Games with Coupled Propagated Constraints in General Topology Optical Networks

Yan Pan and Lacra Pavel

Abstract—We consider games with coupled utilities and constraints towards optimizing channel optical signal-to-noise ratio (OSNR) in optical networks with arbitrary topologies. By fully using the flexibility that channel powers are adjustable at optical switches, we partition the network into stages where each stage is a link. We formulate a partitioned Nash game composed of link Nash games where each link Nash game is played towards minimizing channel OSNR degradation across the link. By breaking quasi-closed loops and selecting one link as the start, links/games can be placed sequentially in a ladder-nested form. The partition is simple and scalable and leads to a three-level hierarchical algorithm towards computation of Nash equilibria.

I. INTRODUCTION

Game theoretical approaches have been used for network congestion control [4] as well as for network power control [2], [3], [16] and channel OSNR optimization in optical networks [11], [13]. Game theoretical problem formulations and equilibria computational approaches have been areas of recent interest, one of which is a procedural method proposed in [14] based on an extension of duality to a game theoretical framework. The setting of the construction in [14] uses the two-argument system cost function and relaxes also the constraints into a two-argument form. Thus the problem is enlarged into a constrained optimization problem in a space of twice the dimension followed by projection back into a one dimension (with a fixed-point solution). Moreover, for convex constraints, duality leads to hierarchical decomposition into a lower-level game with no coupled constraints and an optimization problem for Lagrangian prices.

In a game formulation in optical networks where channel utility is related to maximizing channel OSNR, and players’ actions are channel powers, one coupled constraint, the link capacity constraint, which arises because of the nonlinearity threshold [1], has to be considered. For a single link, the coupled constraint is convex and has been considered based on duality in [14]. In optical networks with multi-link or mesh topologies, coupled constraints are propagated along links and convexity is not ensured. Meanwhile channel powers are adjustable at optical switches [1]. For multi-link topologies, [11] has proposed a partition approach by partially using the flexibility of channel power adjustments and the multi-link structure has been partitioned into stages. Each stage has a single sink structure. A partitioned Nash game has been formulated composed of ladder-nested stage Nash games. Each stage Nash game is played towards minimizing channel OSNR degradation across each stage.

However, no iterative algorithms have been developed in [11] for the computation of Nash equilibria. Moreover, it requires caution to apply this approach to general topologies, where typically there exist fully or partially closed loops being formed by channel optical routes [1]. One link may be the intermediate on one channel optical route and the end on another channel optical route. Thus it is not immediate to achieve single sink structures as in [11] for general topologies with closed loops. This also leads to another issue of concern which is the interconnection among stages after partition. Therefore direct application of the partition approach proposed in [11] is not realizable.

This paper extends the study on games with coupled propagated constraints in optical networks from single link or multi-link topologies to arbitrary topologies. We propose a more natural partition approach by fully using the flexibility of channel power adjustments for optical networks with arbitrary topologies. Specifically we are interested in mesh topologies with quasi-ring structures in which loops are not fully closed from one channel to itself. We partition the network into stages composed of single links. By breaking closed loops and selecting one link as the start, links can be placed sequentially in a ladder-nested form. We present a partitioned Nash game framework to solve channel OSNR maximization problems with link capacity constraints in optical networks. The partitioned Nash game is formulated composed of link Nash games and each link Nash game is played towards minimizing channel OSNR degradation across each link. An iterative algorithm is developed for the computation of Nash equilibria and its convergence is proved. The set of OSNR degradation minimization problems on links is related to the end-to-end channel OSNR maximization problem.

This paper is organized as follows. Section II reviews duality results for games [14]. In Section III, we describe the optical network and provide the channel OSNR model. We formulate a partitioned game problem in Section IV. A computation method towards solving the problem is introduced in Section V. Simulation and experimental results are presented in Sections VI and VII, respectively, followed by conclusions in Section VIII.

II. BACKGROUND

We review results for solving general Nash games in [14]. Consider an $m$-player Nash game, each player minimizing its individual cost $J_i : \Omega \rightarrow \mathbb{R}, i \in \mathcal{M}, \mathcal{M} =$
Let $\Omega = \prod_{i\in M} \Omega_i$, $\Omega_i = [u_{min}, u_{max}]$. Let $u = [u_1, \ldots, u_m]^T \in \Omega$ with $u_i \in \Omega_i$ be the vector of player actions. This game is subject to $g_l(u) \leq 0$, $l = 1, \ldots, L$, or $g(u) \leq 0$ with $g(u) = [g_1(u), \ldots, g_L(u)]^T$, where $g_l : \Omega \to \mathbb{R}$ is the constraint function. The overall action set $\Omega \subset \mathbb{R}^m$ is coupled, given as $\Omega = \{ u \in \Omega | g(u) \leq 0 \}$. A vector $u = (u_{-i}, u_i)$ is called feasible if $u \in \Omega$. For every given $u_{-i}$, a projection action set is defined for each $i \in M$,

$$\Omega_i(u_{-i}) = \{ \xi \in \Omega_i | g(u_{-i}, \xi) \leq 0 \}.$$

Such a Nash game is denoted by $GAME(M, \Omega_i, J_i)$. A vector $u^* = (u^*_{-i}, u^*_i) \in \Omega$ is called a Nash equilibrium (NE) solution of $GAME(M, \Omega_i, J_i)$ if

$$J_i(u_{-i}, u^*_i) \leq J_i(u_{-i}, u^*_i, x_i), \ \forall x_i \in \Omega_i(u^*_{-i}), \ \forall i \in M$$

every given $u^*_{-i}$.

An NE solution can also be defined by using the concept of system-like cost function first introduced in [5]. This is a two-argument function $\tilde{J} : \Omega \to \mathbb{R}$ called the Nash game (NG) cost function, defined as

$$\tilde{J}(u; x) := \sum_{i=1}^{m} J_i(u_{-i}, x_i), \ \forall x \in \Omega.$$  (1)

This augmented function defined on a space of twice the dimension of the original game is instrumental in what follows. Our aim of using $\tilde{J}$ is that of finding a solution of the original Nash game with coupled constraints by solving a constrained optimization problem for $\tilde{J}$ and searching for a fixed-point solution. Firstly, the NG cost function is separable in the second argument $x$ for every given $u$, i.e., each component cost function in $\tilde{J}(u; x)$ is decoupled in $x$. Similarly, the constraints $g$ can be augmented into a separable two-argument form, $\tilde{g}$,

$$\tilde{g}(u; x) = \sum_{i=1}^{m} g_i(u_{-i}, x_i),$$  (2)

thus enlarging the search set. NG-feasibility is equivalent to $\tilde{g}(u; u) \leq 0$. A Nash game with coupled constraints is related to a constrained minimization of $\tilde{J}$, (1), (2), with respect to the second argument $x$, that admits a fixed-point solution (Lemma 1 in [14]). A solution $u^*$ to this constrained minimization satisfies

$$\tilde{J}(u^*; x) \leq \tilde{J}(u^*; x), \ \forall x \in \Omega, \ \tilde{g}(u^*; x) \leq 0,$$  (3)

with $\tilde{g}(u^*; u^*) \leq 0$. Proposition 1 in [14] shows that individual components of $u^*$ constitute an NE solution. As in standard optimization [6], a two-argument Lagrangian can be defined for $\tilde{J}$ and $\tilde{g}$,

$$\tilde{L}(u; x; \mu) = \tilde{J}(u; x) + \mu^T \tilde{g}(u; x),$$  (4)

