On the Generation of Virtual Holonomic Constraints for Mechanical Systems With Underactuation Degree One

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***Abstract*— This paper introduces the notion of a virtual constraint generator (VCG) for underactuated mechanical systems with underactuation degree one. The VCG is a control system whose state space is the configuration manifold of the mechanical system, and whose orbits are all possible regular virtual holonomic constraints of the mechanical system. Leveraging this new tool, we propose to pose the design of virtual constraints as a suitable control specification for the VCG. We take a first step in this direction by presenting two such control designs aimed at producing virtual holonomic constraints whose associated constrained dynamics correspond to a through-motion with bounded speed and oscillatory behaviour, respectively. We apply our results to the control of an acrobot mounted on a cart, with the goal of designing constraints that make the robot "kneel" under an obstacle while traversing the real line, or oscillate back and forth under it.**

I. Introduction

Over the past fifteen years, virtual constraints have become a prominent tool for motion control of underactuated robots. The pioneering work in [1], [2], [3] demonstrated how to induce complex motions in walking robots by imposing, via feedback control, suitable constraints on the robot's configuration variables. Such constraints are called virtual holonomic constraints (VHCs). Early work in [2] proposed constraints based on observations about human behaviour, but soon thereafter, researchers in the area of bipedal locomotion began searching the space of virtual constraints using Bézier polynomial parametrizations in combination with constrained optimization methods (e.g., [1], [4], [3]). Today, numerical optimization methods are the dominant approach to search the space of virtual constraints.

The key requirement of a VHC is that, when the relation defining the constraint is viewed as an output function of the robot, the output in question should yield vector relative degree $\{2, \ldots, 2\}$ (e.g., [3]). In [5], this property is referred to as regularity of the VHC. Numerical optimization methods may not produce regular constraints.

In this paper, for mechanical systems with underactuation degree one, we introduce a way to systematically explore the entire space of regular VHCs by means of a control system which we call the *virtual constraint generator* (VCG). While the state space of the robot is the tangent bundle of the configuration manifold, the state space of the virtual constraint generator is the configuration manifold itself. The VCG has $n-1$ control inputs and its orbits, for arbitrary smooth control signals, form the space of all possible regular VHCs for the mechanical system, up to reparametrization. Armed with the VCG, in this paper we take the first steps towards the conversion of VHC generation problem into control specifications for the VCG. Specifically, we investigate the following two problems. Design regular VHCs inducing constrained dynamics whose orbits correspond, respectively, to traversal of the VHC curve in one direction with bounded speed, and closed orbits encircling a given configuration. We demonstrate our results with the example of an acrobot mounted on a cart. For this system, we use the VCG to design constraints making the cart either traverse the track or oscillate back and forth, while kneeling below an obstacle placed at the origin of the track.

The notion of VCG introduced in this paper improves and generalizes an idea introduced in [6]. In [7], the authors design regular VHCs inducing stable limit cycles in the constrained dynamics. The philosophy driving the present paper is very much in line with that of [7].

This paper is organized as follows. Section II reviews VHCs and the Lagrangian dynamics they induce. Section III introduces the virtual constraint generator. Section IV formulates the two constraint generation problems investigated in this paper, and Section V presents an idea allowing one to convert these problems into output specifications for the VCG. Sections VI and VII present solutions to the problems mentioned above, and Section VIII applies the results to the cart-acrobot system. Finally, Section IX draws some conclusions.

Notation. We denote by $\mathbb{S}^1$ the set of real numbers modulo 2π, naturally diffeomorphic to the unit circle. If $x \in \mathbb{R}$, then $[x]_{2\pi} \in \mathbb{S}^1$ denotes x modulo 2π. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function, and $x \in \mathbb{R}^n$, we denote by df_x the Jacobian matrix of f at x. If $g_1, \cdots, g_m$ is a collection of vector fields on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function, then by $L_g f(x)$ we denote the function $\mathbb{R}^n \to \mathbb{R}^{k\times m}$, $x \mapsto [df_x g_1(x) \cdots df_x g_n(x)]$. If M is a smooth manifold and $p \in M$, T_pM is the tangent space to M at p and TM is the tangent bundle of M.

II. Preliminaries on virtual holonomic constraints

Consider a simple mechanical system with n degrees of freedom and $n-1$ actuators, with generalized coordinates $q = (q_1, \ldots, q_n)$, where each q_i is either a real number or a variable in $\mathbb{S}^1$, the unit circle. This would typically correspond to a robot with prismatic and revolute joints.

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The configuration manifold $\mathcal{Q}$ of the system is a generalized cylinder, and the dynamics are described by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla P(q) = B(q)\tau, \tag{1}$$

where $D : \mathcal{Q} \to \mathbb{R}^{n\times n}$ is the mass matrix, assumed to be everywhere symmetric and positive definite, $C : T\mathcal{Q} \to \mathbb{R}^{n\times n}$ is the Coriolis matrix, $P : \mathcal{Q} \to \mathbb{R}$ is the potential, and $B : \mathcal{Q} \to \mathbb{R}^{n\times n-1}$ is the input matrix. The vector $\tau \in \mathbb{R}^{n-1}$ contains the control inputs. All functions in (1) are assumed to be smooth. We also assume that $B(q)$ has a smooth left-annihilator $B^{\perp} : \mathcal{Q} \to \mathbb{R}^{1\times n}$ which has rank 1 everywhere.

