values of a, b and k is the overall system BIBO stable?

(iii) In the stable case, find a formula for the overall tracking error

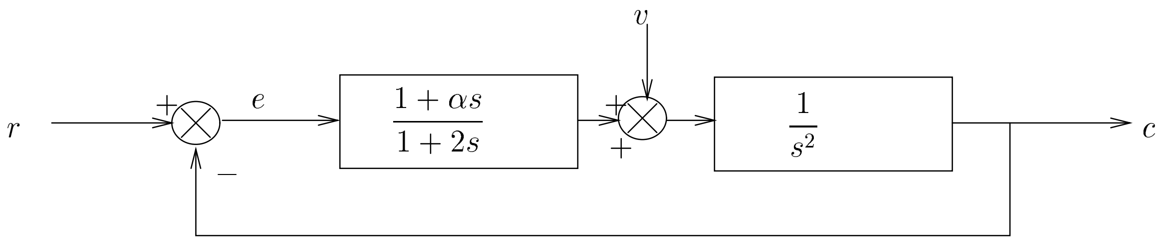
$$z(t) := r(t) - y(t)$$

when $r(t) = t$ $(t \geq 0)$ and $t \to \infty$.

(iv) In the stable case, suppose $r(t)$ is a pure sinusoid of frequency ω rad sec^{-1}. If $\omega \gg 1$ find a reasonable approximate formula for the steady-state amplitude and phase of y. Assuming a and b are small, for what (approximate) value of k is the amplitude gain (from r to y) unity at $\omega = 10$? What is then the (approximate) phase shift?

(v) Suppose the system is restructured by running the feedback branch from y to the comparator $\otimes$, rather than from c. Without making any calculations, describe what you think the effect on stability would be.

43.



In the feedback control system shown, $c(t)$ is the controlled output, $r(t)$ the reference input, $v(t)$ a disturbance input, $e(t)$ the tracking error, and α a parameter of the controller.

(i) Assuming that c and all its derivatives are initially zero, derive an expression for $\hat{e}(s)$, in the form

$$\hat{e}(s) = R(s)\hat{r}(s) + V(s)\hat{v}(s),$$

R and V being rational functions of s.

(ii) For what values of α is the system stable?

(iii) If $v(t) \equiv 0$ and the system is stable, for what class of polynomials $r(t)$ is it true that $e(t) \to 0$ as $t \to \infty$.

(iv) If $r(t) \equiv 0$ and the system is stable, for what class of polynomials $v(t)$ is it true that $e(t)$ is bounded on $0 \leq t < \infty$?

(v) If $r(t) = \sin t$, $v(t) = t^3$, and the system is stable, what is the approximate behavior of $e(t)$ for *large* t?

(vi) If α is such that the system is just at the 'stability boundary', what class of sinusoids $r(t)$ or $v(t)$ could cause $e(t)$ to oscillate unboundedly as $t \to \infty$? How fast would the envelope of this oscillation grow?

FACT: A polynomial of form $s^3 + a_1 s^2 + a_2 s + a_3$, where the a_i are real, has all its roots in Re $s < 0$ iff $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$.

44. (i) State the Laplace inversion formula for functions $f \in PS \cap \text{EXP}[0, \infty)$.

(ii) Derive the Laplace inversion formula from the Fourier integral representation theorem for functions $g \in PS_1\,(-\infty, \infty)$.

(iii) Find the function $f(t)$ $(0 \le t < \infty)$ whose Laplace transform is

$$e^{-2s} \sum_{k=-\infty}^{\infty} \gamma_k (s - 2k\pi i)^{-1}.$$

Here

$$\gamma_k := \int_0^1 e^{-2k\pi i x} J_0(x)dx,$$

$J_0(x)$ being the Bessel function of the first kind of order zero (a nice function).

45. Let f be a periodic signal with period T, defined for $-\infty < t < \infty$. Assume that f is represented by its complex exponential Fourier series.

(i) Suppose f is advanced in time by an amount τ. Call the result f_{new}. How are the Fourier coefficients of f_{new} related to those of f?

(ii) Derive an expression for the r.m.s. value of f in terms of its Fourier coefficients.

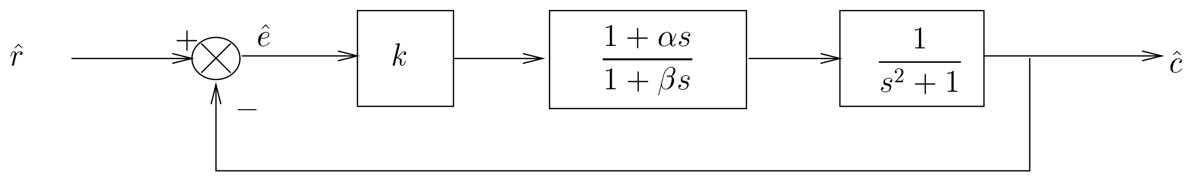
(iii) Write an expression for the power of f contained in frequencies $|\omega|$ higher than some arbitrary frequency Ω.

(iv) Assume the r.m.s. value of f is 2. Suppose you know that the Fourier coefficients of f satisfy the inequality

$$|c_n| \le \frac{5}{1 + |n|^{3/2}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Estimate a value of ω (as in part (iii)) such that at most 1% of the power of f is contained in frequencies higher than ω (in absolute value).