For $u^*$, a fixed-point solution to the minimization of $\tilde{L}$ over $x \in \Omega$, i.e., such that $\tilde{L}(u^*; u^*; \mu) \leq \tilde{L}(u^*; x; \mu), \ \forall x \in \Omega$. In the Lagrangian optimality condition, $u^*$ is obtained by first minimizing the augmented Lagrangian function $\tilde{L}(u; x; \mu)$ with respect to the second argument $x$, which gives $x = \phi(u)$ for every given $u$. The next step involves finding a fixed-point solution $u^*$ of $\phi$ by setting $x = u$, i.e., solving $u = \phi(u)$. We write in a compact notation

$$\tilde{L}(u^*; u^*; \mu) = \left[ \min_{x \in \Omega} \tilde{L}(u; x; \mu) \right]_{x = u^*}.$$  (5)

Such an $u^*$ is a solution to (3) and hence an NE solution if the complementary slackness condition in Proposition 2 in [14] holds. Note that $u^*$ thus obtained depends on $\mu$, $u^*(\mu)$. Then a dual cost function $D(\mu)$ is defined as

$$D(\mu) := \tilde{L}(u^*; u^*; \mu),$$  (6)

and the dual optimal value is defined as

$$D^* = \max_{\mu \geq 0} D(\mu).$$  (7)

The primal and dual optimal solution pairs are characterized by Proposition 2 in [14]. The separability in the second argument of both NG cost function and constraints ensures that $D(\mu)$ in (6) can be decomposed.

**Proposition 1 (Proposition 2, [14]):** For a Nash game with cost $J_i$ and constraints $g_l$, $l = 1, \ldots, L$, where $J_i$ and $g_l$ are continuously differentiable and convex, the dual cost function $D(\mu)$ (6), (4), can be decomposed as

$$D(\mu) = \sum_{i=1}^{m} \left[ \min_{x_i \in \Omega_i(u^*_{-i}, u^*_i)} L_i(u_{-i}, x_i, \mu) \right]_{x_i = u_i} = \sum_{i=1}^{m} L_i(u^*_{-i}, u^*_i, \mu),$$  (8)

where $L_i(u_{-i}, x_i, \mu) = J_i(u_{-i}, x_i) + \mu^T g_i(u_{-i}, x_i)$ and $u^*(\mu) = [u^*_i(\mu)] \in \Omega$ is a fixed-point solution to the set of minimizations on the right-hand side (RHS) of (8).

The duality approach offers a natural way to hierarchically decompose a Nash game with coupled convex constraints into a lower-level Nash game without coupled constraints and a higher-level system optimization problem. In effect the interpretation is that a procedural method for finding a solution to a Nash game with coupled constraints can be based on solving a modified game with no coupled constraints and an optimization problem.

**III. MODEL**

We consider an optical network (Fig. 1(a)) that is defined by a set of links $L = \{1, \ldots, L\}$ connecting optical switches. The optical switches allow channels in the network to be added, dropped or routed. They also provide the flexibility of channel power adjustments [1]. A link $l \in L$ is composed of $N_l$ optical amplified spans and each span includes an optical fiber followed by an optical amplifier (Fig. 1(b)). A set of channels, $M = \{1, \ldots, m\}$, corresponding to a set of wavelengths, are transmitted across the network from transmitters (Tx) to receivers (Rx) by intensity modulation and wavelength-multiplexing [1]. We denote by $M_l$ the set of channels transmitted over link $l \in L$. Also, we denote by $R_l$ the optical route (a set of links) of channel $i$ from its associated Tx to the corresponding Rx. We denote the optical fiber, from each Tx to the optical switch that it is connected to, as a virtual optical link (VOL). The set of VOLs, $L_v$, is equivalent to $M$. We denote by $L' = L \cup L_v = \{1, \ldots, L'\}$ with $L' = L + m$, the set of all links and VOLs. We define
two connection matrices: a channel transmission matrix \( A = [A_{i,l}]_{L \times m} \) and a system connection matrix \( B = [B_{k,l}]_{L' \times L} \), with

\[
A_{i,l} = \begin{cases} 
  1, & \text{channel } i \text{ uses } l \\
  0, & \text{otherwise,}
\end{cases}
\]

and

\[
B_{k,l} = \begin{cases} 
  1, & k \text{ and } l \text{ are connected with } k \rightarrow l \\
  0, & \text{otherwise.}
\end{cases}
\]

Let \( A = [A_l] \) with \( A_l = [A_{l,1}, \ldots, A_{l,m}]^T \) and \( B = [B_l] \) with \( B_l = [B_{l,1}, \ldots, B_{l,L'}]^T \).

Next we introduce the following notations for channels. For each channel \( i \in \mathcal{M} \), we denote by \( u_i/L_i \) and \( p_i/L_i \) the channel signal/noise power at Tx and the channel signal/noise power at Rx, respectively. We let \( u = (u_{i,-}, u_i) = [u_{1,1}, u_{m}]^T \) denote the vector form. On each link \( l \), we denote by \( u_{i,l}, n_{i,l}^\text{in} \) and \( p_{i,l}, n_{i,l}^\text{out} \) the channel signal/noise power at the input and the channel signal/noise power at the output, respectively. There is a physical bound set \( \Omega_i = [u_{\text{min}}, u_{\text{max}}] \) with \( u_{\text{max}} > u_{\text{min}} > 0 \) for each \( u_i \) and \( n_{i,l}^\text{in} \).

Recall that channel powers are adjustable at optical switches. We denote by \( \gamma_{l,i} \in [\gamma_{\text{min}}, \gamma_{\text{max}}] \) with \( \gamma_{\text{max}} > \gamma_{\text{min}} > 0 \), the adjustment parameter for channel \( i \) on link \( l \). Let \( \gamma_l = [\gamma_{l,1}, \ldots, \gamma_{l,m}]^T \) and \( u_l = (u_{l,-}, u_l) = [u_{l,1}, \ldots, u_{l,m}]^T \).

**Lemma 1:** The vector form of the input signal power on link \( l \) is

\[
u_l = \text{diag}(\gamma_l) \cdot V_l \cdot p,
\]

where \( \text{diag}(v) \) denotes a diagonal matrix whose diagonal entries are elements of vector \( v \), \( V_l \) is the matrix direct sum of \( V_{l,i} \), \( i = 1, \ldots, m \), \( V_l = \bigoplus_{i=1}^m V_{l,i} \) with

\[
V_{l,i} = [A_{l,i}B_{l,1}A_{1,i}, \ldots, A_{l,i}B_{L',1}A_{L',1}],
\]

and

\[
p = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix} \quad \text{with} \quad p_i = \begin{bmatrix} p_{i,1} \\ \vdots \\ p_{i,l} \\ \vdots \\ p_{i,L'} \end{bmatrix} \quad \text{and} \quad p_{i,l} = \begin{bmatrix} p_{i,l} \\ \vdots \\ p_{i,l} \end{bmatrix}
\]

\( l \in \mathcal{L} \).