A *regular virtual holonomic constraint* (VHC) ([3], [5]) for system (1) is a relation $h(q) = 0$ in which $h : \mathcal{Q} \to \mathbb{R}^{n-1}$ is a smooth function whose Jacobian dh_q has full rank $n-1$ for all $q \in h^{-1}(0)$, and is such that the $(n-1)\times(n-1)$ matrix $dh_q D^{-1}(q)B(q)$ is invertible for all $q \in h^{-1}(0)$. The latter condition ensures that system (1) with output $y = h(q)$ has vector relative degree $\{2,\ldots,2\}$. The rank condition on the Jacobian ensures that the set $h^{-1}(0)$ is a regular embedded curve (a one-dimensional closed embedded submanifold) in $\mathcal{Q}$. From now on, we will use the term VHC to mean a regular VHC. The vector relative degree property[1] makes it easy to enforce VHCs, as one may simply use input-output feedback linearization to make the output $y = h(q)$ converge to zero exponentially fast [3], [5] and, in so doing, asymptotically stabilize the *constraint manifold*

$$\Gamma = \{(q,\dot{q}) : h(q) = 0, dh_q\dot{q} = 0\}. \tag{2}$$

In particular, there is a unique smooth feedback $\tau^{\star} : \Gamma \to \mathbb{R}^{n-1}$ rendering Γ invariant, so no matter what feedback one chooses to asymptotically stabilize Γ, the dynamics on Γ are uniquely defined.

Geometrically, VHCs for systems with underactuation degree one are one-dimensional closed embedded submanifolds $\mathcal{C}$ of $\mathcal{Q}$ satisfying the transversality condition

$$(\forall q \in \mathcal{C})\ T_q\mathcal{C} \oplus \mathrm{Im}(D^{-1}(q)B(q)) = T_q\mathcal{Q}. \tag{3}$$

It is easily seen [5], [9] that condition (3) is indeed equivalent to the vector relative degree property, when $\mathcal{C}$ is a level set of a smooth function h.

A VHC *in parametric form*, or *parametric* VHC, is a parametrized embedded curve $q = \sigma(s)$ such that $\sigma : \mathbb{I} \to \mathcal{Q}$ is smooth, and $\mathbb{I}$ is either $\mathbb{R}$ (for an open curve) or $\mathbb{S}^1$ (for a closed curve). Moreover, σ is required to satisfy the transversality condition (3), which now takes the form

$$(\forall s \in \mathbb{I})\ \mathrm{span}\{\sigma'(s)\} + \mathrm{Im}\left(D^{-1}(\sigma(s))B(\sigma(s))\right) = T_{\sigma(s)}\mathcal{Q}. \tag{4}$$

In (4), and throughout this paper, we use prime to denote differentiation with respect to the variable s.

Any VHC $h(q) = 0$ admits a parametrization $q = \sigma(s)$ as above, using which one can show (see, e.g., [10])[2] that the closed-loop dynamics on the constraint manifold Γ in (2) are described by

$$\ddot{s} = \Psi_1(s) + \Psi_2(s)\dot{s}^2,\ (s,\dot{s}) \in \mathbb{I}\times\mathbb{R}, \tag{5}$$

$$\Psi_1(s) = -\left.\frac{B^{\perp}\nabla P}{B^{\perp}D\sigma'(s)}\right|_{q=\sigma(s)}, \tag{6}$$

$$\Psi_2(s) = -\left.\frac{B^{\perp}D\sigma'' + \Sigma_{i=1}^{n} B_i^{\perp}{\sigma'}^{\top}Q_i\sigma'}{B^{\perp}D\sigma'(s)}\right|_{q=\sigma(s)}, \tag{7}$$

where $(Q_i)_{jk} = (\partial_{q_k}D_{ij} + \partial_{q_j}D_{ik} - \partial_{q_i}D_{jk})/2$. We refer to (5) as the *reduced dynamics* associated with the VHC $q = \sigma(s)$.

The meaning of the reduced dynamics in (5) is this. Consider system (1) with the unique feedback $\tau^{\star} : \Gamma \to \mathbb{R}^{n-1}$ rendering the constraint manifold Γ in (2) invariant. Then, all solutions of the closed-loop system on Γ have the form $(q(t),\dot{q}(t)) = \big(\sigma(s(t)), \sigma'(s(t))\dot{s}(t)\big)$, where $(s(t),\dot{s}(t))$ is a solution of the reduced dynamics in (5).

Now we briefly discuss the Lagrangian structure of the reduced dynamics, a topic investigated in detail in [11]. Define the functions

$$M(s) = \exp\left(-2\int_0^s \Psi_2(\tau)\,d\tau\right) \tag{8}$$

$$V(s) = -\int_0^s \Psi_1(\tau)M(\tau)\,d\tau, \tag{9}$$

which we will refer to as the *virtual mass* and *virtual potential*, respectively. If $\mathbb{I} = \mathbb{R}$ (i.e., the set $\sigma(\mathbb{I})$ is an open curve), the reduced dynamics in (5) are always Euler-Lagrange, with a Lagrangian given by $\mathcal{L} = (1/2)M(s)\dot{s}^2 - V(s)$. Associated with these reduced dynamics is the *total virtual energy*

$$E(s,\dot{s}) = \frac{1}{2}M(s)\dot{s}^2 + V(s) \tag{10}$$

which is an integral of motion of the reduced dynamics.

On the other hand, if $\mathbb{I} = \mathbb{S}^1$ (i.e., the set $\sigma(\mathbb{I})$ is a closed curve), then the reduced dynamics are Euler-Lagrange if and only if M and V are 2π-periodic (see [11]), in which case the Lagrangian is again $\mathcal{L} = (1/2)M(s)\dot{s}^2 - V(s)$.

III. VIRTUAL CONSTRAINT GENERATOR

In this paper we present the idea of *virtual constraint generator* (VCG), a generalization and improvement of a notion originally presented in [6]. By virtual constraint generator we mean a control system on $\mathcal{Q}$ whose solutions are all possible parametric VHCs, i.e., all possible embedded curves $q = \sigma(s)$ satisfying the transversality condition (4).

Consider the smooth distribution $\Delta : \mathcal{Q} \to T\mathcal{Q}$, defined as $\Delta(q) = \mathrm{Im}(D^{-1}(q)B(q))$. This distribution is regular and has rank $n-1$ on $\mathcal{Q}$. One may think of Δ as an assignment, to each point $q \in \mathcal{Q}$, of an hyperplane $\Delta(q)$ with origin at q. Geometrically, the problem of generating VHCs for system (1) can be thought of as the problem of generating all curves that are everywhere transversal to the distribution $\Delta(q)$, as illustrated in Figure 1.

