

University of Toronto

Department of Electrical and Computer Engineering

ECE410F – Control Systems Fall 2008

EXPERIMENT 2

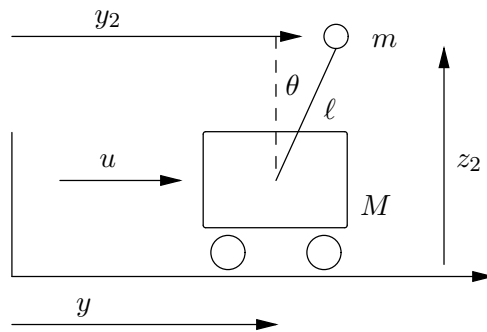
CONTROL OF AN INVERTED PENDULUM ON A CART, PART I

1 Purpose

The purpose of this experiment is to familiarize you with modelling the dynamics of the inverted pendulum using simulink, performing linearization, designing and evaluating different control laws based on pole placement, and verifying your design. It provides you with some understanding of the control of the inverted pendulum, and prepares you for labs 3 and 4 where you will design a real-time controller for a real inverted pendulum in the Control Laboratory.

2 Introduction

An inverted pendulum on a cart is often used as an illustrative model for control system design. The following figure provides the description of the physical system.



The differential equations governing the dynamics of the inverted pendulum can be shown to be given by

$$\ddot{y} = \frac{u + ml\dot{\theta}^2 \sin \theta - mg \cos \theta \sin \theta}{M + m \sin^2 \theta}$$
$$\ddot{\theta} = \frac{(M + m)g \sin \theta - u \cos \theta - ml\dot{\theta}^2 \sin \theta \cos \theta}{l(M + m \sin^2 \theta)}$$

The derivation of these equations can be found for example in my 1999 ECE356S course notes, Chapter 1, which you can download from

<http://www.control.utoronto.ca/~kwong/>

It is well-known from one's common experience of trying to balance a broom with one's finger that the inverted pendulum on a cart is a difficult system to control. The design specifications are:

- The inverted pendulum is balanced at its vertical position;
- The cart is asymptotically positioned at the origin.
- Good transient response is obtained. This is deliberately vague, and better transient response is often obtained by tuning control parameters.

3 Preparation: Building the Simulink Model

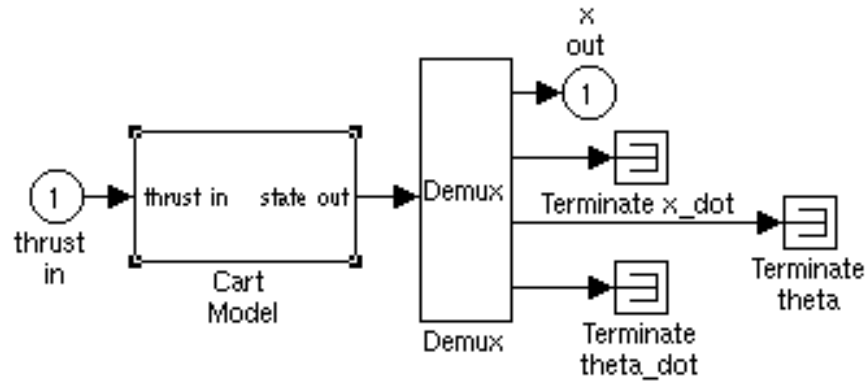
We shall first build a simulink model of the nonlinear system, and then linearize it to get a linear approximation.

1. Take the state vector $x = [y \dot{y} \theta \dot{\theta}]^T$. Write down the nonlinear differential equation for x in the form