**Proof:** For channel \( i \), its signal power launched into link \( l \) is transmitted either from one of previous links of link \( l \) or one of virtual links (Txs). Let \( k' \in \mathcal{L}' \) be such a link. Then \( B_{k',l} = 1, A_{k',i} = 1 \) and

\[
u_{l,i} = A_{l,i} \gamma_{l,i} (B_{k',l}A_{k',i}, p_{k',i}).
\]

Note that there is one and only one \( k' \) such that both \( B_{k',l} \) and \( A_{k',i} \) are non-zero. So the above equation can be rewritten as

\[
u_{l,i} = A_{l,i} \gamma_{l,i} (B_{k',l}A_{k',i}, p_{k',i} + \sum_{k \in \mathcal{L}' \setminus k' \neq k'} B_{k',l}A_{k,i} p_{k,i})
\]

\( = A_{l,i} \gamma_{l,i} \sum_{k \in \mathcal{L}'} B_{k,l} A_{k,i} p_{k,i} \) \quad (10)

Recall that \( \mathcal{L}' = \{1, \ldots, L'\} \). Then

\[
u_{l,i} = \gamma_{l,i} \sum_{k \in \mathcal{L}'} B_{k,l} A_{k,i} p_{k,i}
\]

\( = \gamma_{l,i} \cdot V_{l,i} \cdot p_l \)

\( \text{denoted by } \Omega_l. \)

**Remark 1:** The adjustment parameters affect both input signal and noise power simultaneously. Similarly to (10), the input noise power \( n_{i,l}^\text{in} \) is given as

\[
n_{i,l}^\text{in} = A_{l,i} \gamma_{l,i} \sum_{k \in \mathcal{L}'} (B_{k,l} A_{k,i} n_{k,i}^\text{out}),
\]

where

\[
n_{k,i}^\text{out} = \begin{bmatrix} n_{k,i}^\text{out} \\ 0 \end{bmatrix} \quad k \in \mathcal{L}\setminus\mathcal{L}' \quad n_{i}^\text{out} = \begin{bmatrix} n_{i}^\text{out} \\ 0 \end{bmatrix} \quad k \in \mathcal{L}'.
\]

The OSNR of channel \( i \in \mathcal{M} \) at the output of link \( l \in \mathcal{R}_i \) is defined as \( \text{OSNR}_l = [n_i]^\text{out} \). The end-to-end (Tx-to-Rx) OSNR of channel \( i \) is defined as \( \text{OSNR}_i = [n_i]^\text{out} \). The following provides the framework for modeling channel OSNR in optical networks. An optical amplified span \( s \) on link \( l \) is composed of an optical fiber with low coefficient, \( L_{s,i} \), which is wavelength independent, and an optical amplifier with gain \( G_{l,s,i} \). The optical amplifier introduces amplified spontaneous emission (ASE) noise, denoted by \( \Delta \text{ASE}_{l,s,i} \). Both the gain and ASE noise are wavelength dependent.

We assume that all spans on each link \( l \) have equal length and all optical amplifiers are operated in automatic power control (APC) mode with the same total power target \( P_{l}^\text{th} \) and have the same gain spectral shape [10]. Then by using an optical filter, the gain of an optical amplifier \( G_{l,s,i} \) can be decomposed as \( G_{l,s,i} = G_{l,s} f_{l,s} \), where \( G_{l,s} \) is the gain value for channel \( i \) on the spectral shape of the optical amplifier on link \( l \) and \( f_{l,s} \) is the loss of the filter at span \( s \) on link \( l \), adjusted to achieve the target \( P_{l}^\text{th} \) [7].
Lemma 2 (Lemma 1, [11]): The OSNR of channel $i \in \mathcal{M}$ at the output of link $l \in \mathcal{R}_i$ is
\[
\text{OSNR}_{l,i} = \frac{u_i}{n_i^0 + \sum_{j \in \mathcal{M}} A_{i,j} \Gamma_{i,j} u_j},
\] (13)
where $\Gamma_l = [\Gamma_{i,j}]$ is the link system matrix of link $l$ with
\[
\Gamma_{i,j} = \sum_{n=1}^{N_l} \left( \frac{G_{l,i} G_{l,j}}{G_{l,l}} \right)^n \frac{A S E_{l,i,j}}{P^0_l}, \quad \forall i, j \in \mathcal{M}_l.
\]
Remark 2: The OSNR of channel $i \in \mathcal{M}$ at Rx, $\text{OSNR}_{l,i}$, has been given in [13] (Lemma 2). The end-to-end OSNR model has a similar mathematical form as (13) but with a richer network system matrix $\Gamma$:
\[
\text{OSNR}_i = \frac{u_i}{n_i^0 + \sum_{j \in \mathcal{M}} \Gamma_{i,j} u_j},
\] (14)
where $\Gamma = [\Gamma_{i,j}]$ is with $\Gamma_{i,j} = \sum_{l \in \mathcal{R}_i} \sum_{n \in \mathcal{N}_i} \alpha_{n,l,i,j} \xi^{n-1} \Gamma_{i,j}$, where $\mathcal{R}_i$ is the set of links on $\mathcal{R}_i$ before link $l$ and $H_{i,l,k}$ is the span transmission parameter.

Based on (10), (12) and (13), a recursive OSNR model for channel $i \in \mathcal{M}_l$ can be obtained:
\[
\frac{1}{\delta Q_{l,i}} := \frac{1}{\text{OSNR}_{l,i}} = \frac{1}{\text{OSNR}_{l',i}} = \sum_{j \in \mathcal{M}} A_{i,j} \Gamma_{i,j} u_j, \quad \forall l', i \in \mathcal{M}_l
\] (15)
where link $l'$ is the link precedent to link $l$ for channel $i$. The OSNR of channel $i$ at Tx is $\text{OSNR}_{0,i} = \frac{n_i^0}{u_i}$. Typically $n_i^0$ is negligible compared to $u_i$ such that $\text{OSNR}_{0,i} \approx 0$. Then we have approximately for the end-to-end OSNR of channel $i \in \mathcal{M}$,
\[
\frac{1}{\text{OSNR}_i} \approx \sum_{l \in \mathcal{R}_i} \left( \frac{1}{\delta Q_{l,i}} \right).
\] (16)
It follows that $\frac{1}{\delta Q_{l,i}}$ measures the OSNR degradation of channel $i$ across each link $l$. Thus if we aim to maximize $\text{OSNR}_l$ from Tx to Rx, we can consider minimization of each $\frac{1}{\delta Q_{l,i}}$ between links.

