A Class of Position Controllers for Underactuated VTOL Vehicles

Ashton Roza, Manfredi Maggiore

Abstract

This paper proposes a unified framework for stabilizing a hover equilibrium for a class of vertical take-off and landing (VTOL) aircraft using a cascade control architecture which yields almost global stability in the absence of wind gusts. The control design process is partitioned into two decoupled parts: the design of a bounded position controller for a fully actuated point-mass particle, and the design of an attitude stabilizer. The position feedback is then used in an outer loop to provide reference signals for the inner loop attitude stabilizer. A number of sample feedbacks in the proposed controller class are presented, some of which include integral action to compensate for an unknown mass.

I. Introduction

The most basic control specification for autonomous vehicles is that of position control, which requires stabilizing a desired position and orientation in space of the vehicle. In this paper we investigate the problem of making a VTOL aircraft hover at a desired position in space for almost any initial condition, assuming that the aircraft is not affected by wind gusts. The aircraft in question is propelled by a nonzero thrust vector along a single body axis, and incorporates some mechanism to induce torques about all three body axes. A typical example is a quadrotor helicopter. The challenge in the position control problem arises from the fact that the VTOL aircraft under consideration is underactuated, having six degrees-of-freedom (three for rotation and three for translation) and four control inputs (the magnitude of the thrust and three torques).

The position control problem has been researched with vigour over the past decade. In [5], Hua *et al.* developed a general framework for attitude, position, and velocity control of the same class of vehicles considered in this paper. Using a backstepping approach, they proposed a position control design that is developed in steps through a series of simpler controllers (i.e., from thrust direction control, to velocity control, to position control). The multi-stage design idea in [5] is appealing and intuitive. Different manifestations of this idea are found in the work of many researchers. Of particular note is the work of A. Tayebi and collaborators in [6], [7], [8], where a two-stage design strategy is developed whereby an outer translational control loop assigns a desired thrust vector, treating the vehicle as a point mass. An inner attitude control loop then applies a torque input that aligns the thrust vector to the desired axis, while simultaneously controlling the vehicle's heading. This latter stage is designed using the backstepping technique. In [6], the two-stage design approach is used to derive a position controller that does not

rely on linear and angular velocity measurements, and in [7] it is used to design an adaptive position controller that compensates for a constant disturbance force. The two-stage approach of Tayebi and collaborators relies on unit quaternion representations of attitude. In [9], a similar two-stage approach is developed for rotation matrix representations. All approaches described above achieve almost-global position stabilization, i.e., asymptotic stabilization with a domain of attraction which has full measure in the state space of the closed-loop system. The majority of approaches in the literature provide *local* asymptotic stabilization either because they cannot guarantee that combining separately designed controllers for the translational and rotational systems in cascade will guarantee almost global stability of the overall system, or because they represent attitude using local parametrizations such as Euler angles. For instance, see [10], [11], or [12].

In this paper we present a two-stage control architecture to solve the position control problem almost globally. The proposed architecture is inspired by the work of A. Tayebi and collaborators in [6], [7], [8]. Indeed, the control design is partitioned into two stages, a translational control stage for a point-mass system and an attitude control stage. But rather than relying on specific intermediate control designs for the individual control stages tied together through the backstepping technique, we identify *general classes* of point-mass controllers and attitude stabilizers that can be combined to form almost-globally stabilizing controllers for the thrust-propelled vehicle. In the proposed framework, the control design is modular, in that the design of either one control stage is independent of the design of the other stage. This is achieved by avoiding the use of backstepping, and it has a number of advantages when compared to existing approaches. First, decoupling the design of the two control stages has the effect of reducing the complexity of the control design process, and the final feedback is intuitive and structured. Second, the modularity of the approach allows one to leverage the rich literature on point-mass position control and on attitude stabilization, without having to develop ad-hoc designs. For instance, one may easily compare the performance of different attitude stabilizers by simply swapping them in the architecture. One can incorporate integral action in the position control stage to gain robustness against an unknown mass without having to redesign the attitude controller. Some of the ideas presented here were explored in the context of co-axial helicopters in [13].

II. Preliminaries

We now review preliminary notions and stability definitions used in this paper. We let $v \cdot w$ denote the Euclidean inner product between vectors v, $w \in \mathbb{R}^3$, and by $\|v\| := (v \cdot v)^{1/2}$ the Euclidean norm of v. We let $\{e_1, e_2, e_3\}$ denote the natural basis of $\mathbb{R}^3$. If $x \in \mathbb{R}^3$, we denote

$$x^\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix},$$

so that $x^\times y = x \times y$ for all $x, y \in \mathbb{R}^3$. Vice versa, given a skew-symmetric matrix $M = -M^\top \in \mathbb{R}^{3\times 3}$, we denote $M_\times := [M_{32} \ M_{13} \ M_{21}]^\top$. If $x \in \mathbb{R}^3$, we let $x^\wedge := (0, x) \in \mathbb{R} \times \mathbb{R}^3$. We denote $SO(3) := \{R \in \mathbb{R}^{3\times 3} : R^{-1} = R^\top, \det(R) = 1\}$. If Γ is a closed subset of a Riemannian manifold $\mathcal{X}$, and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a distance metric on $\mathcal{X}$, we denote by $\|x\|_\Gamma := \inf_{y \in \Gamma} d(x, y)$ the point-to-set distance of $x \in \mathcal{X}$ to Γ. If $\epsilon > 0$, we let $B_\epsilon(\Gamma) := \{x \in \mathcal{X} : \|x\|_\Gamma < \epsilon\}$. By $\mathcal{N}(\Gamma)$ we denote a generic neighbourhood of Γ in $\mathcal{X}$. We will let S^1 denote the set of real numbers

modulo 2π. Finally, if A and B are two sets, we denote by $A\backslash B$ the set-theoretic difference of A and B. We use bold characters to denote state-dependent reference values: $\mathbf{R}$ is an attitude reference and $\boldsymbol{\omega}$ is an angular velocity reference. The tilde mark on top of a symbol denotes an error between an actual state and its corresponding reference: $\tilde{\mathbf{R}}$ is an attitude error and $\tilde{\omega}$ is an angular velocity error.