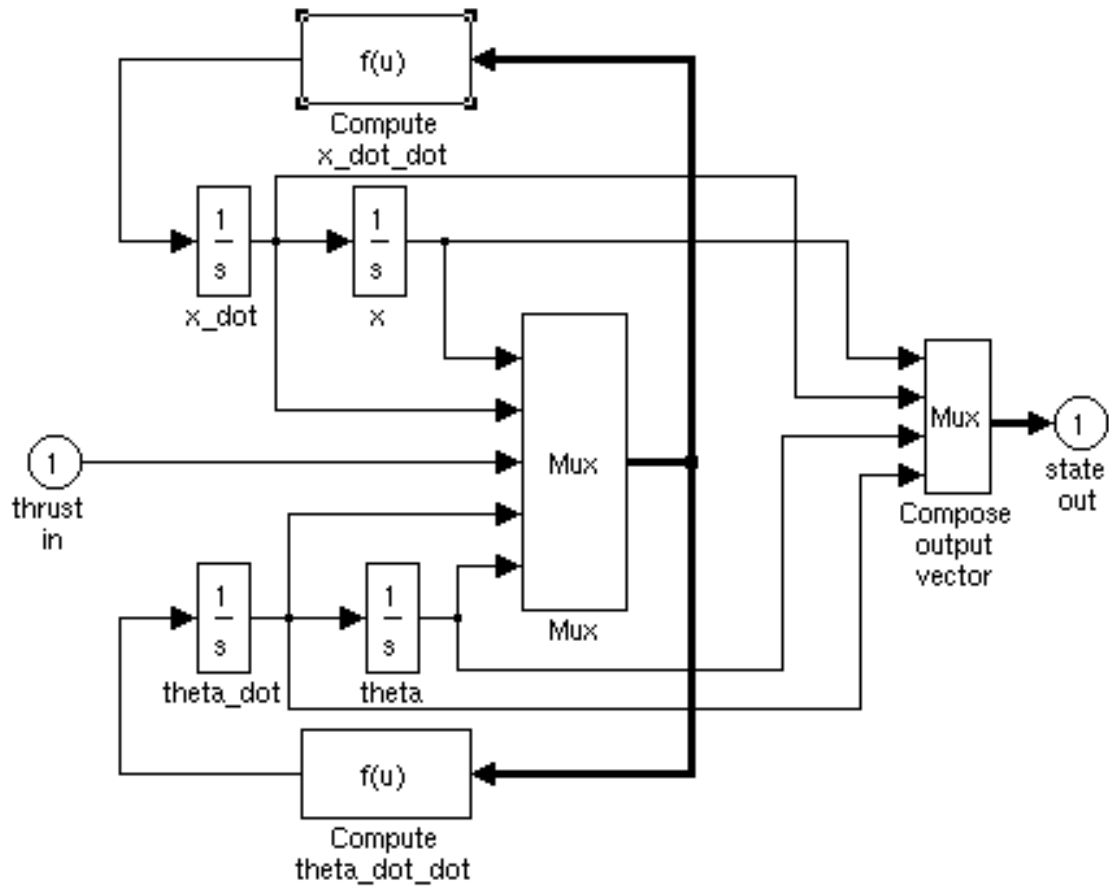
$$\dot{x} = f(x, u). \quad (1)$$

The function f on the right hand side has 4 components. Two components should contain nonlinear functions.

2. Set up your simulink model in the form described by the following figure:



where “thrust in” is the input u of the nonlinear differential equation, and “state out” is the solution of the nonlinear state equation. This block corresponds to the nonlinear state equation and is a subsystem of the simulink model. It can be built in the form described by the following figure:



Fcn blocks from the User Defined Function Library are used to specify the nonlinear functions in the system (denoted $f(u)$ in the figure). Since the nonlinear functions are complicated, use of the Fcn blocks is strongly recommended. Based on these suggestions, set up the complete simulink model corresponding to the inverted pendulum on a cart system. Save your complete model in simulink as, say, invertpen.mdl. Write down the 2 nonlinear functions and the complete $f(x, u)$ in the space below:

3. Check that

$$f(0,0) = 0$$

so that $x = 0, u = 0$ is an equilibrium solution of the system, corresponding to the pendulum perfectly still and balanced at the origin. Linearize the systems about $(0,0)$ to get a linear model for the inverted pendulum of the form

$$\dot{x} = Ax + Bu \tag{2}$$

Determine the eigenvalues of A in terms of the physical parameters $M, m, l,$ and g . Show that A is unstable (i.e., has at least one eigenvalues with nonnegative real part). Write down $A, B,$ and the eigenvalues of A in the space below:

4. Read through the rest of the lab description and heed especially the recommendation in Section 4.2 to prepare matlab .m files for design calculations.
5. Make a copy of your preparation and hand it in to your TA at the beginning of your lab session.

4 Experiment

4.1 Building the Linearized Model:

1. For one of the real inverted pendulums in the lab, we have the following physical parameters:
 - $M = 0.455$ Kg
 - $m = 0.210$ Kg
 - $l = 0.303$ m
 - $g = 9.78$ m/s^2

Substitute in the values of the physical parameters and based on (2), write down the linearized system matrices A and B in the space below:

2. Build the Simulink model for the linearized system.
3. Determine the open loop transfer function from u to y which corresponds to the cart position. Determine the poles and zeros of the open loop transfer function. Using the expressions you obtained earlier for the eigenvalues of A , determine also the numerical values of the eigenvalues. Write down these values in the space below:

In the same space, based on the values of the poles and zeros and eigenvalues, explain why the inverted pendulum on a cart is difficult to control.