IV. PROBLEM FORMULATION

We are interested in optimizing channel OSNR at Rx via game-theoretic approaches with link capacity constraints,
\[
\sum_{i \in \mathcal{M}_l} u_{i,l} \leq P^0_l, \quad \forall l \in \mathcal{L}.
\] (17)
Due to the power propagation along links and among networks, $u_{i,l}$ is a function of $u$. The forgoing (17) can be rewritten as
\[
g_l(u) = \sum_{i \in \mathcal{M}_l} u_{i,l} - P^0_l \leq 0, \quad \forall l \in \mathcal{L}.
\] (18)
Furthermore, it follows from the end-to-end OSNR model (14) that regulating channel power at Tx leads to achieve a satisfactory channel OSNR at Rx. However, it has been shown in [11] that the convexity of $g_l(u)$ in (18) is not always ensured with respect to $u$, which leads to the action space of any game formulations not necessarily convex. Thus apparently this end-to-end OSNR maximization problem is intractable.

Inspired from (16), the end-to-end OSNR maximization problem can be approximately translated to a set of minimization problems of OSNR degradation across each link. Specifically, instead of maximization of $\text{OSNR}_l$, from end to end (Tx to Rx), we can consider minimization of each $\frac{1}{\delta Q_{l,i}}$ between links. Moreover, recall that in optical networks, channel powers are adjustable not only at Tx but also at each optical switch. Thus we propose a partition approach by fully using the flexibility of channel power adjustments. We partition the network into stages composed of single links. By breaking closed loops formed by channel optical routes and selecting one link as the start, links can be placed sequentially in a ladder-nested form. This can be realized because we consider only mesh topologies with quasi-ring structures. We present a partitioned Nash game framework to solve channel OSNR maximization problems with link capacity constraints in optical networks. The partitioned Nash game is formulated composed of link Nash games and each link Nash game is played towards minimizing channel OSNR degradation. The set of OSNR degradation minimization problems on links is related to the OSNR maximization problem from Tx to Rx. Compared to the partition approach proposed in [11], this partition simplifies the structure of each stage and makes it regular and scalable. Moreover, it works for topologies in optical networks with closed loops.

We construct the link Nash game framework. Consider the link capacity constraint on link $l \in \mathcal{L}$:
\[
g_l(u_i) = \sum_{i \in \mathcal{M}_l} A_{i,l} u_i - P^0_l \leq 0, \quad \forall l \in \mathcal{L}.
\] (19)
It follows that such a partition ensures that the convexity of link capacity constraints on each link is automatically satisfied. We let $\Omega_l$ denote the action space on link $l$, $\Omega_l = \{ u_i \in \Omega | g_l(u_i) \leq 0 \}$, where $\Omega$ denotes the Cartesian product of $\Omega_l$. The action set of individual channel $i \in \mathcal{M}_l$ is defined as the projection of $\Omega_l$ on the direction of channel $i$, namely, $\Omega_{i,l} = \{ \xi \in \Omega_l | g_l(u_{i,l} - \xi) \leq 0 \}$. It can be seen that both $\Omega_l$ and $\Omega_{i,l} = \{ \xi \in \Omega_l | g_l(u_{i,l} - \xi) \leq 0 \}$ are compact and convex.

The link Nash game is played with each channel attempting to minimize its individual cost with respect to its OSNR degradation. We specify a channel cost $J_{i,l}$ for channel $i$ on each link $l$, defined as a difference between a pricing function $A_{i,l} u_{i,l}$ and a utility function $U_{i,l}$, namely,
\[
J_{i,l} = A_{i,l} u_{i,l} - \beta_{i,l} U_{i,l}, \quad \forall l \in \mathcal{L},
\] (20)
with
\[
U_{i,l} = \ln \left( 1 + \frac{A_{i,l} u_{i,l}}{1/\delta Q_{l,i} - \Gamma_{i,l}} \right),
\] (21)
where $\beta_{i,l} > 0$ is the channel parameter and $\alpha_{i,l} > 0$ is for scalability. Substituting (15) into (20) yields
\[
J_{i,l} = A_{i,l} u_{i,l} - \beta_{i,l} \ln \left( 1 + \alpha_{i,l} A_{i,l} u_{i,l} / X_{i,l} \right),
\] (22)
where

\[ X_{l,-i} = \sum_{j \in M:j \neq i} A_{l,j} \Gamma_{l,j} u_{l,j}. \]

It follows that \( J_{l,i} \) is continuously differentiable in its arguments and convex in \( u_{l,i} \). We denote such a link Nash game by \( GAME(M, \Omega_{l,i}, J_{l,i}) \).

We exploit the partitioned Nash game. By partitioning and selecting one link as the start, links can be sorted sequentially with the interpretation that across all links, \( \sum_{l \in \mathbb{R}} J_{l,i} \) is related to the overall OSNR degradation for channel \( i \) on its route \( \mathcal{R}_i \). Solutions of all link Nash games are interconnected. The explanation is given as follows. Recall that \( \gamma_{l,i} \) is the adjustable parameter for channel \( i \) on stage \( l \). Let \( u_{l,i}^* = [u_{l,i}^*] \) be a Nash equilibrium (NE) solution of \( GAME(M, \Omega_{l,i}, J_{l,i}) \). The corresponding signal power vector at the output of link \( l \) is \( p_l^* = [p_{l,i}^*] \) and the corresponding augmented output power vector defined in Lemma 1 is \( p^* \).

It is noted that for those channel \( i \not\in M_l \), we randomly set values of \( u_{l,i}^* \) and \( p_{l,i}^* \). By using (9) in Lemma 1, optimal \( \gamma_{l,i} \) can be obtained by solving the corresponding component-wise equation in \( u_{l,i}^* = \text{diag}(\gamma_{l,i}^*) \cdot v_l^* \).

Finally, the partitioned Nash game admits a solution if each \( \gamma_{l,i}^* \in [\gamma_{min}, \gamma_{max}] \). Next, we show how to use Lagrangian extension and duality developed in [14] to obtain an NE solution of \( GAME(M, \Omega_{l,i}, J_{l,i}) \). Then we propose an iterative hierarchical algorithm for computation of equilibria of the partitioned Nash game.

V. DECOMPOSITION AND ALGORITHM

Note that in \( GAME(M, \Omega_{l,i}, J_{l,i}) \), the individual cost function \( J_{l,i} \) is generally defined for each channel \( i \in M \). If channel \( i \) is not transmitted on link \( l \), i.e., \( i \not\in M_l \), then \( A_{l,i} = 0 \) and \( J_{l,i} = 0 \), which means that the decision of channel \( i \) \( i \not\in M_l \) does not affect the decisions made by other channels \( j \in M_l \). Thus \( GAME(M, \Omega_{l,i}, J_{l,i}) \) is equivalently a reduced Nash game played among \( M_l \) channels, denoted by \( GAME(M_l, \Omega_{l,i}, J_{l,i}) \). Furthermore, the existence of an NE solution of \( GAME(M, \Omega_{l,i}, J_{l,i}) \) is guaranteed by Theorem 4.4 in [5]. In the next we use the mark "\( \sim \)" to indicate the associated reduced vector in \( GAME(M_l, \Omega_{l,i}, J_{l,i}) \). For example, \( \bar{u}_l \) is the reduced vector obtained by removing from \( u_l \) those elements \( u_{l,i} \), \( i \not\in M_l \). Sometimes we also write \( \bar{u}_l \) as \( \bar{u}_l = (u_{l,-i}, u_{l,i}) \).