46. Consider the control system with block diagram shown below. Assume $k > 0$, $\beta > 0$.



(i) Calculate the transfer function

$$\hat{h}(s) := \hat{e}(s)/\hat{r}(s)$$

(ii) For what range of the parameter α is the system BIBO stable?

(iii) Suppose the input is

$$r(t) = 1 + 2\sin t - \cos t, \quad t \geq 0$$

Assuming that the system is stable, discuss the behavior of $e(t)$ as $t \to \infty$.

(iv) Let $\alpha = \beta = 1$ and $k = 3$. Assume that the input is

$$r(t) = 1, \quad t \geq 0.$$

Find $c(t)$, $t \geq 0$.

(v) In part (iv), show how the initial value $c(0+)$ could be computed directly from the transform $\hat{c}(s)$.

47. Let f be the signal defined by

$$f(t) = t, \quad 0 \leq t < 1$$

and by periodic extension for $t < 0$ and $t \geq 1$. Assume that t is measured in seconds.

(i) Calculate the complex exponential Fourier series (CEFS) of f.

(ii) Consider a filter F whose differential equation is

$$T^2 \frac{d^2 g}{dt^2} + 2T \frac{dg}{dt} + g = f(t)$$

T is a positive parameter. Calculate the steady state response g of F in the form of a CEFS.

(iii) How would you interpret the parameter T? Derive a reasonable lower bound on T to guarantee that the power of g carried by frequencies greater (in magnitude) than 10 Hz. is at most 1% of the d.c. power of g.

48. Let f, g be defined for $-\infty < t < \infty$ and let

$$h(t) := f(t)g(t), \quad -\infty < t < \infty.$$

(i) Assuming all the relevant transforms and integrals exist, derive a formula for the Fourier transform $\hat{h}$ in terms of $\hat{f}$ and $\hat{g}$.

(ii) Illustrate your result in (i) for the case where h is a time-gated sinusoid.

49. Consider the meandering nonperiodic signal

$$f(t) = e^{it} + e^{-it\sqrt{2}}, \quad -\infty < t < \infty.$$

Assume that t is measured in seconds.

(i) Calculate the (formal) Fourier spectrum $\hat{f}(\omega)$ as a linear combination of δ-functions. Plot.

(ii) Suppose that f is sampled every 2.513274 seconds. Calculate the spectrum of the sampled signal s. Plot $\hat{s}(\omega)$ for $|\omega| \leq 2.0 \text{ sec}^{-1}$.

(iii) Suppose that the Nyquist-Shannon interpolation formula is used in a misguided attempt to reconstruct f from s. What output signal f_0 will the "D-to-A converter" actually produce in the present case? Write $f_0(t)$ as an explicit sum of phasors.

(iv) Suppose f is time-gated by the window function

$$w(t) = \begin{cases} 1, & |t| \leq T \\ 0, & |t| > T \end{cases}.$$

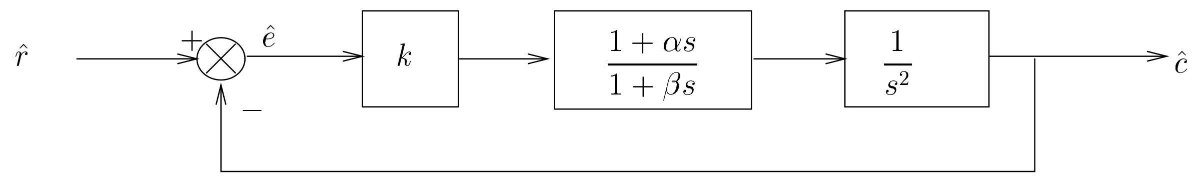
Calculate the spectrum of the gated signal

$$g(t) = f(t)w(t), \quad -\infty < t < \infty.$$

(v) If, in part (iv), $T = 31.4159$ sec., calculate a reasonable bandwidth for g (i.e. a number B such that $\hat{h}(\omega) \simeq 0$ for $|\omega| > B$).

(vi) On the bandwidth assumption of part (v), what sampling rate (samples/sec.) would just suffice to reconstruct g by the Nyquist-Shannon formula with negligible error?

50. Consider the control system with block diagram shown. Assume $k > 0$, $\beta > 0$.



(i) Calculate the transfer function

$$\hat{h}(s) := \hat{e}(s)/\hat{r}(s).$$

(ii) For what range of values of α is the system (as described by $\hat{h}$) stable?

(iii) Suppose the system is stable and that $r(t) = t^2$, $t \geq 0$. Find $e_\infty := \lim_{t\to\infty} e(t)$.

(iv) For what range of k will it be true that $|e_\infty| < 0.1$?

(v) If $r(t) = t^3$, $t \geq 0$, what can be said about $e(t)$ as $t \to \infty$?

51. (i) For the standard Fourier trigonometric series on the interval

$$-1 \leq t \leq 1$$

give both the definitional formula and the concise formula for the Dirichlet kernel corresponding to the 100-th partial sum. In a careful sketch, display the behavior of the kernel in a neighborhood of its point of application (point of symmetry) t^*, specifying the location of the first two zeros to the right of t^* and the value of the kernel at t^*. Also state the value of the integral of the kernel from $t = -1$ to $t = 1$.