Consider a vector field $f^{\pitchfork} : \mathcal{Q} \to T\mathcal{Q}$ that is everywhere transversal to Δ, i.e, such that $\mathrm{span}\{f(q)\} \oplus \Delta(q) = T_q\mathcal{Q}$ for all $q \in \mathcal{Q}$. For instance, the vector field

$$f^{\pitchfork}(q) = (B^{\perp}(q))^{\top} \tag{11}$$

[1] The work by Shiriaev and co-authors in, e.g., [8], does not explictly require the relative degree property, but then there is no a priori guarantee that constrained dynamics are well-defined.

[2] Grizzle and co-authors in [1], [3] develop an equivalent representation of the reduced dynamics on Γ, but use a different state. In place of $\dot{s}$, they use the momentum conjugate to the unactuated variable of the robot.

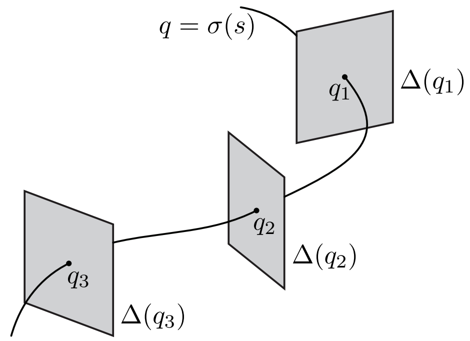

Fig. 1. Illustration of a parametric VHC meeting the transversality requirement (4). In the figure, $\Delta(q)$ is the hyperplane $\mathrm{Im}(D^{-1}(q)B(q))$ with origin at q.

has the required property. The orbits of $f^\pitchfork$ are everywhere transversal to Δ, and therefore they are VHCs. However, these orbits do not exhaust all possible VHCs, for if $g : \mathcal{Q} \to T\mathcal{Q}$ is a smooth vector field such that $g(q) \in \Delta(q)$ for all $q \in \mathcal{Q}$, then the vector field $f^\pitchfork + g$ is also everywhere transversal to Δ, and its orbits are VHCs. In light of this observation, we define the control system

$$\frac{dq}{ds} = f^\pitchfork(q) + D^{-1}(q)B(q)u = f^\pitchfork(q) + \sum_{i=1}^{n-1} g_i(q)u_i, \quad (12)$$

where $u \in \mathbb{R}^{n-1}$ denotes a virtual control input, and $g_i(q)$ is the i-th column of the matrix $D^{-1}(q)B(q)$. By construction, for each $q \in \mathcal{Q}$, the vectors $\{f^\pitchfork(q), g_1(q), \ldots, g_{n-1}(q)\}$ are linearly independent, and they form a basis for $T_q\mathcal{Q}$. As a consequence of this property, system (12) has no controlled equilibria, and therefore all orbits of (12) are regular curves. Moreover, as we now show, the space of solutions of (12) coincides with the space of parametric VHCs up to reparametrization[3].

Theorem 1. *Let $f^\pitchfork : \mathcal{Q} \to T\mathcal{Q}$ be a smooth vector field such that*

$$(\forall q \in \mathcal{Q}) \ f^\pitchfork(q) \oplus \mathrm{Im}(D^{-1}(q)B(q)) = T_q\mathcal{Q}, \quad (13)$$

such as $f^\pitchfork$ in (11)*. Let $\sigma(s)$ be a solution of the differential* (12) *corresponding to a smooth control signal $\bar{u} : \mathbb{R} \to \mathbb{R}^{n-1}$, and assume that the curve $q = \sigma(s)$ is embedded in $\mathcal{Q}$. Then $q = \sigma(s)$ is a parametric* VHC *for system* (1)*. Vice versa, if $q = \sigma(s)$ is a parametric* VHC *for system* (1)*, then there exists a reparametrization $\sigma \circ \mu(\tilde{s})$ that is a solution of* (12) *for a suitable control signal $\bar{u}(\tilde{s})$.*

Proof. Let $\sigma : \mathbb{R} \to \mathcal{Q}$ be a solution of (12) corresponding to a smooth control signal $\bar{u}(s)$. For each $s \in \mathbb{R}$, we have $\sigma'(s) = f^\pitchfork(\sigma(s)) + D^{-1}(\sigma(s)B(\sigma(s)))\bar{u}(s)$. Since $f^\pitchfork$ satisfies (13), it follows that the transversality condition (4) holds, and therefore $q = \sigma(s)$ is a VHC for (1).

Now let $q = \sigma(s)$ be a parametric VHC so that, by definition, the transversality condition (4) holds. We need to find a smooth function $\mu : \mathbb{R} \to \mathbb{R}$ such that $\mu' \neq 0$, and a control signal $\bar{u}$ such that, letting $\tilde{\sigma} = \sigma \circ \mu$ and $s = \mu(\tilde{s})$, it holds that

$$\frac{d\tilde{\sigma}}{d\tilde{s}} = f^\pitchfork(\tilde{\sigma}) + D^{-1}(\tilde{\sigma})B(\tilde{\sigma})\bar{u}(\tilde{s}). \quad (14)$$

By (13), the matrix-valued function $T(q) = [f^\pitchfork(q) \quad D^{-1}(q)B(q)]$ is everywhere nonsingular. Letting $U(s) = T^{-1}(\sigma(s))\sigma'(s)$, we have

$$\sigma'(s) = f^\pitchfork(\sigma(s))U_1(s) + D^{-1}(\sigma(s))B(\sigma(s))U_{2:n}(s),$$

where U_1 is the first component of U and $U_{2:n}$ is the vector containing the last $n-1$ components of U. Since $\sigma(s)$ satisfies (4), the function $U_1(s)$ is nowhere zero. Let $\mu(\tilde{s}) = \int_0^{\tilde{s}} (1/U_1(s))ds$. Then, $\mu' = 1/U_1 \neq 0$, and letting $\tilde{\sigma}(\tilde{s}) = \sigma \circ \mu(\tilde{s})$ it holds that

$$\begin{aligned}\frac{d\tilde{\sigma}}{d\tilde{s}} &= \frac{1}{U_1(\mu(\tilde{s}))}\sigma'(\mu(\tilde{s})) \\ &= f^\pitchfork(\tilde{\sigma}(\tilde{s})) + D^{-1}(\tilde{\sigma}(\tilde{s}))B(\tilde{\sigma}(\tilde{s}))\frac{U_{2:n}(\mu(\tilde{s}))}{U_1(\mu(\tilde{s}))}.\end{aligned}$$