4.2 Control Design Based on the Linearized Model

We now have the following linear model

$$\dot{x} = Ax + Bu$$

where $x^T = [y \ \dot{y} \ \theta \ \dot{\theta}]$. We shall use this model to design the controller. The design methodology is that we first design the controller to meet the specifications for the linearized system, then verify its performance on the original nonlinear inverted pendulum. We shall first use state feedback for control design. Since many of the calculations are repeated with different sets of parameters, it is **strongly recommended** that you write matlab .m script files for the design calculations, **before you go into the lab**.

1. Assume that the state of the linearized pendulum system x is observed. A state feedback law is a controller of the form

$$u = -Kx \tag{3}$$

Substituting into (2), we get the closed loop system

$$\dot{x} = (A - BK)x \tag{4}$$

We know that if the eigenvalues of $A - BK$ lie in the left half plane, i.e., have negative real parts, the closed loop system (4) is stable, resulting in $x \rightarrow 0$ as $t \rightarrow \infty$. Such a **stabilizing** gain K achieves design specs (a) and (b). Design spec (c) is related to how $x \rightarrow 0$, which depends on the location of the closed loop eigenvalues or poles. Choosing the gain K to obtain desired closed loop eigenvalues is therefore referred to as **pole placement** control design.

While you have not yet learned the theory of pole placement, there are Matlab commands which can compute the desired gain K based on specification of the closed loop poles. The Matlab commands **place** or **acker** both can calculate the required K for you. Read about the usage of these commands, and use them to design the state feedback gain K_1 so that the closed loop poles are located at $-1, -2, -3, -4$. Call this controller the SF-controller I.

2. Repeat the design but with the closed loop poles at $-1, -1 \pm i, -2$ to generate K_2 . Call this controller SF-controller II.
3. Suppose the initial cart position $y(0) = 0$, the initial pendulum angle $\theta(0) = 0.035$ (corresponding to approximately 2°), and the initial velocities $\dot{y}(0) = 0, \dot{\theta}(0) = 0.01$. Call this set of initial conditions IC. Simulate the response of the closed loop linearized system under SF-controller I using Simulink, and display the response of the various state components on Simulink scopes. Repeat the simulation for SF-controller II. Plot the responses of the various state components under the 2 controllers. Attach them to the end of the lab report. In the space below, discuss which controller gives the better performance based on your simulations.

4. Now assume that we measure only the cart position y and the pendulum angle θ so that \dot{y} and $\dot{\theta}$ are not available. Although the full state x is no longer available, a reasonable strategy is to use a good approximation of the state x for feedback. You will learn more sophisticated techniques for generating good estimates of the state later in the course. For now, note that the physical interpretation of the missing state variables suggests that we can get good approximations to \dot{y} and $\dot{\theta}$ by doing approximate differentiation of y and θ . Let the differentiating filter have transfer function $D(s) = \frac{\alpha s}{s + \alpha}$ so that $\frac{\alpha s}{s + \alpha} y$ approximates \dot{y} for large α , say 10.

