# MANEUVER REGULATION VIA TRANSVERSE FEEDBACK LINEARIZATION: THEORY AND EXAMPLES

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Abstract: This paper presents a methodology to solve maneuver regulation problems which is based on transverse feedback linearization. Necessary and sufficient conditions for the linearization of dynamics transverse to an embedded submanifold of the state space are presented. Various examples illustrate the main features of this framework.

Keywords: Maneuver regulation, path following, feedback linearization, zero dynamics, output stabilization, approximate feedback linearization.

## 1. INTRODUCTION

The maneuver regulation (or path following) problem entails designing a smooth feedback making the trajectories of a system follow a prespecified path, or *maneuver*. Unlike a tracking controller, a maneuver regulation controller drives the trajectories of a system to a maneuver *up to time-reparameterization*. This difference is crucial in robotics and aerospace applications where the system dynamics impose constraints on the time parameterization of feasible maneuvers.

This paper, together with the work in Nielsen and Maggiore (2004a,b,c), initiates a line of research inspired by the work of Banaszuk and Hauser (1995). There, the authors consider periodic maneuvers in the state space and present necessary and sufficient conditions for feedback linearization of the associated transverse dynamics. Feedback linearization is a natural framework for maneuver regulation, as evidenced by the body of work on path following which employs this approach (see for example Altafini (2002), Gillespie et al. (2001), Hauser and Hindman (1997), Coelho and Nunes (2003)). In all these papers, the maneuver regulation problem is converted to an input output feedback linearization problem with respect to a *suitable* output. This motivates our interest in establishing a general framework for doing this. After reviewing necessary and sufficient conditions for *global* and *local* transverse feedback linearization (TFL) (see also Nielsen and Maggiore (2004a,b,c)), we apply our results to a kinematic unicycle (Section 5.1), a rear wheel drive car (Section 5.2) and a trailer system (Section 5.3). We show that the latter system *is not* transversely feedback linearizable but its transverse dynamics possess a robust relative degree and hence approximate linearization is possible.

The following notation is used throughout the paper. We denote by $\Phi_t^v(x)$ the flow of a smooth vector field $v$. We let $\mathrm{col}(x_1, \dots, x_k) := [x_1 \ \dots \ x_n]^\top$ and, given two column vectors $a$ and $b$, we let $\mathrm{col}(a, b) := [a^\top \ b^\top]^\top$. Given a smooth distribution

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$D$, we let $\mathrm{inv}\,(D)$ be its involutive closure (the smallest involutive distribution containing $D$) and $D^\perp$ be its annihilator. For brevity, the term *submanifold* is used in place of *embedded submanifold of* $\mathbb{R}^n$ throughout.

## 2. PROBLEM FORMULATION

Consider the smooth dynamical system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

defined on $\mathbb{R}^n$, with $h : \mathbb{R}^n \to \mathbb{R}^p$ ($p \geq 2$) of class $C^r$ ($r \geq 1$), and $u \in \mathbb{R}$. Given a smooth parameterized curve $\sigma : \mathbb{D} \to \mathbb{R}^p$, where $\mathbb{D}$ is either $\mathbb{R}$ or $S^1$, the maneuver regulation problem entails finding a smooth control $u(x)$ driving the output of the system to the set $\sigma(\mathbb{D})$ and making sure that the curve is traversed in one direction. When $\mathbb{D} = S^1$, $\sigma(\mathbb{D})$ is a periodic curve. Banaszuk and Hauser (1995) provide a solution to this problem in the special case when $\mathbb{D} = S^1$ and $h(x) = x$. Notice that one particular instance of maneuver regulation is the case when a controller is designed to make $y(t)$ asymptotically track a specific time *parameterization* of the curve $\sigma(t)$ (Hauser and Hindman, 1995). Thus asymptotic tracking and maneuver regulation are closely related problems. In some cases, however, it may be undesirable or even impossible to pose a maneuver regulation problem as one of tracking (consider, for instance, the problem of maneuvering a wheeled vehicle with bounded translational speed by means of steering). We impose geometric restrictions on the class of curves $\sigma(\cdot)$.

*Assumption 1.* The curve $\sigma : \mathbb{D} \to \mathbb{R}^p$ enjoys the following properties

(i) $\sigma$ is $C^r$, ($r \geq 1$)
(ii) $\sigma$ is regular, i.e., $\|\dot{\sigma}\| \neq 0$
(iii) $\sigma : \mathbb{D} \to \sigma(\mathbb{D})$ is injective (when $\mathbb{D} = S^1$ we instead require $\sigma$ to be a Jordan curve)
(iv) $\sigma$ is proper, i.e. for any compact $K \subset \mathbb{R}^p$, $\sigma^{-1}(K)$ is compact (automatically satisfied when $\mathbb{D} = S^1$)

Assumption 1 guarantees that $\sigma(\mathbb{D})$ is a submanifold of $\mathbb{R}^p$ with dimension 1.