A. Hierarchical Decomposition

An NE solution of \( GAME(M_l, \Omega_{l,i}, J_{l,i}) \) can be computed by using the Lagrangian extension and decomposition to the game-theoretic framework developed in [14].

We first define several augmented two-argument functions for \( GAME(M_l, \Omega_{l,i}, J_{l,i}) \):

\[ \tilde{J}_l(\bar{u}_l; \bar{x}_l) = \sum_{l \in \mathbb{M}_l} J_{l,i}(u_{l,-i}, x_{l,i}), \]

\[ \tilde{g}_l(\bar{u}_l; \bar{x}_l) = \left( \sum_{l \in \mathbb{M}_l} g_l(u_{l,-i}, x_{l,i}) \right), \]

\[ \tilde{L}_l(\bar{u}_l; \bar{x}_l; \mu_l) = \tilde{J}_l(\bar{u}_l; \bar{x}_l) + \mu_l \tilde{g}_l(\bar{u}_l; \bar{x}_l), \]

\[ D_l(\mu_l) = \tilde{L}_l(\bar{u}_l^*; \bar{u}_l^*; \mu_l), \]

where \( J_{l,i} \) is defined in (22) with \( A_{l,i} = 1 \), the scalar \( \mu_l \) is the Lagrange multiplier and \( \bar{u}_l^* \) is such that \( \bar{u}_l = \bar{u}_l^* \) satisfies

\[ \bar{u}_l = \arg\min_{\bar{x}_l \in \Omega_l} \tilde{L}_l(\bar{u}_l; \bar{x}_l; \mu_l) \]

with \( \Omega_l \) denoting the Cartesian product of \( \Omega_i \), \( i \in M_l \).

An NE solution of \( GAME(M_l, \Omega_{l,i}, J_{l,i}) \) can be found by solving a constrained minimization of \( \tilde{J}_l(\bar{u}_l; x_{l,i}) \) with respect to \( \bar{x}_l \), which admits a fixed-point solution \( \bar{u}_l^* \). Individual components of \( \bar{u}_l^* \) constitute an NE solution. In the Lagrangian optimality condition, \( \bar{u}_l^* \) is obtained by first minimizing \( \tilde{L}_l(\bar{u}_l; x_{l,i}; \mu_l) \) with respect to \( \bar{x}_l \). The next step involves finding a fixed-point solution \( \bar{u}_l^* \) by setting \( \bar{x}_l = \bar{x}_l^* \).

Since \( \bar{u}_l^* \) depends on \( \mu_l \), an NE solution-Lagrange multiplier pair \((\bar{u}_l^*(\mu_l^*), \mu_l^*)\) is obtained if

\[ \mu_l^* g_l(\bar{u}_l^*) = 0 \quad \text{and} \quad \mu_l^* \geq 0. \]

The minimization of \( \tilde{L}_l(\bar{u}_l; \bar{x}_l; \mu_l) \) can be decomposed by Proposition 1. Furthermore, since each link \( l \) is the simplest single-sink structure, \( GAME(M_l, \Omega_{l,i}, J_{l,i}) \) can be naturally decomposed into a lower-level Nash game with no coupled constraints and a higher-level link problem for pricing (Lagrange multiplier). The lower-level Nash game is obtained with the following individual cost function,

\[ \tilde{L}_{l,i}(\bar{u}_{l,-i}, x_{l,i}; \mu_{l,i}) = J_{l,i}(\bar{u}_{l,-i}, x_{l,i}) + \mu_{l,i} x_{l,i}, \quad \forall i \in M_l, \]

where \( J_{l,i} \) is defined in (22) and an uncoupled action set \( \Omega_i \).

We denote this game by \( GAME(M_l, \Omega_i, \tilde{L}_{l,i}) \), whose NE solution is characterized by the following result.

Proposition 2: For each \( \mu_i \geq 0 \), \( GAME(M_l, \Omega_i, \tilde{L}_{l,i}) \) admits an NE solution if \( a_{l,i} \in \tilde{L}_{l,i} \) satisfies

\[ \sum_{j \in M_l:j \neq i} \Gamma_{l,j} < a_{l,i}, \quad \forall i \in M_l. \]

The inner NE solution \( \bar{u}_{l,i}^*(\mu_i) \) is unique, given as

\[ \bar{u}_{l,i}^*(\mu_i) = \tilde{\Gamma}_i^{-1} \cdot \bar{b}_i(\mu_i), \]

where \( \tilde{\Gamma}_i = [\tilde{\Gamma}_{l,j}] \) and \( \bar{b}_i(\mu_i) = [b_{l,i}(\mu_i)] \) are defined as

\[ \tilde{\Gamma}_{l,j} = \begin{cases} \alpha_{l,j}, & j = i, \\
\Gamma_{l,j}, & j \neq i \end{cases}, \quad \forall i \in M_l, \quad \bar{b}_i(\mu_i) = \frac{a_{l,i} \tilde{\beta}_{l,i}}{\alpha_{l,i} + \mu_i}. \]

Proof: Following an argument similar to the one in the proof of Theorem 3 of [13], we can rewrite \( \tilde{L}_{l,i} \) (24) as

\[ \tilde{L}_{l,i} = (a_{l,i} + \mu_i) u_{l,i} - \beta_{l,i} \ln \left( 1 + \frac{a_{l,i} u_{l,i}}{\sum_{j \in M_l:j \neq i} \Gamma_{l,j} u_{l,j}} \right). \]

It can be seen that for any given \( \mu_i \geq 0 \), \( \tilde{L}_{l,i} \) is jointly continuous in all its arguments and

\[ \frac{\partial^2 \tilde{L}_{l,i}}{\partial u_{l,i}^2} > 0, \quad \forall u_{l,i} \in \Omega_i. \]

It follows that \( \tilde{L}_{l,i} \) is strictly convex in \( u_{l,i} \). Recall that for each \( i \in M \), \( \Omega_i \) is a closed, bounded and convex subset of \( \mathbb{R} \). Then by Theorem 4.3 in [5], \( GAME(M_l, \Omega_i, \tilde{L}_{l,i}) \) admits an NE solution.
An inner NE solution can be found by solving the necessary conditions $\frac{\partial E_i}{\partial u_i} = 0$. From (28), we obtain

$$a_{l,i} u_{l,i}^*(\mu_l) + \sum_{j \in M_l, j \neq i} \Gamma_{l,i,j} u_{l,j}^*(\mu_l) = \frac{a_{l,i} \bar{b}_l(\mu_l)}{\alpha_l} + \mu_l, \quad \forall i \in M_l,$$