(ii) A piecewise-smooth, real-valued function f has a jump discontinuity of magnitude 5 at $t = 0.2$, and is slowly varying but continuous on the intervals

$$-1 \leq t < 0.2, \qquad 0.2 < t \leq 1$$

Suppose the function is represented on the interval $-1 \leq t \leq 1$ by its Fourier series, and let f_N be the N-th partial sum of this series. With a clear sketch, display the behavior of f_{100} in a small neighborhood of $t = 0.2$, specifying the approximate amount of Gibbs overshoot and undershoot, and the approximate locations of the first two maxima or minima on either side of $t = 0.2$.

(iii) For the situation in part (ii), suppose f_{100} is replaced by the corresponding Fejér partial sum, say $\hat{f}_{100}$. How are the coefficients of f_{100} modified to form those of $\hat{f}_{100}$? Sketch the behavior of $\hat{f}_{100}$. Specify the location of the first zero of the derivative of $\hat{f}_{100}$ to the right of $t = 0.2$, and the approximate value of $\hat{f}_{100}$ at that point. Give a brief intuitive explanation for the manner in which the behavior of $\hat{f}_{100}$ differs from that of f_{100}.

52. For a certain complex-valued periodic signal $f(t)$ with base frequency 5 Hz., it is known that f and its first derivative are continuous, while its second derivative is piecewise smooth, for $-\infty < t < \infty$. Let c_n $(n = 0, \pm 1, \pm 2, \ldots)$ be the coefficients of the complex exponential Fourier series of f. It is known that $c_0 = 0.2i$.

(i) For an estimate of the type

$$|c_n| \leq K/|n|^p \text{ all } n \neq 0$$

what is the best (i.e. largest) possible value of p that can be inferred from what you have been told about f?

Hint: Iterate the calculation used in proving the Riemann-Lebesgue Lemma.]

(ii) In (i) suppose that $K = 10$. Find a conservative but reasonable estimate of that frequency Q Hz. such that

$$\frac{\text{power in } f \text{ at frequencies higher than } Q}{\text{power in } f \text{ at zero frequency}} \leq 10^{-7}$$

53. In this question, I denotes a fixed but arbitrary subinterval of the real line. $L_2(I)$ denotes the family of all complex-valued functions defined and finitely square-integrable on I.

1. Define what is meant by the norm and the scalar product for functions in $L_2(I)$. If the interval I has finite length, define the mean square value and the root mean square (r.m.s.) value of a function in terms of its norm.
2. Define what is meant by mean-square convergence of a sequence of functions $\{f_k | k = 0, 1, 2, ...\}$ in $L_2(I)$, to a limit $f \in L_2(I)$.

3. Let
$$\Phi := \{\phi_n | n = 0, 1, 2, ...\}$$
denote an infinite system of functions $\phi_n \in L_2(I)$. Define what is meant by the statement that Φ is an orthonormal (ON) system on I.

For the remainder of the question, assume that Φ is ON on I.

4. Define what is meant by the statement that Φ is complete.
5. Let $f \in L_2(I)$. Define what is meant by the Fourier series (FS) of f with respect to Φ, and give a formula for the Fourier coefficients of f.
6. Let $f \in L_2(I)$. Describe the "minimizing property" of the Fourier coefficients of f.
7. Let
$$f_N := \sum_{n=0}^{N} c_n \phi_n$$
be the Nth partial sum of the Fourier series of f with respect to Φ. Describe the 'Pythagorean' and geometric relationships that hold among the functions f, f_N and $e_N := f - f_N$. Illustrate with a sketch.
8. If for all $f \in L_2(I)$ the FS of f converges to f in mean square, show that Φ is complete.
9. State Bessel's inequality for a function $f \in L_2(I)$. What conclusion can be drawn about the behavior of the Fourier coefficients of f for large n?
10. Give a concrete example of a system Φ, and a function $f \in L_2(I)$, for which Bessel's inequality is strict. [Here and in 11, adopt whatever indexing of your system Φ is notationally convenient.]
11. Give a concrete example of a system Φ for which Bessel's inequality becomes equality for all $f \in L_2(I)$.
12. In 10 and 11 respectively, what can be said about the completeness of Φ? Explain.
13. Assume that Φ is complete. Let $f \in L_2(I)$, let f_N be as in 7, let
$$\hat{f}_N := \sum_{n=0}^{N-1} (1 - n/N) c_n \phi_n$$
be the corresponding Fejér sum, and let $e_N := f - f_N$, $\hat{e}_N := f - \hat{f}_N$. Show carefully that
$$\|\hat{e}_N\|^2 = \|e_N\|^2 + \sum_{n=1}^{N} (n|c_n|/N)^2$$
If $|c_n| \leq 1/n$ for all $n > 1000$, what can be inferred about the behavior of $\hat{e}_N$ as $N \to \infty$? Justify your answer with care.

54. 1. Explain, preferably in both intuitive and mathematical terms, why a nontrivial signal that is *strictly* time-limited cannot be *strictly* band-limited.