The following stability definitions are taken from [14]. Let $\Sigma : \dot{\chi} = f(\chi)$ be a smooth dynamical system with state space a Riemannian manifold $\mathcal{X}$, and let $\phi(t, \chi_0)$ denote its local phase flow. Let $\Gamma \subset \mathcal{X}$ be a closed set which is positively invariant for Σ, i.e., such that for all $\chi_0 \in \Gamma$, $\phi(t, \chi_0) \in \Gamma$ for all $t \geq 0$ for which $\phi(t, \chi_0)$ is defined.

Definition 2.1: The set Γ is *stable* for Σ if for any $\epsilon > 0$ there exists a neighbourhood $\mathcal{N}(\Gamma) \subset B_\epsilon(\Gamma) \subset \mathcal{X}$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma)$ and all $t \geq 0$ for which $\phi(t, \chi_0)$ is defined, $\phi(t, \chi_0) \in B_\epsilon(\Gamma)$. Γ is *attractive* for Σ if there exists neighbourhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that $\lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0$ for all $\chi_0 \in \mathcal{N}(\Gamma)$. The *domain of attraction of* Γ is the set $\{\chi_0 \in \mathcal{X} : \lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$. Γ is *globally attractive* for Σ if it is attractive with domain of attraction $\mathcal{X}$. Γ is *locally asymptotically stable (LAS)* (or just asymptotically stable) for Σ if it is stable and attractive. Γ is *globally asymptotically stable (GAS)* for Σ if it is stable and globally attractive. Γ is *almost-globally asymptotically stable (AGAS)* for Σ if the set Γ is LAS for Σ with domain of attraction $\mathcal{X}\backslash N$ where $N \subset \mathcal{X}$ is a set of Lebesgue measure zero. $\triangle$

Definition 2.2: Let $\Gamma_1 \subset \Gamma_2$ be two closed subsets of $\mathcal{X}$ which are positively invariant for Σ. Γ_1 is *globally asymptotically stable relative to* Γ_2 if it is LAS when initial conditions are restricted to lie in Γ_2, and its domain of attraction contains Γ_2. Γ_2 is *locally stable near* Γ_1 if for all $\chi \in \Gamma_1$, for all $c > 0$, and all $\epsilon > 0$, there exists $\delta > 0$ such that for all $\chi_0 \in B_\delta(\Gamma_1)$ and all $t > 0$, if $\phi([0, t], \chi_0) \subset B_c(\chi)$ then $\phi([0, t], \chi_0) \subset B_\epsilon(\Gamma_2)$. Γ_2 is *locally attractive near* Γ_1 if there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma_1)$, $\|\phi(t, \chi_0)\|_{\Gamma_2} \to 0$ as $t \to \infty$. $\triangle$

Theorem 2.3 (Reduction Theorem [15]): Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed sets that are positively invariant for Σ, and suppose Γ_1 is compact. Consider the following conditions: *(i)* Γ_1 is LAS relative to Γ_2; *(i)′* Γ_1 is GAS relative to Γ_2; *(ii)* Γ_2 is locally stable near Γ_1; *(iii)* Γ_2 is locally attractive near Γ_1; *(iii)′* Γ_2 is globally attractive; *(iv)* all trajectories of Σ are bounded.

Then, the following implications hold: *(i)* $\wedge$ *(ii)* $\implies$ Γ_1 is stable. *(i)* $\wedge$ *(ii)* $\wedge$ *(iii)* $\iff$ Γ_1 is LAS. *(i)′* $\wedge$ *(ii)* $\wedge$ *(iii)′* $\wedge$ *(iv)* $\implies$ Γ_1 is GAS.

The result above is the basic tool used in this paper to solve the position control problem by decoupling translational and rotational control designs. Its significance will become apparent in Sections IV and V. See, in particular, Remark 4.7. The paper [14] generalizes the theorem above to the case when Γ_1 is unbounded, and presents a reduction-based backstepping design technique that does not rely on Lyapunov functions. That technique is used in this paper.

III. Position Control Problem

Consider the vehicle class depicted in Figure 1, where $\mathcal{I}$ and $\mathcal{B} = \{b_1, b_2, b_3\}$ represent, respectively, an inertial and a body frame. The b_1 axis represents the vehicle's heading, while the b_3 axis represents the propulsion direction. The thrust vector has constant direction in the body frame, but its magnitude, u, can be freely controlled. The vehicle incorporates some actuation mechanism inducing a torque τ about the three body axes that can be freely assigned.

The actual physical inputs (e.g., rotor speeds) will depend on the vehicle design and the actuation mechanism used to induce torques. An example of this vehicle type is the quadrotor helicopter depicted in Figure 1, see for instance [2], [16]. Each rotor produces a thrust force f_i
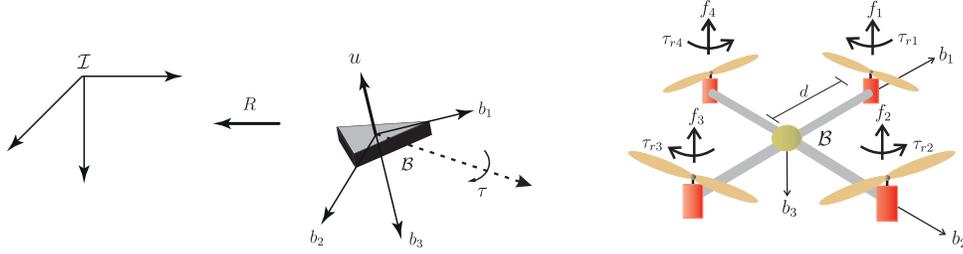
Fig. 1. On the left-hand side, the vehicle class under consideration. On the right-hand side, a quadrotor helicopter is an example of a vehicle in the class discussed in this paper.

parallel to the b_3 axis, and a reaction torque τ_{ri} of the motor that drives it. Using the development from Castillo, Lozano, and Dzul in [12], the relationship between the control inputs and the motor torques is given by $u = c(\tau_{r1} + \tau_{r2} + \tau_{r3} + \tau_{r4})$, $\tau_1 = cd(-\tau_{r2} + \tau_{r4})$, $\tau_2 = cd(\tau_{r1} - \tau_{r3})$ and $\tau_3 = (\tau_{r1} - \tau_{r2} + \tau_{r3} - \tau_{r4})$, where c is a physical constant and d is depicted in Figure 1.