Denoting $\bar{u}(\tilde{s}) = U_{2:n}(\mu(\tilde{s}))/U_1(\mu(\tilde{s}))$, the function $\tilde{\sigma}(\tilde{s})$ satisfies (14), as required. □

Remark 1. If $\sigma(s)$ is a periodic solution of the VCG (12) with period $T > 0$, then the VHC is $q = \sigma \circ \pi_T(s)$, where $\pi_T : \mathbb{R} \to \mathbb{S}^1$ is the smooth covering map $s \mapsto [2\pi s/T]_{2\pi}$. For simplicity, this step was omitted in the statement and proof of Theorem 1.

Remark 2. From the proof of Theorem 1 it follows that the effect of scaling the vector field $f^\pitchfork$ by a nonzero smooth real-valued function is to reparametrize the VHCs resulting from the integral curves of the VCG.

IV. Virtual constraint generation problems

In Theorem 1 we have shown that the VCG (12) is a control system whose orbits are all possible VHCs for system (1). The line of inquiry of this paper is to convert virtual constraint generation problems into control specifications for the VCG. In what follows, as a first step towards a more comprehensive theory, we focus on two basic VHC generation problems. We discuss them first by means of an example.

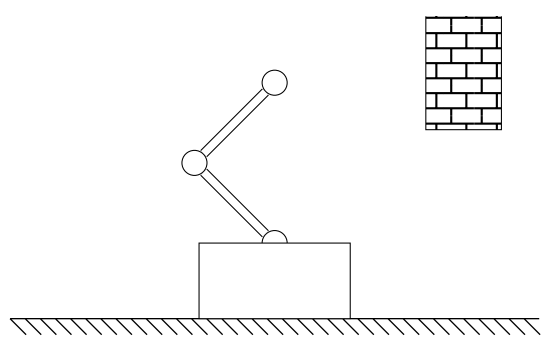

Fig. 2. The cart-acrobot

Consider the cart-acrobot system depicted in Figure 2, a double-pendulum on cart. The system has three degrees-of-freedom (the position of the cart and the pendulum angles) and two actuators, the force applied to the cart and the torque applied to the second joint. There is an obstacle

[3]Here, a reparametrization of a curve $\sigma(s)$ is the curve $\tilde{\sigma}(\tilde{s}) = \sigma(\mu(\tilde{s}))$, where $\mu : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\mu' \neq 0$.

descending from above, which we wish the system to pass under. Represent this obstacle as the set $\mathcal{O} \subset \mathcal{Q}$, such that $q \in \mathcal{O}$ if and only if some part of the cart-acrobot intersects the obstacle.

The objective is to synthesize parametric constraints $q = \sigma(s)$ such that $\sigma(\mathbb{R}) \cap \mathcal{O} = \varnothing$, so that the robot will not hit the obstacle when its state is on the constraint manifold, and the integral curves of the closed-loop system on the constraint manifold, $(q(t), \dot{q}(t)) = (\sigma(s(t)), \sigma'(s(t))\dot{s}(t))$, exhibit one of two qualitative properties:

1) A *through-motion*, where the cart, beginning on one side of the obstacle, passes under it to the other side, and then traverses the remainder of the track with bounded speed, and
2) A *cyclical motion*, where the system repetitively passes back-and-forth under the obstacle.

We will return to the cart-acrobot example in Section VIII. Now, inspired by the problem just outlined, we formulate two general constraint generation problems.

Problem 1. *Design a parametric* VHC $\sigma : \mathbb{R} \to \mathcal{Q}$ *for system* (1) *such that, for all initial conditions* $(s(0), \dot{s}(0))$ *of the resulting reduced dynamics* (5)*, the following properties hold:*

(i) $s(t) \to \infty$ *as* $t \to \infty$*, and*
(ii) $(\exists a > 0)(\forall t \in \mathbb{R})\, \|(d/dt)\sigma(s(t))\| \leq a$.

In other words, the parametric VHCs generated in the context of Problem 1 should induce integral curves $(q(t), \dot{q}(t)) = (\sigma(s(t)), \sigma'(s(t))\dot{s}(t))$ on the constraint manifold with the property that $\sigma(s(t))$ traverses the constraint curve in one direction, and the speed $\|\dot{q}(t)\|$ is bounded.

Now we turn to cyclical motions. In a sense, these are dual to the through motions we seek in Problem 1. Instead of having the system carry out a motion in a particular direction, we want the system to carry out a repetitive motion. First, a definition.

Definition 1. Consider a parametric VHC $q = \sigma(s)$, and let $s^\star \in \mathbb{R}$. An orbit of the reduced dynamics (5) is an *oscillation around* $s^\star$ if it is either a closed orbit encircling the point $(s^\star, 0)$, or an equilibrium at $(s^\star, 0)$.

Problem 2. *Design a parametric* VHC $\sigma : \mathbb{R} \to \mathcal{Q}$ *for system* (1) *such that all orbits of the reduced dynamics in* (5) *are oscillations.*

The goal of motion planning is to find *one trajectory* meeting certain specifications. In contrast, Problems 1 and 2 seek to determine closed-loop dynamics yielding *whole families* of trajectories meeting certain specifications. The trajectories in question are integral curves on the constraint manifold of system (1) with a controller enforcing a suitably designed VHC.