Equivalently, in a matrix form, this is written as

$$\bar{\Gamma}_l \cdot \bar{u}_l^*(\mu_l) = \bar{b}_l(\mu_l),$$

(29)

where matrix $\bar{\Gamma}_l$ and vector $\bar{b}_l(\mu_l)$ are defined in (27). Therefore a unique solution of (29) exists if the matrix $\bar{\Gamma}_l$ is invertible. Hence, $\bar{\Gamma}_l$ is a positive-entry matrix. If (25) holds, then $\bar{\Gamma}_l$ is strictly diagonally dominant. From Gersgorin’s Theorem in [9], it follows that $\bar{\Gamma}_l$ is invertible and a unique solution of (29) exists. Furthermore, it is an inner NE solution of $GAME(M_l, \Omega_l, L_{l,i}).$

B. Algorithm

We propose an iterative hierarchical algorithm based on the NE solution of the lower-level Nash game and coordination at the higher level. For given $\mu_l \geq 0$, $\bar{u}_l^*(\mu_l)$ in Proposition 2 is a solution to the Lagrangian optimality condition. Recall that $(\bar{u}_l^*(\mu_l^*), \bar{\mu}_l^*)$ is an NE solution-Lagrange multiplier pair if (23) is satisfied. By using (19), $\mu_l^* \geq 0$ can be obtained by solving

$$\mu_l^* \left( \sum_{i \in M_l} u_{l,i}^*(\mu_l^*) - P_l^0 \right) = 0.$$  

(30)

Based on these, an iterative hierarchical algorithm is proposed for both link pricing and channel power adjustment.

Channel Algorithm: On each link $l$ with given price $\mu_l(t)$, the following distributed iterative channel algorithm is used for each channel $i \in M_l$:

$$u_{l,i}(n+1) = \frac{\beta_{l,i}}{\mu_l(t)} \frac{\bar{u}_{l,i}(n)}{\alpha_{l,i}} \left( \frac{1}{OSNR_{l,i}(n)} - \frac{1}{OSNR_{l,i}^*} - \Gamma_{l,i} \right),$$

(31)

where link $l'$ is the precedent of link $l$. Note that both $OSNR_{l,i}$ and $\mu_l(t)$ are invariant during the channel iteration on link $l$. Moreover the price $\mu_l(t)$ is same for all channel $i$. Each channel $i$ on link $l$ updates its signal power $u_{l,i}$ based on the feedback information, i.e., its OSNR at the output of link $l$, $OSNR_{l,i}$, and fixed parameters, $\mu_l(t)$ and $OSNR_{l,i}$.

Proposition 3: If for all $i \in M_l$, $a_{l,i}$ in $L_{l,i}$ (24) are selected such that (25) is satisfied. Then for each $\mu_l \geq 0$, channel algorithm (31) converges to the inner NE solution $\bar{u}_l^*(\mu_l)$, (26).

Proof: By Proposition 2, if on link $l$, $a_{l,i}$ is selected satisfying (25), then for each given $\mu_l$, an inner NE solution $\bar{u}_l^*(\mu_l)$, (26), exists and is unique in the sense of inerness. The solution for channel $i \in M_l$ is given as

$$u_{l,i}^* = \frac{\beta_{l,i}}{\alpha_{l,i}} + \frac{1}{a_{l,i}} \sum_{j \in M_l, j \neq i} \Gamma_{l,i,j} u_{l,j}^*.$$  

(32)

The rest of the proof follows directly from the proof of Lemma 4 in [15]. We state here for completeness. Let $e_{l,i}(n) := u_{l,i}(n) - u_{l,i}^*(\mu_l)$. The corresponding vector form is $\bar{e}_l(n) = [\ldots, e_{l,i}(n), \ldots]^T$. We also define

$$\|\bar{e}_l(n)\|_{\infty} := \max_{i \in M_l} |e_{l,i}(n)|.$$  

We can show that under condition (25), $\|\bar{e}_l(n + 1)\|_{\infty} \leq C_0 \|\bar{e}_l(n)\|_{\infty}$, where $0 \leq C_0 < 1$ and $\|\bar{e}_l(n)\|_{\infty} \leq C_0^\infty \|\bar{e}_l(0)\|_{\infty}$, such that the sequence $\{\bar{e}_l(n)\}$ converges to 0. Therefore channel algorithm (31) converges to the inner NE solution $\bar{u}_l^*(\mu_l)$, (26).

Link Algorithm: The link algorithm is a gradient projection algorithm [6], developed based on (30). On each link $l$, after every $N$ iterations of the channel algorithm (31), the new link price $\mu_l$ is generated at each iteration time $t$ as

$$\mu_l(t + 1) = \left[ \mu_l(t) + \eta \left( \sum_{i \in M_l} u_{l,i}(\mu_l)(t) - P_l^0 \right) \right]^+,$$  

(33)

where $\eta > 0$ is a step-size and $[z]^+ = \max\{z, 0\}$. Practically, $N$ is sufficiently large such that each channel power converges to its solution. The proof for the convergence of this link algorithm is sketched next. We show that the step-size $\eta$ has an explicit upper-bound for the convergence. For notation simplicity, we rewrite (33) as

$$\mu_l(t + 1) = \left[ \mu_l(t) + \eta \left( s(\mu_l(t)) - P_l^0 \right) \right]^+.$$  

(34)

It follows from (26) that $s(\mu)$ is a strictly decreasing function with respect to $\mu$. Then the derivative of $s(\mu)$ is negative, i.e., $s'(\mu) < 0$. Thus there exists a unique solution $\mu^* > 0$ for $s(\mu) - \Delta P_l^0 = 0$, i.e., $s(\mu^*) - P_l^0 = 0$. Moreover, $s(\mu)$ is strictly convex. It follows that $s(\mu) > P_l^0$ when $\mu < \mu^*$, and $s(\mu) < P_l^0$ when $\mu > \mu^*$. We define a function

$$\theta(\mu) := \left\{ \begin{array}{ll} \frac{\mu - \mu^*}{P_l^0 - s(\mu)} & \text{when } \mu \neq \mu^*, \\ 1 - \frac{1}{s'(\mu^*)} & \text{when } \mu = \mu^*. \end{array} \right.$$  

(35)

Lemma 3: The function $\theta(\mu)$ is positive, continuous and increasing with $\lim_{\mu \to 0} \theta(\mu) = 0$ and $\lim_{\mu \to \infty} \theta(\mu) = \infty$.