Returning to the general vehicle class, we define the state $\chi := (x, v, R, \omega) \in \mathcal{X} := \mathbb{R}^3 \times \mathbb{R}^3 \times$ SO(3) $\times \mathbb{R}^3$, where $x \in \mathbb{R}^3$ is the vehicle position expressed in frame $\mathcal{I}$; $v \in \mathbb{R}^3$ is the vehicle's linear velocity expressed in frame $\mathcal{I}$; $R \in$ SO(3) is the attitude of the vehicle (the columns of R are the vectors b_1, b_2, b_3 expressed in frame $\mathcal{I}$); $\omega \in \mathbb{R}^3$ is the vehicle angular velocity expressed in frame $\mathcal{B}$. We model the vehicle dynamics using the Newton-Euler equation:

$$\dot{x} = v, \ m\dot{v} = mge_3 - uRe_3 = mge_3 + T \tag{1}$$

$$\dot{R} = R\,\omega^\times, \ J\dot{\omega} + \omega \times J\omega = \tau. \tag{2}$$

In the above, m is the mass of the vehicle, J is its symmetric inertia matrix expressed in frame $\mathcal{B}$, and $T = \text{col}(T_1, T_2, T_3) = -Re_3u$ is the thrust vector. The vehicle has four control inputs: u, the magnitude of the thrust vector, and $\tau := \text{col}(\tau_1, \tau_2, \tau_3)$, the vector of applied torques expressed in frame $\mathcal{B}$. The model in (1)-(2) neglects disturbances and dissipative effects that are present in specific vehicle types. In Section V we discuss how our approach can be used to handle state-dependent uncertainties.

Position Control Problem (PCP): Consider system (1), (2), and a desired position $x^\star \in \mathbb{R}^3$ and attitude $R^\star \in$ SO(3) such that $R^\star e_3 = e_3$. Letting $\chi^\star = (x^\star, 0, R^\star, 0)$, design a smooth feedback of the form

$$\dot{z} = \varphi(\chi_{\text{tran}}), \ u = \psi_1(\chi, z), \ \tau = \psi_2(\chi, z) \tag{3}$$

with $z \in \mathbb{R}^p$, $\chi_{\text{tran}} := (x, v, z)$ such that for some $z^\star \in \mathbb{R}^p$, the state $(\chi, z) = (\chi^\star, z^\star)$ is an AGAS equilibrium of the closed-loop system. $\triangle$

The condition $R^\star e_3 = e_3$ expresses the obvious requirement that, at a hovering equilibrium, the vehicle's b_3 axis is parallel to the third inertial axis. It is not a restriction in PCP. The statement of PCP allows for a dynamic feedback controller for flexibility of design, but dynamic feedback control is not required.

IV. Two-Stage Control Architecture

In this section we present a class of feedback controllers solving PCP. The main result is Theorem 4.6. The proposed control design has two nested feedback loops, shown in the block

diagram of Figure 2. The outer loop views system (1) as a fully actuated point-mass system with three control inputs $T = \mathrm{col}\,(T_1, T_2, T_3)$, and produces a thrust controller. The inner loop is an attitude controller for the rotational subsystem (2) which orients the thrust vector T of the vehicle to match the desired thrust assigned by the outer loop controller. To do that, from the desired thrust direction we extract a desired attitude with the property that the propulsion axis associated with this attitude is parallel to the desired thrust force. Then, the attitude controller regulates the actual attitude of the vehicle to the desired one.
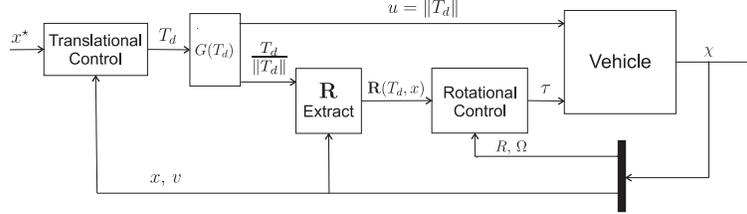
Fig. 2. Block diagram of position control system. The outer loop assigns a desired thrust vector T_d. The inner loop converts the desired thrust direction $T_d/\|T_d\|$ into a reference attitude $\mathbf{R}(T_d, x)$, which is then used by an attitude controller to assign the vehicle torques.

A similar two-stage design philosophy has been used in the literature, particularly in the work of Tayebi and collaborators [6], [7], [8], as well as in [9]. These papers present specific position and attitude control designs that are tied together through the technique of backstepping. Backstepping requires the knowledge of Lyapunov functions, and it may lead to controllers that have complex expressions. More importantly, when using backstepping, position and attitude control designs are tied together. One may not change one design without affecting the other. We now present a control architecture that is modular, in that it completely decouples position and attitude control designs, and it identifies general *classes* of position and attitude controllers that can be combined to solve PCP.