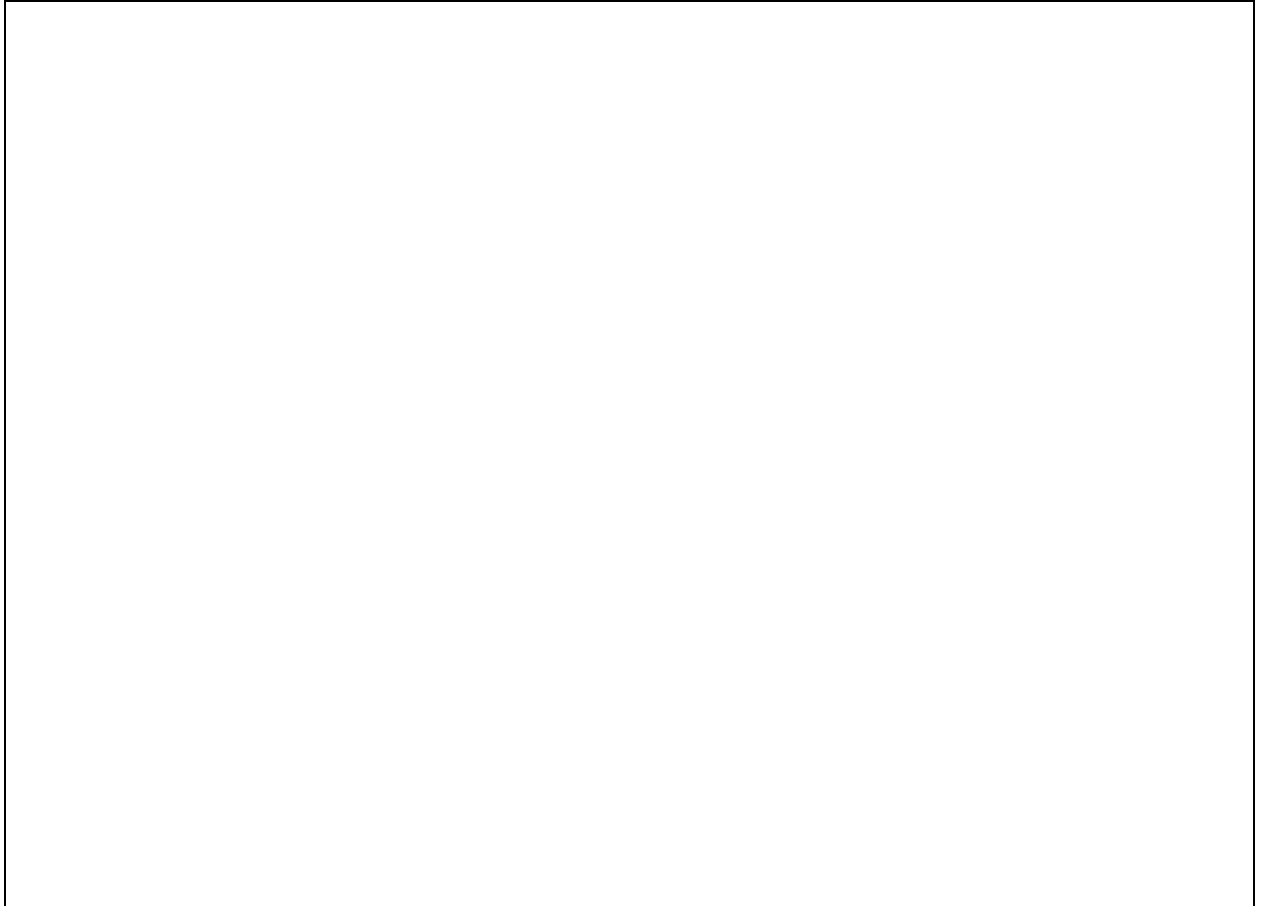
Using $D(s)$, we can now generate \hat{y} and $\hat{\theta}$, the approximations to \dot{y} and $\dot{\theta}$ from y and θ . Put $\hat{x} = [y \ \hat{y} \ \theta \ \hat{\theta}]^T$. We can now use the control law

$$u = -K_1 \hat{x} \quad (5)$$

Call this the OF-controller I. Similarly, OF-controller II is given by

$$u = -K_2 \hat{x} \quad (6)$$

Simulate the responses of the linearized pendulum under OF-controllers I and II using Simulink for the same initial conditions IC as before. Display your results on 4 Simulink scopes. **Demonstrate your results** for the 2 SF-controllers and the 2 OF-controllers to your TA. Again, discuss the behaviour of the closed loop system under OF-controllers I and II in the space below:



Do your results change if α , the parameter of the differentiating filter, is decreased or increased?

4.3 Verification of Control Design on the Nonlinear Inverted Pendulum

After carrying out the design using a linear approximation to a nonlinear system, you need to verify that the design is satisfactory for the nonlinear system.

1. Add SF-controller I to your simulink diagram of the nonlinear inverted pendulum. Simulate the response of true nonlinear continuous time system under initial conditions IC using simulink and view the state response on the simulink scope. Is the response satisfactory? Is it different from the linear approximation?
2. Add also the differentiating filter to your simulink diagram and implement OF-controller I. Repeat the simulation. Again record the system state response.
3. Now gradually increase the initial pendulum angle, initial angular velocity, initial cart position and velocity, still using OF-controller I. Are the design specs still achieved? Does the system

respond differently from the linear approximation as you make IC farther and farther away from the equilibrium $[0 \ 0 \ 0 \ 0]^T$?