From now on, without loss of generality we let $f^{\pitchfork} : \mathcal{Q} \to T\mathcal{Q}$ in the VCG be as in (11), so that the VCG becomes

$$q' = (B^\perp(q))^\top + D^{-1}(q)B(q)u. \tag{15}$$

V. Viewing $\Psi_1(s)$ as an output signal of the VCG

In this section we make an observation that will allow us to map Problems 1 and 2 into control specifications for the VCG in (12), namely the fact that the function $\Psi_1(s)$ in the reduced dynamics (5) can be regarded as an output trajectory of the VCG (15) with output $\tilde{\Psi}_1 : \mathcal{Q} \to \mathbb{R}$ given by

$$\tilde{\Psi}_1(q) = -\frac{B^\perp(q)\nabla P(q)}{B^\perp(q)D(q)(B^\perp(q))^\top}. \tag{16}$$

More precisely, let $q = \sigma(s)$ be a VHC in parametric form for the mechanical system (1). By Theorem 1, after possibly reparametrizing σ, we may assume that $\sigma(s)$ is an integral curve of (15). Recall that the function $\Psi_1(s)$ in the reduced dynamics is given by

$$\Psi_1(s) = -\left.\frac{B^\perp \nabla P}{B^\perp D\sigma'(s)}\right|_{q=\sigma(s)}.$$

Using the fact that $\sigma'(s)$ satisfies (15) we get

$$B^\perp(\sigma(s))D(\sigma(s))\sigma'(s) = \left[B^\perp(q)D(q)(B^\perp(q))^\top\right]_{q=\sigma(s)},$$

from which it follows that $\Psi_1(s)$ in (6) can be expressed as $\Psi_1(s) = \tilde{\Psi}_1(\sigma(s))$. Thus $\Psi_1(s)$ is an output trajectory of the VCG (15) with output $\tilde{\Psi}_1$, as claimed.

In light of the above, requiring $\Psi_1(s)$ in the reduced dynamics to have certain properties corresponds to posing an output specification for the VCG (12) with output function $\tilde{\Psi}_1$ in (16). In the next two sections, we leverage this insight to solve Problems 1 and 2.

VI. Solution to Problem 1

In this section we map Problem 1 into a control specification for the VCG (15). Recall that, when system (1) is subject to a feedback $\tau^\star : \Gamma \to \mathbb{R}^{n-1}$ rendering the constraint manifold Γ invariant, all integral curves of the closed-loop system on Γ have the form $(q(t), \dot{q}(t)) = (\sigma(s(t)), \sigma'(s(t))\dot{s}(t))$. Problem 1 requires that $s(t) \to \infty$ as $t \to \infty$ and that $\|\sigma'(s(t))\dot{s}(t)\|$ is bounded. Since our expressions only allow us to determine dynamic properties in terms of s and $\dot{s}$, we cannot *a priori* guarantee that $\|\sigma'(s(t))\dot{s}(t)\|$ is bounded directly from the reduced dynamics. The following lemma provides this guarantee under mild conditions.

Lemma 1. *Consider the* VCG *in* (15)*. Assume that there exists* $A > 0$ *such that for each* $q \in \mathcal{Q}$, $\|B(q)\|$, $\|B^\perp(q)\|$, $\|D^{-1}(q)\| < A$*. Let* $\bar{u} : \mathbb{R} \to \mathbb{R}^{n-1}$ *be a bounded differentiable control signal, and* $\sigma(s)$ *be a solution of the* VCG (15) *under this control. Then for any differentiable signal* $s : \mathbb{R} \to \mathbb{R}$*, such that* $\|\dot{s}(t)\|$ *is bounded,* $\|\sigma'(s(t))\dot{s}(t)\|$ *is also bounded.*

In light of the lemma, we replace requirement (ii) in Problem 1 by the simpler requirement that $|\dot{s}(t)|$ be bounded, so we can now focus our attention on designing dynamics for s.

Suppose we only generate constraints which are not closed curves. Then they are automatically Euler-Lagrange, and the Lagrangian is given by $(1/2)M(s)\dot{s}^2 - V(s)$. We can leverage our intuition about Lagrangian systems to design dynamics for the constraint. In the case where the mass $M(s) = m$ is constant, the reduced dynamics are simply

$$m\ddot{s} = -\frac{\partial V}{\partial s} = \Psi_1(s) \tag{17}$$

which is identical to Newton's second law.

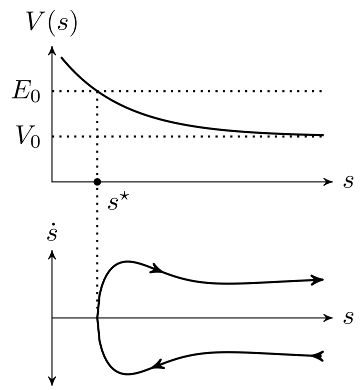

Fig. 3. E-L Dynamics with an unbounded and strictly decreasing potential

To ensure that $s(t)$ traverses the entire real line from left to right, we would like the potential function $V(s)$ to be strictly decreasing and such that $V(s) \to \infty$ as $s \to -\infty$. This will ensure that each solution $s(t)$ of (17) is strictly increasing after a finite amount of time. In light of (9), the properties just stated are equivalent to having $\Psi_1 > 0$ and $\int_0^{-\infty} \Psi_1(\tau) d\tau = -\infty$. Further, in order to ensure that $|\dot{s}(t)|$ is bounded, we would like that $V(s) \to V_0$ as $s \to +\infty$ or, in light of (9), $\int_0^{\infty} \Psi_1(\tau) d\tau < \infty$. A potential function possessing all required properties is depicted in Figure 3. That the properties of $V(s)$ discussed above and illustrated in Figure 3 yield reduced dynamics meeting the requirements of Problem 1 is clear when the virtual mass is constant. In what follows we establish that this holds more generally when the virtual mass varies, subject to mild assumptions on the virtual mass.