Definition 4.1 (Point-Mass Position Control Class, PMPC*):* A pair (φ, T_d) is a controller of class PMPC, written $(\varphi, T_d) \in$ PMPC, if $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^p \to \mathbb{R}^p$ and $T_d : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^p \to \mathbb{R}^3$ are smooth maps, and the dynamic smooth feedback

$$\dot{z} = \varphi(\chi_{\text{tran}}), \ T = T_d(\chi_{\text{tran}}) \tag{4}$$

where $\chi_{\text{tran}} := (x, v, z)$, satisfies the following properties: *(i)* There exist constants $C_1, C_2 > 0$ such that $\|\varphi(\chi_{\text{tran}})\| \leq C_1 + C_2\|\chi_{\text{tran}}\|$. Moreover, $\sup(e_3^\top T_d(\chi_{\text{tran}})) < 0$ and $\sup\|T_d(\chi_{\text{tran}})\| < \infty$. *(ii)* There exists $z^\star \in \mathbb{R}^p$ such that the equilibrium $\chi_{\text{tran}} = (x^\star, 0, z^\star)$ is GAS for the point-mass system (1) with dynamic feedback (4). *(iii)* There exists $\epsilon > 0$ such that for any piecewise continuous function $\rho : \mathbb{R} \to \mathbb{R}^3$ such that $\sup\|\rho\| < \epsilon$ and $\rho(t) \to 0$ as $t \to \infty$, letting $T = T_d(\chi_{\text{tran}}) + \rho(t)$, all solutions of the system (1)-(4) are bounded. If the feedback is static, so that $\chi_{\text{tran}} = (x, v)$, we will write $(\varnothing, T_d) \in$ PMPC. $\triangle$

Definition 4.2 (Attitude Control Class, AC*):* A smooth function $\tau_d : \mathrm{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ is a feedback of class AC, written $\tau_d \in$ AC, if, letting $\tau = \tau_d(R, \omega)$, the point $(R, \omega) = (I, 0)$ is an AGAS equilibrium for the rotational subsystem (2). $\triangle$

Definition 4.3 (Attitude Extraction Map, $\mathbf{R}$): Let $\mathcal{U} := \{T \in \mathbb{R}^3 : e_3^\top T < 0\}$. A smooth function $\mathbf{R} : \mathcal{U} \times \mathbb{R}^3 \to \mathsf{SO}(3)$ is an *attitude extraction map* if it satisfies:

$$\text{(a)} \ (\forall (T, x) \in \mathcal{U} \times \mathbb{R}^3) \ \|T\| \mathbf{R}(T, x) e_3 = -T,$$

$$\text{(b)} \ \mathbf{R}(-mge_3, x^\star) = R^\star, \tag{5}$$

where $x^\star$, $R^\star$ are defined in the statement of PCP. $\quad\triangle$

Remark 4.4: Identity (a) in (5) guarantees that when $R = \mathbf{R}(T_d, x)$ and $u = \|T_d\|$, the resulting thrust vector T in (1) coincides with T_d. Identity (b) in (5) guarantees that the attitude extraction map $\mathbf{R}$ returns the desired equilibrium orientation $R^\star$ when the vehicle hovers at the desired equilibrium position $x^\star$. The set $\mathcal{U}$ in Definition 4.3 is introduced to guarantee the existence of map $\mathbf{R}$ with the required properties, as we shall see in Section VI. $\quad\triangle$

Definition 4.5 (Position Controller Class, PC): Let $(\varphi, T_d) \in \mathsf{PMPC}$, $\tau_d \in \mathsf{AC}$, and $\mathbf{R}$ be an attitude extraction map. Denote $\chi_{\text{tran}} := (x, v, z)$, $\chi_{\text{rot}} := (R, \omega)$, and $\bar\chi := (\chi_{\text{tran}}, \chi_{\text{rot}})$. A feedback of class PC is defined as

$$\dot z = \varphi(\chi_{\text{tran}})$$

$$u = \|T_d(\chi_{\text{tran}})\|$$

$$\tau = \tau_d(\tilde{\mathbf{R}}, \tilde\omega) - \tilde\omega \times J\tilde\omega + \omega \times J\omega$$

$$\quad - J\left(\tilde\omega^\times \tilde{\mathbf{R}}^{-1}\omega - \tilde{\mathbf{R}}^{-1}\dot\omega\right), \tag{6}$$

where $\omega(\bar\chi) := \left[(\mathbf{R}(T_d(\chi_{\text{tran}}), x))^{-1}\dot{\mathbf{R}}(\bar\chi)\right]_\times$, $\tilde{\mathbf{R}}(\bar\chi) := (\mathbf{R}(T_d(\chi_{\text{tran}}), x))^{-1}R$, $\tilde\omega(\bar\chi) := \omega - \tilde{\mathbf{R}}^{-1}(\bar\chi)\omega(\bar\chi)$ and $\dot{\mathbf{R}}$, $\dot\omega$ are the time derivatives of the maps $\bar\chi \mapsto \mathbf{R}(T_d(\chi_{\text{tran}}), x)$ and $\bar\chi \mapsto \omega(\bar\chi)$ along (1), (2) setting $u = \|T_d(\chi_{\text{tran}})\|$. $\quad\triangle$

The structure of the attitude control τ in (6) is intuitive. It is an attitude tracker for the reference signal $\mathbf{R}(T_d, x)$ provided by the outer translational control loop. The first term $\tau_d(R, \omega)$ is an almost global attitude stabilizer for the upright attitude $R = I$. The remaining terms act to transform the attitude stabilizer into an attitude tracker. Consider the diffeomorphism $(x, v, R, \omega) \mapsto (x, v, \tilde{\mathbf{R}}, \tilde\omega)$. One can show that, in new coordinates, the closed-loop system reads

$$\dot x = v, \ m\dot v = mge_3 - u\mathbf{R}(T_d(\chi_{\text{tran}}), x)\tilde{\mathbf{R}}e_3 \tag{7}$$

$$\dot{\tilde{\mathbf{R}}} = \tilde{\mathbf{R}}\tilde\omega^\times, \ J\dot{\tilde\omega} + \tilde\omega \times J\tilde\omega = \tau_d(\tilde{\mathbf{R}}, \tilde\omega). \tag{8}$$

Theorem 4.6 (Main Result): Any feedback of class PC solves PCP for system (1), (2).