2. The signal $f(t) = \sin(2\pi t)$ $(-\infty < t < \infty)$ is sampled at times $t = n/3$, $n = ... - 1, 0, 1, ...$ Later (in the next world) a reconstructed version f_{NS} is created using the Nyquist-Shannon interpolation formula.
Write a formula for $f_{NS}(t)$.
What would you expect the r.m.s. reconstruction error to be?

3. Under the same conditions as in 2, suppose instead that reconstruction is done by piecewise-linear interpolation, giving a signal f_{PL}.
Compare f and f_{PL} by means of a sketch.
State how to calculate the r.m.s. reconstruction error (RMSE), expressing your answer in terms of a suitable integral.
By inspection, roughly estimate the ratio RMSE/(r.m.s. value of f) within order of magnitude.

4. Let $T > 0$. Suppose f (as in 2) is multiplied by the window function

$$w_T(t) = \begin{cases} 1, & -T \le t \le T \\ 0, & |t| > T \end{cases}$$

to give the windowed signal $f_T(t)$ $(-\infty < t < \infty)$.
Calculate the Fourier transform $\hat{f}_T$ of f_T.
For $T = 50$, sketch $\hat{f}_T(\omega)$ in the vicinity of $\omega = 2\pi$.
With $T = 50$, suppose f_T is sampled and reconstructed as in 2. What would you expect the effect to be on r.m.s. reconstruction error (calculated over the interval $[-T, T]$), compared to the result in 2 for f?
With $T = 50$, if f_T were sampled and reconstructed as in 3, and the r.m.s. error calculated over $[-T, T]$, how would the result compare with that of 3 for f?
Answer the last two questions again, but now with $T = 1$.

55. For the half-wave rectified sine wave

$$f(t) = \begin{cases} 0, & -\pi \le t \le 0 \\ \sin t, & 0 \le t \le \pi \end{cases}$$

show that the Fourier series partial sum $f_N(t)$ represents $f(t)$ with absolute error $|f(t) - f_N(t)|$ at most $1/(N\pi)$.

For the full-wave rectified sine wave $f(t) = |\sin t|$ show that the error is at most $2/(N\pi)$.

For the triangular pulse of half-width $a \le \pi$,

$$f(t) = \begin{cases} 1 - |t|/a, & |t| \le a \\ 0, & a \le |t| \le \pi \end{cases}$$

show that the absolute error is at most $4/(N\pi a)$. Taking $a = \pi/2$, choose the least N such that the absolute error does not exceed 0.01; then compute and plot $e_N(t) := f(t) - f_N(t)$ for $-\pi \le t \le \pi$.

56. A periodic signal f is defined by

$$f(t) = \begin{cases} 2-t, & 0 \le t \le 2 \\ 0, & 2 \le t < 3 \end{cases}$$

and by periodic extension for all $t < 0$ and $t \ge 3$. Answer questions 1 to 3 about f.

1. Calculate the
   - (i) average value (= average 'd.c. component') of f
   - (ii) energy of f on the interval $0 \le t \le 3$
   - (iii) average power of f
   - (iv) r.m.s. value of f
   - (v) L_2-norm of f on the interval $0 \le t \le 6$
   - (vi) total mean square value of the oscillatory components of f (i.e. the 'a.c. power' of f)
2. (i) On what closed subintervals of the interval $-1 \le t \le 3$ does the Fourier series of f (FS) converge uniformly? [A closed subinterval is one that includes both its endpoints. State your answer in the form: "Interval is $t_1 \le t \le t_2$", giving suitable ranges for t_1, t_2. There may be more than one such family of intervals.]
   - (ii) Give a brief intuitive explanation of why FS cannot converge uniformly on any interval of the form $6 - \delta \le t \le 6 + \delta$, where $\delta > 0$.
   - (iii) To what value does FS converge at $t = 0$?

3. Suppose the CEFS of f is

$$\sum_{n=-\infty}^{\infty} c_n e^{in\alpha t}$$

   - (i) State the value of α.
   - (ii) Write an integral formula for c_n, but don't work it out yet.
   - (iii) Evaluate the formula for c_n far enough to obtain an estimate for c_n of the form

$$|c_n| \approx K/|n|^p$$

     approximately valid for all large $|n|$. [Thus, identify K and p.]
   - (iv) Let e_N be the error when CEFS is used to approximate f with the partial sum of terms from $n = -N$ to $n = +N$. Using the result of (iii), estimate a value of N high enough to guarantee that $\mathrm{rms}(e_N)/\mathrm{rms}(f) \le 0.01$.

57. (i) State Parseval's formula for Fourier transforms.

Consider the signal

$$h(t) = \begin{cases} e^{-\alpha t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

where α is a positive parameter.

(ii) Calculate $\|h\|^2$, where $\|\cdot\|$ denotes norm on $L_2(-\infty, \infty)$.

(iii) Calculate the Fourier transform $\hat{h}(\omega)$, $-\infty < \omega < \infty$. Sketch the graph of $|\hat{h}(\omega)|$ vs. ω for (1) small α and (2) large α. State briefly how the graphs illustrate "time-bandwidth duality".

(iv) Find a simple estimate for the amount of energy of h contained in frequencies ω with $|\omega| > \Omega$, where $\Omega >> \alpha$. Call this the 'tail energy' of h.