Lemma 2. *Suppose that the function $\Psi_1(s)$ in the reduced dynamics* (5) *satisfies $\Psi_1 > 0$, $\int_0^{\infty} \Psi_1(\tau)\, d\tau < \infty$, and $\int_0^{-\infty} \Psi_1(\tau)\, d\tau = -\infty$. Assume further that there exist two positive constants M_1, M_2 such that the function $M(s)$ in* (8) *satisfies $0 < M_1 < M(s) < M_2$ for all $s \in \mathbb{R}$. Then, for all initial conditions $(s(0), \dot{s}(0)) = (s_0, \dot{s}_0) \in \mathbb{R} \times \mathbb{R}$ of the reduced dynamics* (5)*, $s(t) \to \infty$ and $\dot{s}(t)$ is bounded.*

Lemmas 1 and 2 convert the requirements of Problem 1 on integral curves on the constraint manifold into requirements on the function $\Psi_1(s)$ which, as shown in Section V, can be viewed as an output signal of the VCG in (15) with output function $\tilde{\Psi}_1(q)$. We may therefore reformulate Problem 1 as a control specification for the VCG, as follows.

Problem 3 (reformulation of Problem 1)**.** *For the* VCG *in* (15) *with output $\tilde{\Psi}_1(q)$ given in* (16)*, design a smooth feedback $\bar{u}(q)$ such that for each initial condition in a suitable set, the output signal $\Psi_1(s) = \tilde{\Psi}_1(\sigma(s))$ corresponding to the solution $\sigma(s)$ of* (15) *enjoys the following properties:*

(i) $\Psi_1(s) > 0$ *for all* $s \in \mathbb{R}$,
(ii) $\Psi_1(s) \to 0$ *exponentially as* $s \to \infty$, *and*
(iii) there exists a positive constant ε such that $\Psi_1(s) > \varepsilon > 0$ for all $s \leq 0$.

Property (ii) ensures that $\int_0^{\infty} \Psi_1(\tau)\, d\tau < \infty$, while property (iii) guarantees that $\int_0^{-\infty} \Psi_1(\tau)\, d\tau = -\infty$. The next proposition presents a solution to this problem.

Proposition 1. *Consider the* VCG *in* (15) *with output $\tilde{\Psi}_1$ given in* (16)*. Assume that there exist positive constants A and $\varepsilon^\star$ such that for each $q \in \mathcal{Q}$, $\|B^\perp(q)\|$, $\|B(q)\|$, $\|D^{-1}(q)\| < A$, and $\operatorname{rank}(L_g\tilde{\Psi}_1) = 1$ for all $q \in U = \{q \in \mathcal{Q} : 0 \leq \tilde{\Psi}_1(q) < \varepsilon^\star\}$. Let $K > 0$ and $\varepsilon \in (0, \varepsilon^\star)$, and consider the feedback*[4]

$$\bar{u}(q) = -(L_g\tilde{\Psi}_1)^+(L_f\tilde{\Psi}_1 - K\tilde{\Psi}_1(\tilde{\Psi}_1 - \varepsilon)). \tag{18}$$

Then, for each initial condition $q(0) \in \{q \in \mathcal{Q} : 0 < \tilde{\Psi}_1(q) < \varepsilon\}$, the integral curve of the VCG *in* (15) *with feedback* (18) *is a parametric* VHC *for system* (1) *meeting the conditions of Problem 1.*

Proof. Since $L_g\tilde{\Psi}_1$ has full rank on U, the feedback (18) is well-defined everywhere in U. Within this set, letting $e = \tilde{\Psi}_1(q)$, we have $e' = Ke(e - \varepsilon)$. This ODE has two equilibria at 0 and ε. The equilibrium $e = 0$ is exponentially stable, while the equilibrium $e = \varepsilon$ is unstable. For all $e(0) \in (0, \varepsilon)$, the image of the integral curve $e(s)$ is contained in the interval $(0, \varepsilon)$. Moreover, $e(s) \to 0$ exponentially as $s \to \infty$, and $e(s) \to \varepsilon$ as $s \to -\infty$. Thus for each $q(0) \in \{q \in \mathcal{Q} : 0 < \tilde{\Psi}_1(q) < \varepsilon\}$, the integral curve $\sigma(s)$ of the VCG in (15) with feedback (18) is contained in the set U where the feedback is well-defined, and the output signal $\Psi_1(s) = \tilde{\Psi}_1(\sigma(s))$ satisfies the hypotheses of Lemma 2. By Theorem 1, $q = \sigma(s)$ is a parametric VHC for system (1). By Lemma 2, for each initial condition of the reduced dynamics in (5), the solution $(s(t), \dot{s}(t))$ is such that $s(t) \to \infty$ and $|\dot{s}(t)|$ is bounded. By Lemma 1, this latter property implies that $\|\sigma'(s(t))\dot{s}(t)\|$ is bounded, proving that the VHC $q = \sigma(s)$ satisfies all the requirements of Problem 1. □

VII. Solution to Problem 2

From classical mechanics it is familiar that for systems with one degree-of-freedom, oscillations occur around local minima of the potential energy. If that minimum is unique, and the potential is unbounded, then all orbits of the systems are oscillations.

The following two lemmas give characterizations of the required conditions on a VHC to guarantee the two cases of oscillatory behaviour described above. The first guarantees the existence of oscillations locally.

Lemma 3. *Consider the reduced dynamics given in* (5)*. Let $s^\star \in \mathbb{R}$ be a point such that $\Psi_1(s^\star) = 0$, and $\Psi_1'(s^\star) < 0$.*

[4]Here, the superscript $+$ denotes the Moore-Penrose pseudoinverse of a matrix $M^+ = M^\top/(MM^\top)$.

Then there exist $e^\star \in \mathbb{R}$ and an interval $I \subset \mathbb{R}$ containing $s^\star$ such that for all initial conditions with $s_0 \in I$ and $E(s_0, \dot{s}_0) < e^\star$, the solution of the reduced dynamics is an oscillation around $s^\star$ in the sense of Definition 1.