*Assumption 2.* There exists a $C^1$ map $\gamma : \mathbb{R}^p \to \mathbb{R}^{p-1}$ such that 0 is a regular value of $\gamma$ and $\sigma(\mathbb{D}) = \gamma^{-1}(0)$. Moreover, the *lift* of $\gamma^{-1}(0)$ to $\mathbb{R}^n$, $\Gamma := (\gamma \circ h)^{-1}(0)$, is a submanifold of $\mathbb{R}^n$.

A sufficient condition for

$$\Gamma = \{x : \gamma_1(h(x)) = \ldots = \gamma_{p-1}(h(x)) = 0\} \tag{2}$$

to be a submanifold of $\mathbb{R}^n$ is that $h$ be *transversal* to $\gamma^{-1}(0)$, i.e., (Guillemin and Pollack, 1974)

$$(\forall x \in \Gamma)\ \ \mathrm{Im}\,(dh)_x + T_{h(x)}\gamma^{-1}(0) = \mathbb{R}^p.$$

The codimension of $\Gamma$ is equal to the codimension of $\gamma^{-1}(0)$ which implies $\dim \Gamma = n - p + 1$ (Consolini and Tosques, 2003). The problem of maneuvering $y$ to $\gamma^{-1}(0)$ is thus equivalent to maneuvering $x$ to $\Gamma$ and can be cast as an output stabilization problem for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y' &= (\gamma \circ h)(x). \end{aligned} \tag{3}$$

In general one may be able to maneuver $x$ to the subset of $\Gamma$ which can be made invariant by a suitable choice of the control input. Accordingly, let $\Gamma^*$ be the largest controlled invariant submanifold of $\Gamma$ under (1) and let $n^* = \dim \Gamma^*$ ($n^* \leq \dim \Gamma = n - p + 1$). Further, let $u^*$ be a smooth feedback rendering $\Gamma^*$ invariant and define $f^* := (f + gu^*)|_{\Gamma^*}$.

*Assumption 3.* $\Gamma^*$ is a closed connected submanifold (with $n^* \geq 1$) and the following conditions hold

(i) $(\exists\, \epsilon > 0)(\forall\, x \in \Gamma^*)\ \ \|L_{f^*}h(x)\| > \epsilon$.
(ii) $f^* : \Gamma^* \to T\Gamma^*$ is complete

In Banaszuk and Hauser (1995), $\Gamma^* = \Gamma = \sigma(S^1)$, and it is assumed that $f(x) \neq 0$ on $\Gamma^*$. Thus in that work Assumption 3 is automatically satisfied (the completeness of $f^*$ follows from the periodicity of $\sigma(S^1)$).

The requirement, in Assumption 3, that $\Gamma^*$ be a closed connected submanifold can be checked using conditions presented in Nielsen and Maggiore (2004a,c). The condition, in Assumption 3, that $\|L_{f^*}h(x)\| > \epsilon$ on $\Gamma^*$ implies that there are no equilibria on $\Gamma^*$ and that, whenever $x \in \Gamma^*$, $\|\dot{y}\| = \|L_{f^*}h(x)\| > \epsilon$. This condition ensures that the output of (1) traverses the curve $\sigma(\mathbb{D})$. We are now ready to formulate the main problems investigated in this paper. The following are a direct generalization of analogous statements found in Banaszuk and Hauser (1995).

Problem 1: *Find, if possible, a single coordinate transformation* $T : x \mapsto (z, \xi) \in \Gamma^* \times \mathbb{R}^{n-n^*}$ *valid in a neighborhood* $\mathcal{N}$ *of* $\Gamma^*$ *such that in* $(z, \xi)$ *coordinates*

*(i)* $\Gamma^* = \{(z, \xi) \in \Gamma^* \times \mathbb{R}^{n-n^*} : \xi = 0\}$
*(ii) The dynamics of system* (1) *take the form*

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi}_1 &= \xi_2 \\ &\ \vdots \\ \dot{\xi}_{n-n^*-1} &= \xi_{n-n^*} \\ \dot{\xi}_{n-n^*} &= b(z, \xi) + a(z, \xi)u \end{aligned} \tag{4}$$

*where $a(z,\xi) \neq 0$ in $\mathcal{N}$.*

The following is the local version of Problem 1.

