Given the system $y(s) = \frac{1}{s^3} u(s)$, find a controller which solves the RSP for constant disturbances / tracking signals.

(a) State Feedback.

In general

For checking stabilizability and detectability : for $\forall \lambda \in sp(A)$.

$\mathrm{rank}\begin{pmatrix} A-\lambda I & B \\ C & 0 \end{pmatrix}$ = Full rank.

$[A-\lambda I,\ B]$: stabilizability. $\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}$: detectability

$$y(s) = \frac{1}{s^3} u(s) \Rightarrow \dot{x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda = 0} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \ 0 \ 0) x.$$

$$\mathrm{rank}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \mathrm{rank}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 4 \Rightarrow \text{stabilizable \& detectable}$$

$\Rightarrow$ sol'n exist for RSP.

For a special case of constant signal

Servo compensator has the form

According to internal model theorem $\Rightarrow$ $\dot{\eta} = 0\eta + \underset{(y_{ref} - y)}{\overset{\parallel}{e}}$.

For state-feedback, we will assume $e = x$.

$\Rightarrow \quad \dot{\eta} = x.$

$\therefore$ We can put the compensator and system together

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \overbrace{\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right)}^{\hat{A}} \begin{pmatrix} x \\ \eta \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}^{\hat{B}} u.$$

$$\eta = \underbrace{(0 \ 0 \ 0 \ 1)}_{\hat{C}} \begin{pmatrix} x \\ \eta \end{pmatrix}.$$

Now our objective is to find a controller.

$$u = (K_0 \ K_1)\begin{pmatrix} x \\ \eta \end{pmatrix} = K_0 x + K_1 \eta \quad \text{s.t.}$$

minimize the cost f^n $J = \int_0^\infty (\eta' \eta + \epsilon u' u) d\tau.$

$\Rightarrow \quad Q = \hat{C}^T\hat{C} \qquad R = \epsilon = I \qquad$ if we let $\epsilon = 1$

Now we need to solve the Riccati Eq^n.

$$\hat{A}'P + P\hat{A} + \underbrace{\hat{C}^T\hat{C}}_{Q} - P\hat{B}\underbrace{R^{-1}}_{1}\hat{B}'P = 0$$

solve for P

$$\Rightarrow \quad u = -R^{-1}B'P\begin{pmatrix} x \\ \eta \end{pmatrix}.$$

$$= \begin{pmatrix} -2.6131 & -3.4142 & -2.6131 & -1 \end{pmatrix}\begin{pmatrix} x \\ \eta \end{pmatrix}$$

$\Rightarrow$ Our State Feedback Controller:

$$u = \begin{pmatrix} -2.6131 & -3.4142 & -2.6131 \end{pmatrix} x - \eta.$$

<u>NB</u> $\quad \dot{\eta} = (y - y_{ref}) \Rightarrow \quad \eta = \int_0^t (y - y_{ref})\, d\tau.$

$$\therefore \quad u = \begin{pmatrix} -2.6131 & -3.4142 & -2.6131 \end{pmatrix} x - \int_0^t (y - y_{ref})\, d\tau.$$

2. If we don't observe internal state, then we need to construct an observer, for system

$$\dot{x} = \overset{A}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} x + \overset{B}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} u$$

$$y = \underset{C.}{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}} x$$

Suppose observer pole @ $(-50, -50, -50)$. $\Uparrow$ pole-placement techniques.

then find Λ ?

$$\Theta = \mathcal{C}(A', C') = (C', A'C', A'^2C') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cancel{\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{pmatrix} = \lambda^3 \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0. \end{matrix}$$

Desired : $(\lambda+50)^3 = \lambda^3 + 3\lambda^2 \times 50 + 3\lambda 50^2 + 50^3$

$$a_1^* = -50^3$$
$$a_2^* = -3 \times 50^2$$
$$a_3^* = -3 \times 50$$

$$\hat{\Theta} = \begin{pmatrix} -a_2 & -a_3 & 1 \\ -a_3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Theta\hat{\Theta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (\Theta\hat{\Theta})^{-1}$$

$$[a_1^* - a_1 \quad a_2^* - a_2 \quad a_3^* - a_3](\Theta\hat{\Theta})^{-1}$$

$$-\Lambda' = [-50^3 \quad -3\times 50^2 \quad -3\times 50]\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$-\Lambda' = [-3\times 50 \quad -3\times 50^2 \quad -50^3]$$

$$\Lambda = \begin{bmatrix} 3\times 50 \\ 3\times 50^2 \\ 50^3 \end{bmatrix}$$

Controller:

$$u = (-2.6131 \quad -3.4142 \quad -2.6131)\hat{x} - \eta$$
$$\dot{\eta} = (y - y_{ref})$$
$$\dot{\hat{x}} = (A - \Lambda C)\hat{x} + \Lambda y + Bu$$