(v) From the result of (iv), calculate a value of Ω (depending on α) which guarantees that the tail energy of h is no larger than 0.01% of the total energy of h.

58. Let $f(t) = e^{-\alpha|t|}$, $-\infty < t < \infty$, where $\alpha > 0$. Define

$$g(t) := \sum_{k=-\infty}^{\infty} f(t + 2k), \qquad -\infty < t < \infty$$

1. Calculate $g(0)$.
2. Show that g is periodic.
3. Calculate the complex exponential Fourier series (CEFS) of g.
4. As usual, define the Nth partial sum of CEFS as

$$g_N := \sum_{n=-N}^{N} \ldots$$

and the corresponding approximation error as $e_N(t) := g(t) - g_N(t)$. Estimate the maximum error magnitude

$$\max(e_N) := \max\{|e_N(t)| \ : \ -\infty < t < \infty\}$$

as a function of N and α, when N is large (i.e. $N\pi >> \alpha$).

5. Estimate a value of N sufficient to guarantee that $\max(e_N)/g(0) < 0.001$. Sketch the dependence of your estimate on α.
6. Calculate the average power of g at zero frequency (i.e. 'd.c. power' of g).
7. Estimate a value of N sufficient to guarantee that the ratio

$$\frac{\text{power of } e_N}{\text{d.c. power of } g}$$

is less than 0.001. Sketch the dependence of your estimate on α.

59. Let $f(t)$, $-\infty < t < \infty$, be a real-valued signal with L_1 and L_2 norms both finite. Consider a linear time-invariant filter (called $\mathbf{H}$, say) with impulse response h, given by

$$h(u) = f(-u + \tau), \qquad -\infty < u < \infty$$

where τ is a fixed real number.

1. When the input to $\mathbf{H}$ is f, let the output signal be $g(t)$, $-\infty < t < \infty$. Calculate the Fourier transform $\hat{g}(\omega)$ in terms of the Fourier transform $\hat{f}(\omega)$.
2. With g as in 1., use the Fourier integral representation of g to calculate the specific output value $g(\tau)$, in terms of the energy of f.
3. With g as in 1. use 2. to show that $g(t)$ is maximized with respect to t ($-\infty < t < \infty$) when $t = \tau$.
4. Under what condition on f is the filter $\mathbf{H}$ causal?
5. Specifically let

$$f(t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ 0, & \text{otherwise} \end{cases}$$

and let $\tau = 2\pi$. Carefully define h and sketch the graphs of f and h.

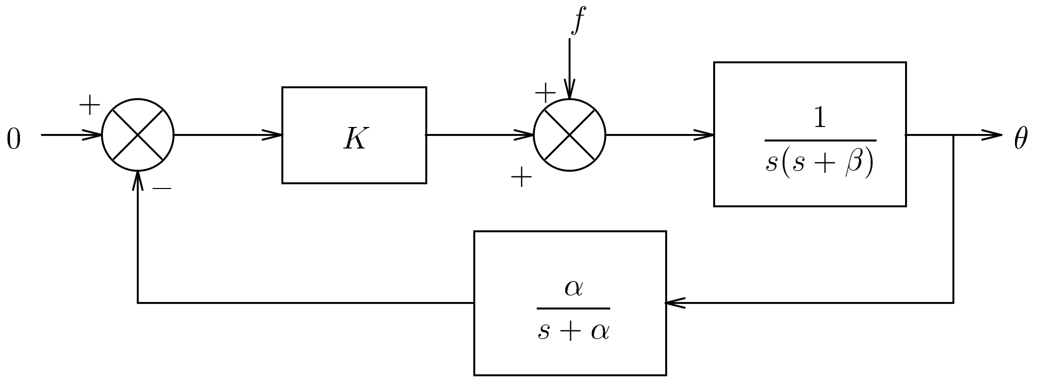
6. With f as in 5. write detailed expressions for $g(t)$, in each of the four ranges

$$t \le \pi, \quad \pi \le t \le 2\pi, \quad 2\pi \le t \le 3\pi, \quad t \ge 3\pi$$

Use the picture to write down, but don't evaluate, the relevant integrals.

7. From the result of 6. just evaluate $g(\pi)$, $g(2\pi)$ and $g(3\pi)$. Sketch the graph of $g(t)$ for $0 \le t \le 4\pi$.
8. Consider the general case when the input to $\mathbf{H}$ is a real-valued signal $r(t)$, $-\infty < t < \infty$, with energy $\|r\|^2 = 1$. Denote the output signal by s. Use Schwarz to show that $s(\tau)$ is a positive maximum with respect to all such r when $r(t) = f(t)/\|f\|$ for all t, $-\infty < t < \infty$.

60.



The system with block diagram displayed represents a regulator whose purpose is to hold the output angle variable θ approximately constant at the 'setpoint' value of 0, in the face of 'disturbances' represented by the input signal f. The parameters α, β, K are positive real numbers.

1. Find the transfer function $\hat{h}(s) := \hat{\theta}(s)/\hat{f}(s)$ (on the assumption that all internal initial values of the system are 0).
2. Find the range of K such that the system is BIBO stable.