The preceding lemma only relies on a local property of Ψ_1, and so only provides local guarantees for the dynamics. The following lemma guarantees the existence of oscillations globally, making only mild assumptions on the global properties of Ψ_1.

Lemma 4. *Suppose that for some $s^\star \in \mathbb{R}$, the function $\Psi_1(s)$ in the reduced dynamics (5) satisfies $\Psi_1(s^\star) = 0$, $(s - s^\star)\Psi_1(s) < 0$ for all $s \neq s^\star$, and $\int_{s^\star}^{\pm\infty} \Psi_1(\tau)\, d\tau = -\infty$. Assume further that there exists a positive constant M_1 such that $M(s) > M_1 > 0$ for all $s \in \mathbb{R}$. Then, all solutions of the reduced dynamics are oscillations around $s^\star$.*

Proposition 2. *Consider the* VCG *in* (15) *with output $\tilde{\Psi}_1$ given in* (16). *Suppose there exist constants $\varepsilon_-^\star < 0 < \varepsilon_+^\star$ such that $\operatorname{rank}(L_g\Psi_1) = 1$ for all $q \in U = \{q \in \mathcal{Q} : \varepsilon_-^\star < \tilde{\Psi}_1(q) < \varepsilon_+^\star\}$. Let $K > 0$, $\varepsilon_- \in (0, \varepsilon_+^\star)$, and $\varepsilon_+ \in (\varepsilon_-^\star, 0)$ and consider the feedback*

$$\bar{u}(q) = -(L_g\tilde{\Psi}_1)^+(L_f\tilde{\Psi}_1 - K(\tilde{\Psi}_1 - \varepsilon_+)(\tilde{\Psi}_1 - \varepsilon_-)). \quad (19)$$

Then, for each initial condition $q(0) \in \{q \in \mathcal{Q} : 0 < \tilde{\Psi}_1(q) < \varepsilon_+\}$, the integral curve of the VCG *in* (15) *is a parametric* VHCs *for system* (1) *meeting the conditions of Problem 2.*

Proof. The proof is very similar to the proof of Proposition 1, and it is therefore sketched. Letting $e = \tilde{\Psi}_1(q)$, we have $e' = K(e - \varepsilon_+)(e - \varepsilon_-)$. Thus for each $q(0) \in \{q \in \mathcal{Q} : \varepsilon_- < \tilde{\Psi}_1(q) < \varepsilon_+\}$, the integral curve $\sigma(s)$ of the VCG (15) with feedback (19) is strictly decreasing and such that $\tilde{\Psi}_1(\sigma(s)) \to \varepsilon_-$ as $s \to \infty$, and $\tilde{\Psi}_1(\sigma(s)) \to \varepsilon_+$ as $s \to -\infty$. The resulting output signal $\Psi_1(s) = \tilde{\Psi}_1(\sigma(s))$ satisfies the assumptions of Lemma 4. This fact and Theorem 1 imply that $q = \sigma(s)$ is a parametric VHC for system (1) whose associated reduced dynamics meet the requirements of Problem 2. □

VIII. Example: Acrobot on Cart

Consider the model of a double pendulum on cart of mass M, where the pendula are modelled as point masses m_1 and m_2 with respective link lengths l_1 and l_2. Measure the angle of the first link θ_1 counter-clockwise from the vertical, while the angular position of the second link θ_2 will be given counter-clockwise relative to the first link. The configuration of this system is then given by the vector $q = (x, \theta_1, \theta_2)$, and the Lagrangian of this system is $\mathcal{L} = \frac{1}{2}\dot{q}^\top D(q)\dot{q} - P(q)$, where the entries of the symmetric mass matrix D are

$$\begin{aligned}
D_{11} &= M + m_1 + m_2\\
D_{12} &= -l_1(m_1 + m_2)\cos(\theta_1) - l_2 m_2 \cos(\theta_1 + \theta_2)\\
D_{13} &= -l_2 m_2 \cos(\theta_1 + \theta_2)\\
D_{22} &= m_1 l_1^2 + m_2(l_1^2 + 2 l_1 l_2 \cos(\theta_2) + l_2^2)\\
D_{23} &= m_2 l_2^2 + m_2 l_1 l_2 \cos(\theta_2)\\
D_{33} &= m_2 l_2^2
\end{aligned}$$

and the potential is $P = m_1 g l_1 \cos(\theta_1) + m_2 g(l_1\cos(\theta_1) + l_2\cos(\theta_1 + \theta_2))$. Let the cart and the second joint of the pendulum be actuated, such that $B = \begin{bmatrix} e_1 & e_3 \end{bmatrix}$. For any $\lambda \neq 0$, the vector $B^\perp = \lambda e_2^\top$ is a rank-one left-annihilator of B. We chooose $\lambda = -1$ because, as we shall see, that will induce VHCs $q = \sigma(s)$ where x increases in s.

The VCG for the cart-acrobot system has the form $q' = (B^\perp)^\top + g_1 u_1 + g_2 u_2$. Due to space limitations, we omit the expressions of g_i, but we remark that in the definition of g_i, we replaced D^{-1} by $\operatorname{adj}(D)$ in order to get simpler expressions. The output function $\tilde{\Psi}_1$ given in (16) is

$$\tilde{\Psi}_1(q) = \frac{2\sin(\theta_1) + \sin(\theta_1 + \theta_2)}{2\cos(\theta_2) + 3}.$$

Calculation of $L_g\tilde{\Psi}_1$ reveals that it satisfies the rank assumptions of Propositions 1 and 2.