Proof: Consider system (1), (2). Since $(\varphi, T_d) \in \mathsf{PMPC}$, $\sup(e_3^\top T_d(\chi_{\text{tran}})) < 0$, so the image of the function $\chi_{\text{tran}} \mapsto T_d(\chi_{\text{tran}})$ is contained in $\mathcal{U}$ (see Definition 4.3), so that the function $\chi_{\text{tran}} \mapsto \mathbf{R}(T_d(\chi_{\text{tran}}), x)$ is well-defined, and the feedback (6) is smooth. Denote $\chi_{\text{tran}}^\star := (x^\star, 0, z^\star)$, $\chi_{\text{rot}}^\star := (R^\star, 0)$, and $\bar\chi^\star := (\chi_{\text{tran}}^\star, \chi_{\text{rot}}^\star)$. Define sets $\Gamma_1 = \{\bar\chi^\star\}$, $\Gamma_2 = \{\bar\chi : \tilde{\mathbf{R}}(\bar\chi) = I, \tilde\omega(\bar\chi) = 0\}$. We first show that $\Gamma_1 \subset \Gamma_2$. Since $(\varphi, T_d) \in \mathsf{PMPC}$, $\chi_{\text{tran}} = \chi_{\text{tran}}^\star$ is an equilibrium of (1) with $T = T_d(\chi_{\text{tran}})$. This implies that $T_d(\chi_{\text{tran}}^\star) = -mge_3$. Using property (b) in (5), we have $\mathbf{R}(T_d(\chi_{\text{tran}}^\star), x^\star) = \mathbf{R}(-mge_3, x^\star) = R^\star$. Therefore, $\tilde{\mathbf{R}}(\bar\chi^\star) = I$. We also have $\tilde\omega(\bar\chi^\star) = -\omega(\bar\chi^\star) = -[(R^\star)^{-1}\dot{\mathbf{R}}(\bar\chi^\star)]_\times = 0$ because $\dot{\mathbf{R}}(\bar\chi^\star)$ is proportional to $\dot\chi_{\text{tran}}|_{\bar\chi = \bar\chi^\star} = 0$. This proves that $\Gamma_1 \subset \Gamma_2$. Recall the error dynamics in (8). Since $\tau_d \in \mathsf{AC}$, the equilibrium $(\tilde{\mathbf{R}}, \tilde\omega) = (I, 0)$ is AGAS for (8), implying that Γ_2 is AGAS provided that the closed-loop system has no finite escape times. Assume for the moment that this is the case, and let $\bar{\mathcal{X}}$ be the domain of attraction of Γ_2. Then, $\bar{\mathcal{X}}$ is positively invariant, and Γ_2 is GAS relative to $\bar{\mathcal{X}}$. For all $\bar\chi \in \Gamma_2$, $R = \mathbf{R}(T_d(\chi_{\text{tran}}, x))$

so that, by property (a) in (5), $-uRe_3 = -\|T_d(\chi_{\text{tran}})\|\mathbf{R}(T_d(\chi_{\text{tran}}), x)e_3 = T_d(\chi_{\text{tran}})$. Therefore, the motion on Γ_2 is governed by $\dot{x} = v$, $\dot{v} = mge_3 + T_d(\chi_{\text{tran}})$, $\dot{z} = \varphi(\chi_{\text{tran}})$. Since $(\varphi, T_d) \in$ PMPC, the equilibrium $\chi_{\text{tran}} = \chi_{\text{tran}}^{\star}$ is GAS for the system above, so that Γ_1 is GAS relative to Γ_2. We now wish to apply the reduction Theorem 2.3. To this end, we need to show that all solutions of the closed-loop system originating in $\bar{\mathcal{X}}$ have no finite escape times (implying that Γ_2 is GAS relative to $\bar{\mathcal{X}}$) and they are bounded. Let $\bar{\chi}(0) \in \bar{\mathcal{X}}$ be arbitrary, and let $\bar{\chi}(t)$ be the corresponding solution of the closed-loop system. The dynamics of the χ_{tran} subsystem can be written as $\ddot{x} = mge_3 + T_d(\chi_{\text{tran}}) + (-\|T_d\|Re_3 - T_d(\chi_{\text{tran}}))$, $\dot{z} = \varphi(\chi_{\text{tran}})$. Here, Re_3 has unit norm and, by assumption, $\sup\|T_d\| < \infty$, so $\ddot{x}$ is bounded. Since $(\varphi, T_d) \in$ PMPC, $\|\varphi(\chi_{\text{tran}})\| \leq C_1 + C_2\|\chi_{\text{tran}}\|$, so the signal $\chi_{\text{tran}}(t) = (x(t), v(t), z(t))$ is defined for all $t \geq 0$. The function $\boldsymbol{\omega}$ depends on (x, v, z, R). Since the first three arguments are defined for all $t \geq 0$, and $R \in$ SO(3), a compact set, $\boldsymbol{\omega}(\bar{\chi}(t))$ is defined for all $t \geq 0$. Since $(\tilde{\mathbf{R}}, \tilde{\omega}) = (I, 0)$ is AGAS for (8), $\tilde{\omega}(\bar{\chi}(t)) = \omega(t) - \boldsymbol{\omega}(\bar{\chi}(t))$ is bounded, implying that $\omega(t)$ is defined for all $t \geq 0$. In conclusion, $\bar{\chi}(t)$ is defined for all $t \geq 0$, so that the closed-loop system has no finite escape times, and Γ_2 is GAS relative to $\bar{\mathcal{X}}$. Now consider property (iii) in Definition 4.1, and let $\rho(t) = -\|T_d(\chi_{\text{tran}}(t))\|R(t)e_3 - T_d(\chi_{\text{tran}}(t))$. By property (a) in (5), and by the global asymptotic stability of Γ_2 relative to $\bar{\mathcal{X}}$, $\rho(t) \to 0$. Since $(\varphi, T_d) \in$ PMPC, this implies that $\chi_{\text{tran}}(t)$ is bounded. Since $\boldsymbol{\omega}$ is a smooth function of (χ_{tran}, R), and the signals $\chi_{\text{tran}}(t), R(t)$ are bounded, $\boldsymbol{\omega}(\bar{\chi}(t))$ is bounded. Finally, the boundedness of $\tilde{\omega}(\bar{\chi}(t))$ and that of $\boldsymbol{\omega}(\bar{\chi}(t))$ imply that $\omega(t)$ is bounded. Having shown that all solutions of the closed-loop system originating in $\bar{\mathcal{X}}$ are bounded, by part (c) of Theorem 2.3 we conclude that Γ_1 is GAS relative to $\bar{\mathcal{X}}$ or, what is the same, the equilibrium $\bar{\chi} = \bar{\chi}^{\star}$ is AGAS for the closed-loop system, proving that the feedback (6) solves PCP. $\blacksquare$