3. Write $\hat{k}(s)$ for the limiting form of $\hat{h}(s)$ as the angle sensor (represented by the block in the feedback return path) becomes 'perfect'. What can be said about the stability of $\hat{k}(s)$?

4. Suppose f satisfies the differential equation and initial conditions

$$d^2 f(t)/dt^2 = 0, \quad t > 0; \qquad f(0) = f_o, \quad f'(0) = f_1$$

Find the Laplace transform $\hat{f}(s)$.

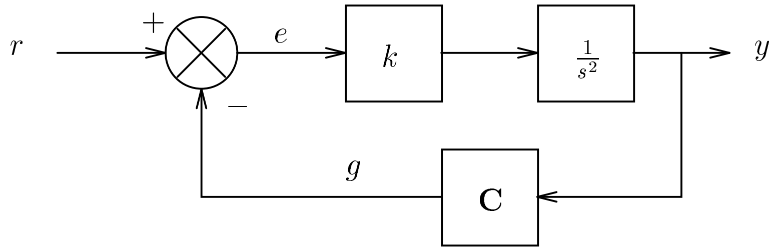
5. With f as in 4. what conditions must hold on the parameters K, α, β, f_o and f_1 to guarantee that $\lim \theta(t)$ $(t \to \infty)$ exists and is finite? In that case, evaluate the limit.

6. With $f \equiv 0$, find the critical frequency at which the system may oscillate spontaneously when it is just on the boundary between stability and instability.

7. For what range of K does the system have a steady-state frequency response? For such K, find the approximate amplitude and phase of the frequency response for very high frequencies ω.

8. Assume $\alpha = 2$, $\beta = 1$, $K = 30$, and that $f \equiv 0$. Verify carefully that $\theta(t)$ may oscillate with an amplitude that grows like e^t as $t \to \infty$, and calculate the frequency of this oscillation.

61.



In the feedback system shown, k is a constant gain, and the compensator block labelled **C** in the return path has transfer function $\hat{c}(s) = (s+\alpha)/(s+\beta)$. The parameters α, β are positive constants.

1. What is the differential equation relating g to y?

2. Determine the impulse response function $c(u)$ of **C**. Provide a formula for the compensator output signal $g(t)$ $(t \geq 0)$ in terms of its initial value $g(0) = \gamma$ and the input $y(t')$ for $0 \leq t' \leq t$.

3. Let **H** denote the feedback system with input signal taken to be r and output e. Calculate the transfer function $\hat{h}(s)$ of **H**. For what range of values of α, β, k is **H** BIBO stable?

4. Find the (critical) frequency of spontaneous oscillation of the system **H** when the parameters are adjusted so that **H** is on the boundary between stability and instability. If the input signal r is a sinusoid at the critical frequency, what is the general form of the output $e(t)$ for large t?

5. Let $r(t) = t^2$, $t \geq 0$. If $\mathbf{H}$ is stable, find $\lim e(t)$ $(t \to \infty)$.

62. Consider a filter with differential equation

$$T\frac{dg(t)}{dt} + g(t) = f(t), \qquad t \geq 0$$

and initial condition $g(0) = 0$.

1. If the input signal is $f(t) = \cos(\omega t)$ $(t \geq 0)$ find the output $g(t)$ for *all* $t \geq 0$. Check your answer when $\omega = 0$.
2. Suppose the input signal is $f(t) = |\sin(t)|$, $t \geq 0$. Let the Fourier series (corresponding to base interval $[0, \pi]$) be

$$f(t) = a_o/2 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)]$$

   Find $a_o/2$, ω_o and b_n.
3. Using the results from 1. and 2., find the 'steady-state' filter output, say $g_\infty(t)$ (in terms of T and the a_n for $n \geq 1$), accurate for sufficiently large t. What would be a reasonable numerical assessment of 'sufficiently large' here?
4. Write expressions for the 'd.c.' power and the average 'a.c.' power corresponding to the solution $g_\infty(t)$ in 3. Given the time constant $T = 3\pi$, and $a_n = -4/(\pi(4n^2 - 1))$, estimate the summation limit required to compute the a.c. power numerically, with error no greater than 10^{-8} in magnitude.

63. Let $f(t)$ $(-\infty < t < \infty)$ be the signal whose Fourier transform is

$$\hat{f}(\omega) = \begin{cases} 1, & |\omega| \leq 1 \\ 0, & |\omega| > 1 \end{cases}$$

1. Calculate f and sketch its graph.
2. Calculate the total energy of f.
3. Suppose f is sampled at times $n\tau$ $(n = ..., -1, 0, 1, ...)$, and call the sampled signal g. Setting $\Omega := \pi/\tau$, provide a formula for the Fourier transform $\hat{g}$, in terms of $\hat{f}$. Sketch the graph of $\hat{g}$ when (i) $\Omega = 1.1$, and (ii) $\Omega = 0.9$.
4. With g as in 3., an attempt is made to recover f from g by passing g through an ideal low-pass filter with gain k and passband $-B \leq \omega \leq B$. Denote the filter output by $h(t)$ $(-\infty < t < \infty)$, with Fourier transform $\hat{h}$. For a general value Ω, write (i) a formula for $\hat{h}$ in terms of $\hat{g}$, and (ii) a formula for $h(t)$ in terms of the samples $f(n\tau)$.
5. For what range of Ω will there exist suitable k and B as in 4. such that the signal h is identical with f? In particular evaluate your formula for $h(t)$ when $B = \Omega = 1$ and $k = \pi$.