Proof: First, the function $\theta(\mu)$ is obviously continuous everywhere except at $\mu^*$ and

$$\lim_{\mu \to \mu^*} \frac{\mu - \mu^*}{P_l^0 - s(\mu)} = \frac{1}{s'(\mu^*)} > 0.$$  

Hence, it is continuous everywhere. Second, it can be easily checked that the function is positive when $\mu > 0$. Moreover, since $s(\mu) \to \infty$ as $\mu \to 0$ and $s(\mu) \to 0$ as $\mu \to \infty$, it follows that

$$\lim_{\mu \to 0} \frac{\mu - \mu^*}{P_l^0 - s(\mu)} = 0 \text{ and } \lim_{\mu \to \infty} \frac{\mu - \mu^*}{P_l^0 - s(\mu)} = \infty.$$  

Third, taking the derivative of $\theta$ with respect to $\mu$, we obtain

$$\theta'(\mu) = \frac{(P_l^0 - s(\mu)) + (\mu - \mu^*)s'(\mu)}{(P_l^0 - s(\mu))^2}.$$  

Recall that $s(\mu)$ is strictly convex. Thus $(P_l^0 - s(\mu)) + (\mu - \mu^*)s'(\mu) > 0$ and therefore $\theta'(\mu) > 0$, which implies $\theta(\mu)$ is increasing. Hence, the conclusion follows.
Next we define a constant $\sigma_0$ as

$$\sigma_0 := \inf_{x \in \mathcal{I}} \frac{x - s^{-1}(2P^0 - s(x))}{P^0 - s(x)},$$

where $\mathcal{I}$ is an interval, $\mathcal{I} = [s^{-1}(2P^0), \mu^*]$ with $\mu^*$ satisfying $s(\mu^*) - P^0 = 0$ and $s^{-1}$ is the inverse function of $s$. For $\mu > \mu^*$, we define a function $\sigma(\mu)$ as

$$\sigma(\mu) := \inf_{x \in \mathcal{I} \cup [\mu^*, \mu]} \frac{x - s^{-1}(2P^0 - s(x))}{P^0 - s(x)}.$$ 

It can be easily seen that $\sigma(\mu)$ exists and is positive and with $\lim_{\mu \to \mu^*} \sigma(\mu) = \sigma_0$. Moreover, $\sigma(\mu)$ is continuous and non-increasing. Hence, if $\sigma_0 > \theta(\mu^*)$, then there is a unique $\mu = \mu_\star$ such that $\sigma(\mu) = \theta(\mu)$. The following result proves the convergence of link algorithm (34).

**Theorem 1:** For sufficient small $\eta$, the link algorithm (34) converges to $\mu_\star$. Specifically,

$$\eta < \begin{cases} \theta(\mu^*), & \sigma_0 \leq \theta(\mu^*) \\ \theta(\mu_\star), & \text{otherwise} \end{cases}$$

where function $\theta(\cdot)$ is defined in (35).

**Proof:** See Appendix.

Fig. 2 depicts the channel algorithm and the link algorithm acting on the system of link $l$. After every $N$ iteration time that each channel runs the channel algorithm, link updates its price. In Fig. 2, $\lceil \frac{n}{N} \rceil$ denotes the floor function of $\frac{n}{N}$ and $u_{\Gamma_l}(n)$ is the total input channel power on link $l$ at iterative time $n$. Note that the channel algorithm is implemented practically in a distributed way.

![Fig. 2. Channel algorithm and link algorithm applied on link $l$](image)

**Remark 3:** Recall that the partition leads to the convexity of coupled constraints propagated along links on each stage (link) naturally. For multi-link topologies, this partition simplifies the partitioned structure compared to the approach proposed in [11]. Moreover, such a partition also helps the development of the link algorithm and makes it simple, scalable and distributed.

**VI. SIMULATIONS**

Simulations are conducted on the quasi-ring topology shown in Fig. 3(a) which is a representative for selected paths extracted from mesh configurations. The quasi-ring topology is a ring-type topology with partially closed loops being formed by channel optical paths [1] as shown in Fig. 3(b). We assume that each link is with same number of amplified spans and all optical amplifiers deployed along links have the same gain spectral shape. The dynamic adjustment parameter $\gamma_{l,i}$ is bounded within $[\gamma_{\min}, \gamma_{\max}] = \{0, 10\}$ for all $l \in \mathcal{L}$ and for all $i \in \mathcal{M}$. Individual channel cost $J_{l,i}$ is with $\alpha_{l,i} = 10 \cdot \Gamma_{l,i}$, $\beta_{l,i} = 1 + 0.1 \cdot i$ and $\sigma_{l,i} = 60 \cdot \Gamma_{l,i}$. The link total power targets are $P_{m} = 1.5$, $P_{1} = 2.5$ and $P_{0} = 2.0$ (mW). The step-size in the link algorithm is $\eta = 0.1$.

![Fig. 3. Quasi-ring topology](image)

Optical routes of channels 1 and 3 are link 1 $\to$ link 2 $\to$ link 3 and link 3 $\to$ link 1 $\to$ link 2, respectively. We notice that each of the links is the intermediate or the end on channel optical routes. We partition this structure and formulate a partitioned Nash game composed of three link Nash games. On each link Nash game, the convexity of link capacity constraints is automatically satisfied. By breaking the closed loop and selecting one link as the start, say, link 3, which is shown in Fig. 4, link Nash games can be played sequentially.

![Fig. 4. Unfolded with the starting link, link 3](image)

The overall recursive process is such that the order of play is: link 3 $\to$ link 1 $\to$ link 2. On link 3, the adjustable parameters for channels 1 and 2 are initially set as $\gamma_{3,1}$ and $\gamma_{3,2}$, respectively, where the superscript $1$ indicates the number of iteration of the game among links. The game on link 3 settles down at $u_{3}^\star(\mu_3^\star)$ with the corresponding channel output power $p_{3}^\star(\mu_3^\star)$. Sequentially, the game on link 1 is played with a solution, $u_{1}^\star(\mu_1^\star)$. The channel output power is $p_{1}^\star(\mu_1^\star)$. Given $p_{3}^\star(\mu_3^\star)$, the adjustable parameter on link 1, $\gamma_{3,3}^\star$, is determined. The game on link 2 is played after that. The corresponding solution is $u_{2}^\star(\mu_2^\star)$ and the channel output power is $p_{2}^\star(\mu_2^\star)$. Then the adjustable parameters on link 2, $\gamma_{2,i}^\star$, $i = 1, 2$, are determined. With the given $p_{2}^\star(\mu_2^\star)$, link 3 determines its adjustable parameters by $\gamma_{3,i}^\star = \frac{u_{3}^\star(\mu_3^\star)}{p_{3}^\star(\mu_3^\star)}$, $i = 1, 2$.

In the simulation, for every $N = 20$ iteration, the link adjusts its price via the link algorithm and then channels...
readjust their powers. Evolutions in time of channel OSNR, total power and link price on each link $l$ are shown in Fig. 5–Fig. 7, respectively. Note that a different starting link can be selected, say, link 1, such that games on links are played sequentially: link $1 \rightarrow$ link $2 \rightarrow$ link 3. Typically we select the starting link as the link where channels are added directly from the transmitters.

![Fig. 5. Link 3](image)

VII. EXPERIMENTAL RESULTS

We use experimental results to validate the applicability of the achieved theoretical results. Experiments are conducted on a 2-link configuration shown in Fig. 8. In this configuration, three channels, corresponding to three wavelengths, 1535.04, 1537.40, 1533.47 (nm), are added on link 1, channel 3 exits at the output of link 1 and channels 1 and 2 exit at the output of link 2. Each link has an optical amplifier and the constant total power targets of two links are 2.5 and 1.5 (mW), respectively.

Since there is no closed loops existing in the configuration, the game on link 1 is played first. For every $N_1 = 10$

iteration, link 1 adjusts its price via the link algorithm and then channels readjust their powers based on new price. Fig. 9 shows the evolutions in iteration time of channel power, total power and link price on link 1.