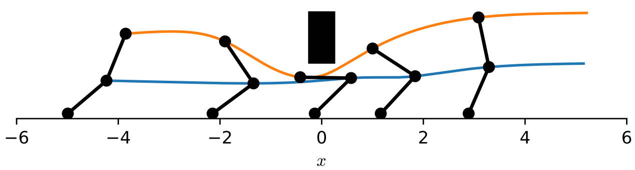

Fig. 4. Configuration of the cart-acrobot along a constraint satisfying the conditions of Problem 1

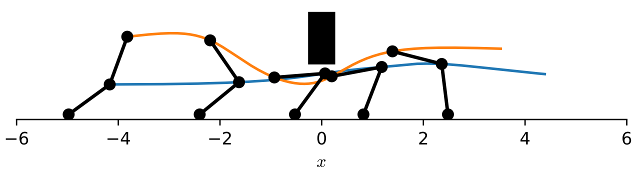

Fig. 5. Configuration of the cart-acrobot along a constraint satisfying the conditions of Problem 2

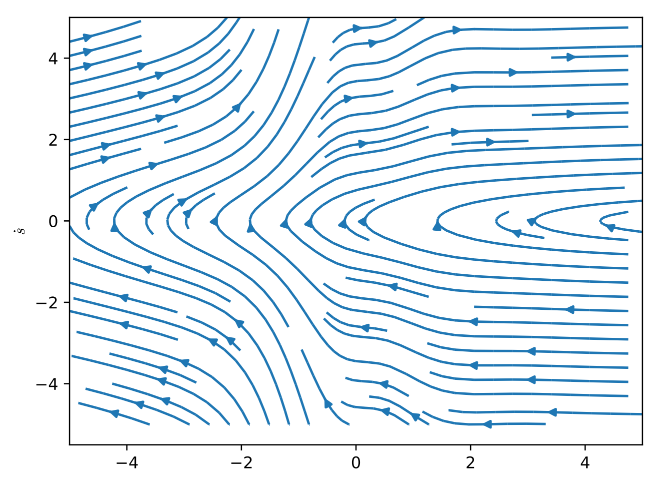

Fig. 6. Phase portrait of the reduced dynamics of the constraint depicted in Figure 4. All integral curves eventually turn around and progress to ∞.

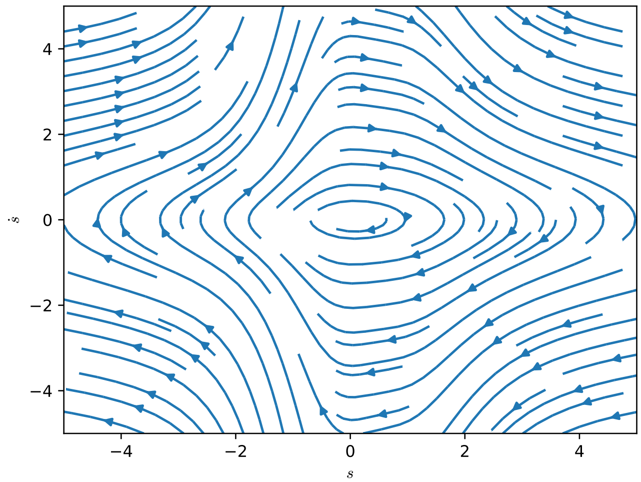

Fig. 7. Phase portrait of the reduced dynamics of the constraint depicted in Figure 5. Orbits clearly circle a point.

The results of this paper help one generate constraints inducing certain qualitative properties in the reduced dynamics, but they do not explicitly provide information about the geometry of the VHC. To illustrate, in the context of the cart-acrobot we would like the pendula to avoid the obstacle. We would also like to prevent the second pendulum from performing full revolutions along the constraint. Finally, we would like that, on the constraint $q = \sigma(s)$, $\sigma_1(s)$ is monotonically increasing, so that x spans the entire real line. We accomplish these objectives in an ad hoc fashion as follows. We define the feedback transformation

$$u = \bar{u}(q) + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (L_g \tilde{\Psi}_1)^\top u_\perp,$$

where $u_\perp$ is a scalar control input to be assigned, and $\bar{u}(q)$ is the feedback (18) (for Problem 1) or (19) (for Problem 2). For any smooth feedback $\bar{u}_\perp(q)$, the above controller has no effect on dynamics of the output $e = \tilde{\Psi}_1(q)$ in the proof of Propositions 1 and 2, and therefore the conclusions of the propositions remain unchanged. For our simulations, $u_\perp$ = constant will yield the latter two properties listed above. Initializing the VCG in a configuration underneath the obstacle allows us to generate constraints avoiding it by relying on the slow evolution of the configuration under the VCG dynamics. The justification of this approach and its generalization will be the subject of future research.

The controller for the VCG presented in Proposition 1 is identical to the controller of Proposition 2 if we let $\varepsilon_- = 0$ and $\varepsilon_+ = \varepsilon$. Table I summarizes the parameters used to generate the corresponding constraints, using the notation of the controller in (19). Figures 4 and 5 show the constraints solving Problems 1 and 2, respectively, while Figures 6 and 7 show phase portraits of the associated reduced dynamics, verifying that the desired properties hold.

Dynamics simulations of the cart-acrobot under controllers enforcing the above constraints were carried out. The system was initialized left of the obstacle, off the constraint. In each case, as the system converged to the constraint, it also carried out the corresponding desired motion. When enforcing the constraint solving Problem 1, the system performed a single pass under the obstacle and proceeded to the right. In contrast, when enforcing the constraint solving Problem 2, it began to repeatedly pass back and forth under the obstacle.

TABLE I

VCG PARAMETERS

	Constraint type	
Parameter	Traversing	Oscillatory
K	1	1
ε_+	0.5	0.5
ε_-	0	-0.5
Initial x	0	0
Initial θ_1	$-\pi/6$	$\pi/6$
Initial θ_2	$2\pi/3$	$-3\pi/4$
$u_\perp$	-0.4	-0.5

IX. Conclusion

For mechanical systems with underactuation degree one, we have introduced the virtual constraint generator, a control system on the configuration manifold whose orbits are all the VHCs of the system, up to reparametrization. We have presented two techniques to generate constraints inducing certain types of constrained dynamics: traversal of the VHC curve in one direction, and oscillations. In future work we will explore from a general perspective the idea presented in Section VIII of using the remaining degrees of freedom of the VCG to impose requirements on the geometry of the VHCs it produces.

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