6. In the case $B = \Omega = 0.9$, and $k = \pi/\Omega$, calculate the energy of h. How does your result differ from the energy of f? Discuss briefly.

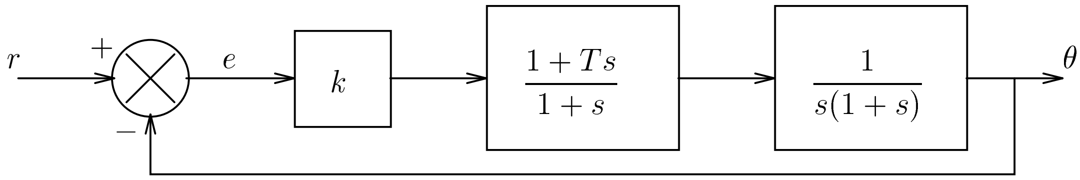
64. Consider a data-processing system $\mathbf{S}$ with input signal f and output signal g, where

$$g(t) = \alpha e^{t-1} f(t-1) + \beta[f(t+1)]^2 + \gamma df(t)/dt + \int_{-\infty}^{\infty} k(t,v) f(v) dv$$

for $-\infty < t < \infty$. Here α, β, γ are real parameters, and the weighting function k is an ordinary real-valued function of t and v. The input signal class $\mathcal{F}$ consists of all real-valued signals f defined on $-\infty < t < \infty$ for which the above derivative and integral exist, and the output signal class $\mathcal{G}$ is chosen large enough to contain all the signals $\mathbf{S}f$. Assume that $\mathcal{F}$ and $\mathcal{G}$ are linear vector spaces.

1. Provide the 'best' condition(s) you can find, on the constants α, β, γ and the function k, which guarantee that $\mathbf{S}$ is linear. Taking a simple instance of $\mathbf{S}$, show that linearity can fail if your condition fails.
2. Repeat 1., for the property of time invariance of $\mathbf{S}$.
3. Repeat 1., for the causality of $\mathbf{S}$.
4. Repeat 1., for the BIBO stability of $\mathbf{S}$. *Hint:* Can you invent a signal f for which $\|f\|_\infty < \infty$, but whose derivative $df(t)/dt$ is an unbounded function of t?

65.



In the feedback system shown, k and T are positive parameters.

1. Calculate the transfer function $\hat{h}(s) = \hat{e}(s)/\hat{r}(s)$. Call the system with this transfer function (i.e. with r as input, e as output) **SYS**. Assume from now on that $r(t) = 0$ for all $t < 0$.
2. Determine the conditions on k and T for **SYS** to be BIBO stable. Write an equation for the stability boundary relating the critical values (k_c, T_c) in the (k, T)-plane (in the first quadrant $k, T > 0$). Display this boundary with a careful sketch, and label the regions of stability and instability.
3. For a suitable range of T (obtained in 2.) determine the critical frequency ω_c at which **SYS** will oscillate, when (k, T) lies on the stability boundary. Express ω_c as a function of T_c, and sketch its graph.
4. Suppose the input is $r(t) = 1 - t$ $(t > 0)$. Calculate $\hat{e}(s)$. Under what condition will it be true that $e(\infty) := \lim e(t)$ $(t \to \infty)$ exists and is finite? Under this condition, calculate $e(\infty)$. Under what condition on T will it be possible to reduce $|e(\infty)|$ to less than $\frac{1}{3}$ by a suitable choice of k? When could $e(\infty)$ be made arbitrarily small?

5. Suppose $r(t) = t^2$ $(t \geq 0)$ and the **SYS** is stable. Without working out $e(t)$ in detail, describe the general behavior of $e(t)$ as $t \to \infty$.

6. Suppose $r(t) = \cos(\omega t)$ $(t \geq 0)$ for some $\omega > 0$. Under what condition would you be safe in assuming that $e(t)$ is also sinusoidal with frequency ω as $t \to \infty$? Under this condition, calculate the approximate amplitude of $e(\cdot)$ as a function of ω, (i) when ω is small, (ii) when ω is very large.

66\. 1. A signal $f(t)$ is defined for $t \geq 0$ by $f(t) = t/\pi$ $(0 \leq t < \pi)$, and by periodic extension for $t \geq \pi$. Calculate the complex exponential Fourier series (CEFS) of f. From the CEFS derive the standard trigonometric Fourier series of f.

2. Consider the linear time-invariant causal filter, say **FIL**, with transfer function $\hat{h}(s) = s(s-1)/(s+1)^2$. Calculate the impulse response function $h(u)$.

3. Write down the complex frequency response function and calculate the amplitude response function $A(\omega)$, of **FIL**. Sketch the graph of A for $0 < \omega < \infty$.

4. Suppose the signal f (as in 1.) is input to **FIL**. Using the previous results, show that the steady-state output of **FIL** can be written as an infinite series of real sinusoids, and specify their amplitudes and frequencies (but not their phases).