Problem 2: *For some $x^0 \in \Gamma^*$, find, if possible, a transformation $T^0 : x \mapsto \left(z^0, \xi^0\right) \in \Gamma^* \times \mathbb{R}^{n-n^*}$ valid in a neighborhood $U^0$ of $x^0 \in \Gamma^*$ such that in $\left(z^0, \xi^0\right)$ coordinates properties (i) and (ii) of Problem 1 are satisfied in $U^0$.*

It is clear that if one can solve Problem 1 or 2, then the smooth feedback

$$u = -\frac{1}{a(z,\xi)}(b(z,\xi) + K\xi). \tag{5}$$

achieves local output stabilization of (3) and hence local stabilization of (1) to $\Gamma^*$ (resp., $\Gamma^* \cap U^0$). In turn, stabilization to $\Gamma^*$ implies, by Assumption 3(i), traveral of $\sigma(\mathbb{D})$ in output coordinates. In light of this, the main focus in Problems 1 and 2 is the output stabilization of (3).

We begin by reviewing necessary and sufficient conditions to solve Problem 1 (Theorem 1) and Problem 2 (Theorem 3). For the sake of brevity we do not include the proofs which can be found in Nielsen and Maggiore (2004c) or Nielsen and Maggiore (2004b) (Problem 1) and Nielsen and Maggiore (2004a) (Problem 2).

## 3. SOLUTION TO PROBLEM 1

*Theorem 1.* Problem 1 is solvable if and only if there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $\Gamma^* \subset \{x \in \mathbb{R}^n : \alpha(x) = 0\}$
(2) $\alpha$ yields a uniform relative degree $n - n^*$ on $\Gamma^*$.

The function $\alpha$ is used to generate the feedback (5) by setting

$$a(T(x)) = L_g L_f^{n-n^*-1}\alpha(x)$$
$$b(T(x)) = L_f^{n-n^*}\alpha(x).$$

The conditions in Theorem 1, although rather intuitive, are difficult to check in practice. In what follows we present sufficient conditions for the existence of a solution to Problem 1 which are easier to check.

*Corollary 1.* If one of the *path constraints* in (2), $\gamma_{\bar{k}} \circ h$, yields a relative degree $n - n^*$ then Problem 1 is solved by setting $\alpha = \gamma_{\bar{k}} \circ h$.

Thus, it may be possible to solve Problem 1 by performing input-output linearization choosing as output one of the path constraints. However (Nielsen and Maggiore (2004c)), Problem 1 may be solvable even when *none* of the path constraints yields a well-defined relative degree.

*Lemma 1.* If there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ which satisfies the conditions of Theorem 1, then for all $x \in \Gamma^*$

$$T_x\Gamma^* + \mathrm{span}\{g, \ldots, ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n. \tag{6}$$

Condition (6) is a generalization of the notion of transverse linear controllability to the case of controlled invariant submanifolds of any dimension. It is useful in deriving checkable sufficient conditions for the existence of a solution to Problem 1. The notion of transverse linear controllability was originally introduced in Nam and Arapostathis (1992) and later used in Banaszuk and Hauser (1995) for transverse feedback linearization. In both papers, $n^* = 1$, $\mathbb{D} = S^1$, and $T_x\Gamma^* = \mathrm{span}\{f^*(x)\}$.

*Theorem 2.* Problem 1 is solvable if

(1) $\Gamma^*$ is parallelizable ($T\Gamma^* \cong \Gamma^* \times \mathbb{R}^{n^*}$)
(2) $T_x\Gamma^* + \mathrm{span}\{g \ldots ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n$ on $\Gamma^*$
(3) The distribution $\mathrm{span}\{g \ldots ad_f^{n-n^*-2}g\}$ is involutive.

## 4. SOLUTION TO PROBLEM 2

The following is an obvious result in the light of Theorem 1.

*Theorem 3.* Problem 2 is solvable if and only if there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ defined in a neighborhood $U^0$ of some $x^0 \in \Gamma^*$ such that

(1) $\Gamma^* \cap U^0 \subset \{x \in U^0 : \alpha(x) = 0\}$
(2) $\alpha$ yields a relative degree $n - n^*$ at $x^0$.

*Lemma 2.* If there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ which satisfies the conditions of Theorem 3, then

$$T_{x^0}\Gamma^* + \mathrm{span}\{g, \ldots, ad_f^{n-n^*-1}g\}(x^0) = \mathbb{R}^n.$$

Let $D = \mathrm{span}\{g \ldots ad_f^{n-n^*-2}g\}$. Theorem 2 proves that involutivity of $D$, together with transverse linear controllability, are sufficient conditions for the existence of a function $\alpha$ satisfying conditions (1) and (2) in Theorem 1 and hence solving Problem 1. When the involutive closure of $D$, $\mathrm{inv}(D)$, is regular at $x^0 \in \Gamma^*$, the next result provides necessary and sufficient conditions to solve Problem 2. These conditions are easier to check than those in Theorem 3.