Remark 4.7: The basic tool enabling the decomposition of PCP into point mass position control and attitude control is the reduction theorem, Theorem 2.3. In the proof of Theorem 4.6, we consider positively invariant sets $\Gamma_1 \subset \Gamma_2$. Γ_2 is the set where the vehicle attitude is equal to the desired attitude signal provided by the translational stage. On Γ_2, the vehicle behaves like a point-mass system subject to the stabilizing control law $(\varphi, T_d) \in$ PMPC. Thus, the hovering equilibrium $\Gamma_1 \subset \Gamma_2$ is GAS relative to Γ_2. The feedback for τ in (6) is designed to almost globally stabilize the set Γ_2. The reduction theorem allows us to deduce that the combination of the translational and rotational control modules makes Γ_1 AGAS for the closed-loop system. $\triangle$

V. Remarks on Two-Stage Control Architecture

The two-stage control architecture presented in the previous section parametrizes a class of feedbacks solving PCP using three modules: a point-mass position controller of class PMPC, an attitude extraction map, and an attitude stabilizer of class AC. The combination of these modules, depicted in Figure 2, results in two nested loops. The outer loop is the feedback of class PMPC producing a reference thrust signal $T_d(\chi_{\text{tran}}(t))$. The inner loop relies on the attitude extraction map to generate reference signals $\mathbf{R}(T_d(\chi_{\text{tran}}(t)), x(t))$ and $\boldsymbol{\omega}(\bar{\chi}(t))$. Then, an attitude controller $\tau(\bar{\chi})$ makes $R(t)$ and $\omega(t)$ track these references. Specifically, the proof of Theorem 4.6 shows that τ is such that the error functions $(\tilde{\mathbf{R}}, \tilde{\omega})$ satisfy (8), the equation of a rotating rigid body subject to the stabilizing torque τ_d.

The two-stage control architecture has a number of useful features: **(1)** The control modules of class PMPC and AC are completely decoupled: the design of either one of them is independent of

the design of the other. **(2)** Being modular, the control design process is intuitive and structured. It can be explained to a non-expert. **(3)** One can replace either one of the control modules without having to redesign the remaining module. This allows one to easily test and compare different feedbacks of class PC. **(4)** The combination of control modules in the final feedback (6) does not require the knowledge of Lyapunov functions for the individual modules. **(5)** In light of the reduction theorem, Theorem 2.3, the proof of Theorem 4.6 remains unchanged if one replaces the equilibrium $\Gamma_1 = \{\bar{\chi}^\star\}$ by an arbitrary compact and controlled invariant subset of $\Gamma_2 = \{\bar{\chi} : \tilde{\mathbf{R}}(\bar{\chi}) = I, \tilde{\omega}(\bar{\chi}) = 0\}$. Since the definition of Γ_1 arises from the control specification for system (1), allowing $\Gamma_1 \subset \Gamma_2$ to be an arbitrary compact set corresponds to changing the control specification for the PMPC class. The result is that the two-stage control architecture can be used to solve other control problems for thrust-propelled vehicles. For instance, suppose the translational dynamics (1) are affected by a state-dependent uncertainty, and that (φ, T_d) is a robust controller that globally practically stabilizes the desired equilibrium $\chi_{\text{tran}} = \chi_{\text{tran}}^\star$ for (1). In other words, for any compact neighbourhood Γ_1 of $\chi_{\text{tran}}^\star$, a parameter in (φ, T_d) can be chosen that makes Γ_1 asymptotically stable for the fully actuated point-mass system (1). Then, the feedback of class PC will recover the properties of the point-mass controller, and make Γ_1 almost globally asymptotically stable for the thrust-propelled vehicle. For another example, suppose we wish to make the vehicle follow a smooth embedded path in three-space. Solving the problem for the point-mass system (1) would correspond to finding a feedback (φ, T_d) stabilizing a compact set Γ_1. Then, the two-stage control architecture will solve the path following problem for the thrust-propelled vehicle. **(6)** One can employ observers in the two-stage architecture (adding a dynamic extension to the rotational subsystem) and exploit the proof technique of Theorem 4.6 to prove asymptotic stability of the desired equilibrium. This is done in [17]. Finally, in [17] it is shown that the result of Theorem 4.6 holds if one uses the quaternion representation of attitude.

VI. Sample Feedbacks Solving PCP

In this section we present specific feedbacks of class PC that solve PCP. Following the two-stage control architecture, we need to design three modules: a point-mass position controller of class PMPC, an attitude extraction map, and an attitude stabilizer of class AC.

A. Sample Feedbacks of Class PMPC

We will present three feedbacks of class PMPC for the point-mass system $\dot{x} = v$, $m\dot{v} = mge_3 + T$.

1) Nested saturation feedback. To meet the requirements in part (i) of Definition 4.1 one can use the nested saturation controller of A. Teel (see [18], [19], [20]),

$$T_d(x, v) = -mge_3 - m\sigma\left(K_1 v + \lambda\sigma\left(\frac{K_2\tilde{x} + K_3 v}{\lambda}\right)\right) \tag{9}$$

where $\tilde{x} = x - x^\star$, K_1, K_2, K_3 are positive definite diagonal matrices, λ is a positive scalar, and $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth saturation function satisfying (i) $\sigma(s) = (\sigma_1(s_1), \sigma_2(s_2), \sigma_3(s_3))$, $s = (s_1, s_2, s_3)$; (ii) $s_i\sigma_i(s_i) > 0$ for all $s_i \neq 0$, $i = 1, 2, 3$; (iii) $\dot{\sigma}(0) = I$; (iv) $|\sigma_i(\cdot)| \leq M_i$ for some $M_i > 0$, $i = 1, 2, 3$, with $M_3 < g$. In particular, we choose $\sigma_i(s_i) = M_i \tanh\left(M_i^{-1} s_i\right)$. The feedback in (9) is static, with $\chi_{\text{tran}} = (x, v)$. We will see in Proposition 6.1 that it belongs to class PMPC.