4. Repeat parts 1, 2, and 3 of this Section using SF-controller II and OF-controller II.

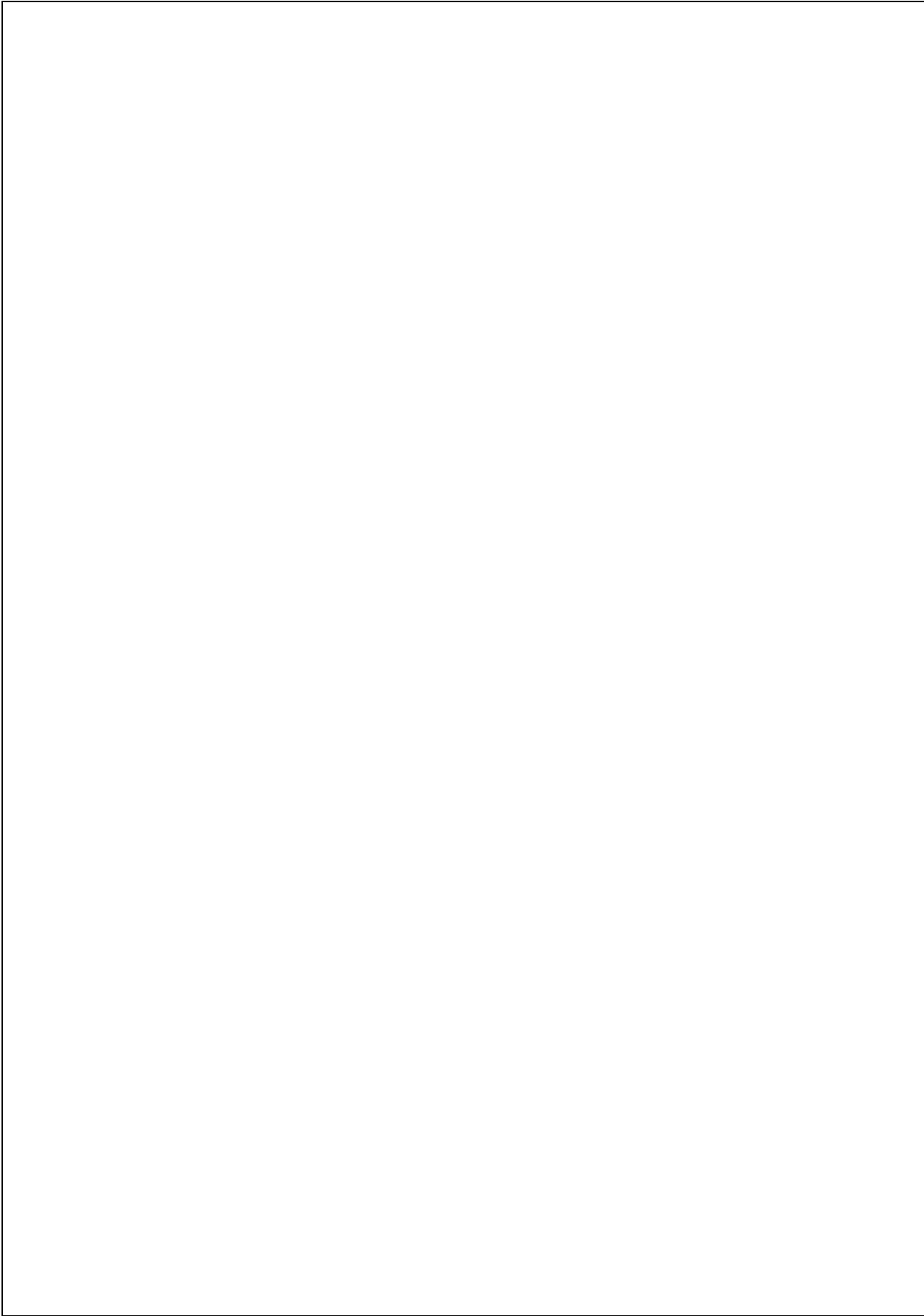
Demonstrate all your results in this section to the TA. Print or sketch your simulation results, label them, and attach them to the end of the report.

One important aspect of the control design carried out in this experiment should be emphasized: the control design is done in a systematic manner. There is no need to guess what the required controller structure is. Note also that the system has 1 input, but **at least 2 outputs** (y and θ). The design techniques you have learned for single-input single-output systems would not be directly applicable.

5 Report

In the space below, discuss concisely the following:

1. The performance of the SF and OF controllers for the linearized system and the nonlinear system;
2. Effects of choice of closed loop pole locations and the parameter for the differentiating filter on the performance of the controllers;
3. Effects of initial conditions on the stability of the closed loop system under the SF and OF controllers.
4. Any other observations you wish to describe or comment on.



- *Name:*
- *Student No.:*

- *Name:*
- *Student No.:*

- *Name:*
- *Student No.:*

- *Lab Section, Date, Time:*

NOTE: You can find a discussion of the inverted pendulum on a cart and its modelling in simulink in *The Student Edition of SIMULINK* by J.B. Dabney and T.L. Harman, from where the two figures in these laboratory notes are taken. However, you should note that the physical constants used in this lab are different from those in the book.