The game on link 2 starts to play after the game on link 1 settles down. For every $N_2 = 7$, iteration, link 2 adjusts its price. The evolutions are shown in Fig. 10.

VIII. CONCLUSIONS

We studied channel OSNR optimization problems with link capacity constraints in optical networks with general topologies. We formulated a partitioned Nash game composed of link Nash games. In the partitioned Nash game, each link Nash game is played towards minimizing channel OSNR degradation across links. The hierarchical decomposition is applicable to each link Nash game. By selecting a starting link, the link Nash game can be played sequentially and the hierarchical decomposition leads to a lower-level Nash game for channels with no coupled constraints and a higher-level problem for link pricing. Computation of equilibria is based...
on an iterative algorithm. We implemented this approach in the quasi-ring topology.

REFERENCES

is always below \( \mu^* \). Note that \( \eta < \sigma_0 \). Then for 
\[ \mu \in [s^{-1}(2P^0), \mu^*) , \]
\[ \eta < \frac{\mu - s^{-1}(2P^0) - s(\mu)}{P^0 - s(\mu)} \]
\[ \eta[(P^0 - s(\mu)) < \mu - s^{-1}(2P^0) - s(\mu)) \]
\[ \mu + s(\mu - P^0) < s^{-1}(2P^0 - s(\mu)) \]
\[ 2P^0 - s(\mu) < s(\mu + s(\mu - P^0)) \]
\[
\]
When \( \mu < s^{-1}(2P^0) \), we have \( 2P^0 - s(\mu) < 0 \). Then \( 2P^0 - s(\mu) < s(\mu + s(\mu - P^0)) \). Hence, for \( \mu < \mu^* \), we have
\[
2P^0 - s(\mu) < s(\mu + s(\mu - P^0)) \]
\[ \Rightarrow s(\mu) > P^0 \Rightarrow s(\mu + s(\mu - P^0)) \]
Thus when \( \mu(2t) \in (\mu^*, \theta^{-1}(\eta)) \), \( s(\mu(2t)) > P^0 \). It implies that \( \mu(2t+1) \) is strictly increasing. Both subsequences \( \mu(2t) \) and \( \mu(2t+1) \) have to converge to \( \mu^* \) by using Lemma 4 in [12] and the discrete-time Lyapunov stability theorem [8].

(b) For \( \mu(0) \geq \theta^{-1}(\eta) \), we show that it goes into \( [\mu^*, \theta^{-1}(\eta)] \) at a finite step by contradiction. Suppose it does not go into the interval \( [\mu^*, \theta^{-1}(\eta)] \). That means it remains greater than \( \theta^{-1}(\eta) \) infinitely. By using the link algorithm (34), it is strictly decreasing and will be smaller than \( \theta^{-1}(\eta) \). This contradicts the assumption. Thus it goes into the interval \( [\mu^*, \theta^{-1}(\eta)] \). Then it converges in the sequence converges to \( \mu^* \).

(c) For \( \mu(0) < \mu^* \); by (1), it jumps to above \( \mu^* \) with one step. Then by (a) and/or (b) it converges to \( \mu^* \).

Case 2 is illustrated in Fig. 11(b). We conclude that the link algorithm (34) converges to \( \mu^* \) given that \( \eta \leq \theta(\mu^*) \) or \( \eta < \theta(\mu) \) if \( \sigma_0 \leq \theta(\mu^*) \).

Case 1: Suppose \( \eta \leq \theta(\mu^*) \) under the condition \( \sigma_0 \leq \theta(\mu^*) \).

Then by Lemma 3 we know that \( \theta^{-1}(\eta) < \mu^* \) and that for \( \mu \geq \theta^{-1}(\eta) \), \( \eta \leq \theta(\mu) \), which implies \( 0 < \mu + \eta(s(\mu) - P^0) \leq \mu^* \) if \( \mu \in (\theta^{-1}(\eta), \mu^*) \) and \( \mu + \eta(s(\mu) - P^0) > \mu^* \) if \( \mu > \mu^* \). That means for initial condition \( 0 \) in \( (\theta^{-1}(\eta), \mu^*) \), it monotonically increases with upper bound \( \mu^* \) and for initial condition \( 0 \) in \( \mu^* \), it monotonically decreases with lower bound \( \mu^* \). So they converge. For initial condition \( 0 < \theta(\mu) \), we obtain \( \eta < \theta(\mu) \) and \( \eta \leq \theta(\mu) \), which means that with one step it jumps one step and then monotonically converges. Case 1 is illustrated in Fig. 11(a).

Case 2: Suppose \( \theta(\mu^*) < \eta < \theta(\mu) \) with \( \theta(\mu^*) < \sigma_0 \).

It follows from \( \theta(\mu^*) < \eta \) and Lemma 3 that

1. For any \( \mu < \mu^* \), \( \eta > \theta(\mu) = \mu - s(\mu - P^0) \) and therefore \( \mu + \eta(s(\mu) - P^0) > \mu^* \), which means that with one step it jumps to above \( \mu^* \);

2. For \( \mu \in (\theta^{-1}(\eta), \infty) \), we have \( \eta < \theta(\mu) = \mu - s(\mu) \) and therefore \( \mu + \eta(s(\mu) - P^0) > \mu^* \), which means that with one step it remains above \( \mu^* \);

3. For \( \mu \in (\mu^*, \theta^{-1}(\eta)) \), we have \( \eta < \theta(\mu) = \mu - s(\mu) \), and therefore \( \mu + \eta(s(\mu) - P^0) < \mu^* \), which means that with one step it jumps to below \( \mu^* \); Furthermore, we have \( \eta > \theta(\mu) = \mu - s(\mu) \), and \( \eta < \theta(\mu) = \sigma(\mu) \), which implies by the definition, for any \( \mu \in (\mu^*, \theta^{-1}(\eta)) \),

\[
\eta < \frac{\mu - s^{-1}(2P^0) - s(\mu)}{P^0 - s(\mu)} 
\]
\[ \eta(\mu - s^{-1}(2P^0) - s(\mu)) \]
\[ \eta(s(\mu) - P^0) < s^{-1}(2P^0 - s(\mu)) \]
\[ \mu + \eta(s(\mu) - P^0) > s^{-1}(2P^0 - s(\mu)) \]
\[ 2P^0 - s(\mu) > s(\mu + s(\mu - P^0)) \]
\[
\]
In discrete-time domain when \( \mu(\bar{t}) \in (\mu^*, \theta^{-1}(\eta)) \), it leads to
\[
\eta[s(\mu(t)) - P^0] > \eta(s(\mu(t+1)) - P^0), \mu(t) \in (\mu^*, \theta^{-1}(\eta)) 
\]
Based on the above, for different initial value of \( \mu \), we conclude:

(a) For \( \mu(0) \in (\mu^*, \theta^{-1}(\eta)) \), \( \mu(2t) \) is always above \( \mu^* \) and is strictly decreasing. From (2), \( \mu(2t+1) \)