2) Nested saturation feedback with integral action. In order to accommodate for imperfect knowledge of the vehicle's mass, it is desirable to design a feedback with integral action. This can be achieved by adding a state representing the integral of the position error $\tilde{x} = x - x^\star$ and applying Teel's nested saturation approach to the resulting triple-integrator:

$$\dot{z} = \tilde{x}$$

$$T_d(x, v, z) = -mge_3 - m\sigma\left(K_1^1 v + \lambda_1 \sigma\left(\frac{K_1^2 \tilde{x} + K_2^2 v}{\lambda_1}\right.\right.$$

$$\left.\left. + \frac{\lambda_2}{\lambda_1}\sigma\left(\frac{\lambda_3}{\lambda_2}\sigma\left(\frac{K_1^3 z + K_2^3 \tilde{x} + K_3^3}{\lambda_3}\right)\right)\right)\right), \tag{10}$$

where σ is as above, and K_i^j are suitable positive definite diagonal matrices and λ_k suitable positive scalars. Note that the feedback (10) is dynamic, being of the form (φ, T_d), with $\chi_{\text{tran}} = (x, v, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ and $\varphi(\chi_{\text{tran}}) = x - x^\star$.

Proposition 6.1: There exist positive definite diagonal matrices $K_1, \ldots, K_3$, K_i^j, $i, j = 1, 2, 3$, and positive scalars λ, λ_k, $k = 1, 2, 3$, such that the feedbacks in (9) and (10) belong to class PMPC.

Proof: For both feedbacks (9) and (10), we have $\sup \|T_d\| < mg + m \max\{M_1, M_2, M_3\}$. Moreover, $e_3^\top T_d = -mg - m\sigma_3(\cdot) < 0$ because $|\sigma_3| < g$. For feedback (10) we have $\|\varphi(\chi_{\text{tran}})\| = \|x - x^\star\| \leq \|x^\star\| + \|\chi_{\text{tran}}\|$. Thus, property (i) in Definition 4.1 holds. By Theorem 5 in [19], there exist positive definite diagonal matrices $\lambda, K_1, \ldots, K_3$, K_i^j, $i, j = 1, 2, 3$, and λ_k, $k = 1, 2, 3$ such that properties (ii) and (iii) of Definition 4.1 hold. ∎

3) Saturated PID controller with antiwindup compensation. The presence of integral action with nested saturation may give rise to undesirably slow transient performance. For better transient performance, one may use the saturated PID controllers with antiwindup compensation of [21]:

$$\dot{\xi} = \tilde{x} - q_e$$
$$\ddot{q}_e = \sigma(\sigma(y_c) + v_1) - y_c$$
$$T_d(x, v, z) = -mge_3 - m\sigma(\sigma(y_c) + v_1) \tag{11}$$
$$z := (\xi, q_e, \dot{q}_e) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3,$$

where σ satisfies (i)-(iv) above[1], $v_1 = K_g \sigma(K_g^{-1} K_q q_e) + K_{qd}(q_e, \dot{q}_e)\dot{q}_e$, K_{qd} is a diagonal smooth positive definite matrix-valued function which is constant in a neighbourhood of the origin and bounded away from zero. Finally, $y_c = K_P(\tilde{x} - q_e) + K_D(v - \dot{q}_e) + K_I \xi$.

Proposition 6.2: Let K_g be a diagonal matrix with entries in the interval $(0, 1)$, and let K_q, K_P, K_D, K_I be diagonal positive definite matrices. Then, feedback (11) belongs to class PMPC.

Proof: Property (i) of Definition 4.1 is satisfied by the same argument presented in the proof of Lemma 6.1. Properties (ii) and (iii) follow from Theorem 1 and Lemma 1 in [21]. ∎

B. Sample Attitude Extraction Maps

Let $b_{1d}(T, x)$ be any smooth function $\mathbb{R}^3 \times \mathbb{R}^3 \to \{v \in \mathbb{R}^3 : v \cdot v = 1\}$ such that, for all (T, x), $b_{1d}(T, x)$ is orthogonal to T and $b_{1d}(-mge_3, x^\star) = R^\star e_1$ (this is the desired heading

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[1]In [21], the authors use the standard saturation function which is globally Lipschitz but not smooth. As the authors of [21] point out, their approach is applicable to a wider class of saturation nonlinearities, and in particular it is applicable to the class identified in this paper.

at the hovering equilibrium). The simplest choice is the vector $b_{1d}(T, x) = (T \times (R^\star e_2))/\|(T \times (R^\star e_2))\|$, but in certain applications one may have specifications for the vehicle's heading during transient, in which case other functions $b_{1d}(T, x)$ would be required. Define $b_{3d}(T) = -T/\|T\|$. Then, the function

$$\mathbf{R}(T, x) := \begin{bmatrix} b_{1d}(T, x) & b_{3d}(T) \times b_{1d}(T, x) & b_{3d}(T) \end{bmatrix} \tag{12}$$

is an attitude extraction map. Note that $\mathbf{R}(T, x)$ is defined as long as $T \neq 0$, so in particular it is smooth on $\mathcal{U} \times \mathbb{R}^3$.

C. Sample Feedbacks of Class AC

Now we need to define attitude controllers τ_d that achieve almost global stabilization of $(R, \omega) = (I, 0)$. There is a vast literature on the subject of attitude stabilization, and our modular design allows one to pick from a multitude of designs. We pick the controller presented in [22],

$$\tau_d(R, \omega) = -K_R \left(\sum_{i=1}^{3} a_i e_i \times R e_i \right) - K_\omega \omega, \tag{13}$$

where K_R, K_ω are symmetric positive definite matrices, and a_1, a_2, a_3 are distinct positive numbers.