5. Write an infinite series for the average power of the steady-state output of **FIL** under input f.

6. Assume that t is measured in sec. To a good approximation, calculate the least frequency $\nu_{\min}$ (in Hz.), such that the total output power in the frequency range $\nu > \nu_{\min}$ is no greater than 0.1% of the total average power of f.

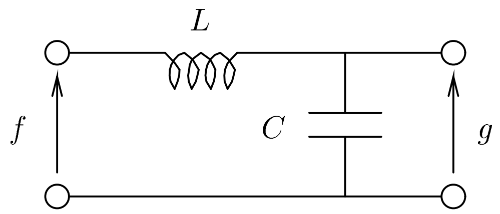
67\. Consider a general system $\mathbf{S} : \mathcal{F} \to \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are suitable classes for the input and output signals respectively, assumed defined for $-\infty < t < \infty$.

1. Provide mathematical definitions of the properties that $\mathbf{S}$ is (i) causal, (ii) time-invariant, (iii) BIBO stable. Note that in case (iii) you will first need to define a suitable norm for signals in $\mathcal{F}$ and $\mathcal{G}$. Along with each mathematical definition, provide a short (one-sentence) intuitive interpretation.

2. Provide an example of a system $\mathbf{S}$ that is stable and time-invariant and non-causal.

3. Repeat 2 for the property unstable and (not time-invariant) and causal.

4. Consider the network **NET** displayed below, with L and C constant, and $LC = 1/4$. The signals f and g are the input and output terminal voltages as shown.



Calculate (by any method) the transfer function $\hat{h}(s)$ and impulse response function $h(u)$ of **NET**.

5. Assuming the initial conditions on **NET** are zero, write an expression for the output $g(t)$ $(t > 0)$ in terms of the input $f(t')$ for $0 \leq t' \leq t$.
6. Fix t in 5. Suppose you'd like to maximize $g(t)$ over the set of input signals f that satisfy $|f(t')| \leq 1$, $0 \leq t' \leq t$. Show exactly how this could be done when $t = \pi$, and sketch the maximizing signal f. More generally, calculate the maximum value of $g(t)$, say $g^*(t)$, when $t = k\pi$ $(k = 1, 2, ...)$. What can you say about the sequence of numbers $g^*(k\pi)$ as $k \to \infty$? What, if anything, does this imply about the BIBO stability of **NET**?

68. Let f be the signal defined on $-\infty < t < \infty$ according to

$$f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ -1, & 1 \leq t < 2 \end{cases}$$

and by periodic extension for $t < 0$ and $t \geq 2$. Suppose t is measured in sec.

1. Specify the base angular frequency of f.
2. Calculate the average ('d.c.') value of f on the interval $1 \leq t \leq 3$.
3. Calculate the energy of f over the interval $-1 \leq t \leq 1$.
4. Calculate the (total) average power of f on the interval $-\infty < t < \infty$.
5. Calculate the 'a.c.' average power of f on the interval $-\infty < t < \infty$.
6. Calculate the L_2-norm of f on the interval $0 \leq t \leq 2$.
7. Calculate the r.m.s. value of f on $-1 \leq t \leq 49$.

69. For the same signal f as in Exercise 68, answer the following.

1. Write explicit integral formulas for the coefficients of the standard Fourier trigonometric series for f (but don't work out these integrals). This series will be called FS.
2. State the value to which FS converges, for each point t in the interval $0 \leq t \leq 2$.
3. Specify on what subintervals of $1 \leq t \leq 3$ FS converges uniformly.
4. Define what it means to say that FS converges to f in mean square.
5. State Parseval's formula for FS.
6. Denote the N^{th} partial sum of FS by f_N, as usual. Explain how you would use a computer to determine the least value of N such that the (total) average power of f_N on $-\infty < t < \infty$ exceeds 99% of the average power of f.

70. Let $f(t) = e^{-\alpha|t-1|}$ $(-\infty < t < \infty)$, and $g(t) = f(t+1)$, where α is a positive real parameter.

1. Calculate $\|g\|$ and show that $\|f\| = \|g\|$, where $\|\cdot\|$ denotes the usual L_2-norm on $(-\infty, \infty)$.

2. Write the explicit integral expression for the Fourier transform $\hat{g}$ of g (but don't work it out). Express $\hat{f}(\omega)$ in terms of $\hat{g}(\omega)$.
3. Write the expression for the Fourier integral representation of f in terms of $\hat{f}$.
4. State Parseval's Theorem for the Fourier integral.
5. It's desired to find Ω such that 0.1% of the total energy of f lies outside the frequency range $-\Omega \leq \omega \leq \Omega$. Provide an explicit integral formula (in terms of $\hat{f}$, assumed known) for calculating Ω (but don't carry out the calculation).
6. Assume Ω in 5 is the bandwidth of f, for practical purposes of Nyquist-Shannon sampling and signal reconstruction. If Ω is expressed in rad/sec, what is the minimum rate at which f should be sampled (samples/sec)?
7. For this signal f, and assuming 16-bit quantization of the amplitude range $[-1, 1]$ into subranges of equal width, estimate the effective time duration of f. So, roughly how many samples would you need to store (in terms of α and Ω)?