Proposition 6.3: Feedback (13) belongs to class AC

Proof: The fact that (13) is of class AC is proved[2] in Theorem 1 of [22]. ∎

VII. Simulation Results

In this section, we present simulation results for a feedback of class PC. The vehicle is specified to travel from an initial to a desired position in $\mathbb{R}^3$. We will look at two cases. In case 1, the vehicle is initially upright and the desired heading is different from the initial heading. In case 2, the vehicle is initially upside-down, and the desired heading is the same as the initial heading. The initial conditions are taken as $x_0 = (1, 1, 1)$ m, $v_0 = (0, 0, 0)$ m/s, $R_0 = I$ (case 1: upright) or $R_0 = \text{diag}(1, -1, -1)$ (case 2: upside-down). The target position is chosen to be $x^\star = (0, 0, 0)$ m, and the target attitude is $R^\star = \begin{bmatrix} e_2 & -e_1 & e_3 \end{bmatrix}$ in case 1, and $R^\star = I$ in case 2. The vehicle mass is chosen to be $m = 2$ Kg. The inertia matrix J is diagonal, with diagonal elements equal to 1.2416 Kg·m^2 as in [16]. All simulations use $(\phi, T_d) \in$ PMPC given in (9), with $\sigma_i(s_i) = 5 \tanh(s_i/5)$, $\lambda = 2/5I$, $K_1 = 2I$, $K_2 = 2/5I$, $K_3 = 1/5I$. The gains are chosen as $K_R = 200I$, $K_\omega = 8I$, $a_1 = 0.9$, $a_2 = 1$, $a_3 = 1.1$. Figure 3 shows the results when disturbances are present. The disturbance includes an additive random noise on the applied force and torque with maximum magnitude of 0.5 N and 0.5 N·m, respectively. We also include random additive noise on the attitude (yaw, pitch and roll) and on the angular velocity measurements with maximum magnitudes of 0.5 rad and 0.5 rad/s, respectively, and include a constant bias of 0.1 rad/s added to the measurement of angular velocity. The translational plots show the vehicle trajectory projected onto the $x_i - y_i$, $y_i - z_i$ and $x_i - z_i$ planes, and the linear speed given by $\sqrt{v_1^2 + v_2^2 + v_3^2}$. The attitude plot shows the three body axes plotted on a unit sphere. For all the results, the vehicle successfully converges to the desired equilibrium point. For the results including disturbances, the vehicle converges to a point in a small neighbourhood of the desired equilibrium point with an oscillating attitude.

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[2]Theorem 1 of [22] requires a_1, a_2, a_3 to be positive integers. This requirement does not seem necessary, hence we are letting a_i be positive real numbers.
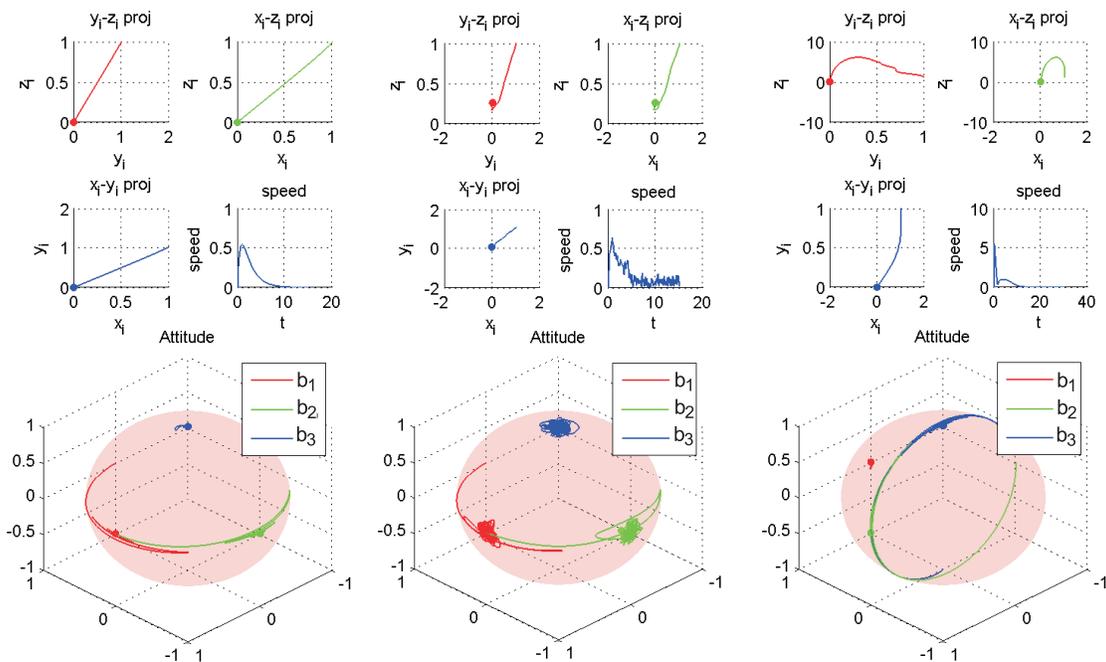
Fig. 3. Position control simulations. Case 1 without disturbances on the left hand side. Case 1 in the presence of random disturbances on the applied force and torque in the middle. Case 2 without disturbances on the right. The attitude plots in the bottom panels show the body axes with respect to the inertial frame (b_1, b_2, b_3) parametrized by time starting from the initial condition $R_0 = I$ (case 1: upright) or $R_0 = \mathrm{diag}(1, -1, -1)$ (case 2: upside-down). The small oscillations in the steady-state attitude in the middle plot are due to the random disturbances.

VIII. Conclusions

We have presented a class of feedbacks and a control architecture solving the position control problem for a class of VTOL aircraft. We have not explicitly addressed issues of robustness against unmodeled uncertainties and measurement errors, although we surmise that the overall robustness of the nested control loop is inherited from the robustness of the feedbacks one selects in the classes PMPC and AC. More research is needed to substantiate this claim. Two of the sample feedbacks of class PMPC presented in this paper include integral action to accommodate for imperfect knowledge of the vehicle's mass. The control approach presented in this paper can easily accommodate observers for attitude estimation, and it can be used with quaternion-based attitude stabilizers (see [17]).

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