*Theorem 4.* Assume that $\mathrm{inv}(D)$ is regular at $x^0 \in \Gamma^*$. Then Problem 2 is solvable if and only if

(1) $T_{x^0}\Gamma^* + \mathrm{span}\{g, \ldots, ad_f^{n-n^*-1}g\}(x^0) = \mathbb{R}^n$
(2) $ad_f^{n-n^*-1}g(x^0) \notin \mathrm{inv}(D)(x^0)$.

*Corollary 2.* If $\dim(\mathrm{inv}\, D) = n$, then Problems 1 and 2 are unsolvable.

*Corollary 3.* Assume that $\mathrm{inv}\, D$ is regular on $\Gamma^*$ and that

(1) $T_x\Gamma^* + \text{span}\{g, \ldots, ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n$ on $\Gamma^*$
(2) $ad_f^{n-n^*-1}g(x) \notin \text{inv}(D)(x)$ on $\Gamma^*$.

Then there exists an open covering $\{U^{(i)}\}$ of $\Gamma^*$ and a collection of transformations $\{T^{(i)}\}$, with $T^{(i)} : x \mapsto (z^{(i)}, \xi^{(i)}) \in \Gamma^* \cap U^{(i)} \times \mathbb{R}^{n-n^*}$ such that $\Gamma^* \cap U^{(i)} = \{\xi^{(i)} = 0\}$ and in $(z^{(i)}, \xi^{(i)})$ coordinates the systems has the form (4).

## 5. EXAMPLES

### *5.1 Kinematic Unicycle*

Consider the kinematic model of a unicycle with fixed translational speed $v \neq 0$

$$\dot{x} = \begin{bmatrix} v\cos x_3 \\ v\sin x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \text{col}\,(x_1, x_2)\,.$$

We now solve the problem of maneuvering the output of the unicycle to *any* curve $\sigma(\mathbb{D})$ satisfying Assumption 1 with $r \geq 2$. We first show, by means of Theorem 2, that the problem is *always solvable*. Later, we show that the function $\alpha$ yielding the solution can always be chosen as the path constraint.

Note that Assumption 2 is not needed. Specifically, we do not need to assume that there exists a submersion $\gamma$ such that $\sigma(\mathbb{D}) = \gamma^{-1}(0)$ because the lift of $\sigma(\mathbb{D})$, $\Gamma$, is always an embedded submanifold of dimension 2 (a generalized cylinder $\Gamma = \sigma(\mathbb{D}) \times \mathbb{R}$) and Assumption 3 is always satisfied. To see why the latter is true, refer to Figure 1 and observe that the nonholonomic constraint of the unicycle yields $\Gamma^* = \{x \in \mathbb{R}^3 : x = (\sigma(t), \text{arctan2}(\dot{\sigma}(t))), t \in \mathbb{D}\}$, where $\text{arctan2} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denotes the smooth arctangent function. Hence, $\Gamma^*$ is a well defined closed submanifold of dimension 1. To show that $f^*$ is complete, assume without loss of generality that $\sigma$ is unit speed, i.e., $\|\dot{\sigma}\| = 1$. Then, the flow of $f^*$, $t \mapsto \Phi_t^{f^*} = (\sigma(vt), \text{arctan2}(v\dot{\sigma}(vt)))$ is well defined for all $t \in \mathbb{D}$.

Next, the conditions of Theorem 2 reduce to

$$T_x\Gamma^* + \text{span}\{g, ad_f g\} = \mathbb{R}^3 \text{ on } \Gamma^*,$$

where $ad_f g = v\sin x_3 \frac{\partial}{\partial x_1} - v\cos x_3 \frac{\partial}{\partial x_2}$. Simple geometric considerations (see Figure 1) show that, for all $x \in \Gamma^*$,

$$T_x\Gamma = \text{span}\{f^*, g\}(x) = T_x\Gamma^* + \text{span}\{g\}(x)$$

and

$$\text{span}\{ad_f g\}(x) = (T_x\Gamma)^\perp.$$

Theorem 2 can thus be applied to conclude that Problem 1 has a solution. Here Theorem 2 allows us to recover the well known fact that unicycles with constant forward velocity can follow any smooth regular curve on the plane.[2]

Now we show that the path constraint defining $\Gamma$ can always be used to define the function $\alpha$ in Theorem 1.

*Lemma 3.* If the curve $\sigma$ satisfies Assumption 1 with $r \geq 2$ then Corollary 1 applies to the unicycle system.

*Proof :* We have already seen that Assumptions 2 and 3 are automatically satisfied. This implies that there exists a map $\gamma : \mathbb{R}^2 \to \mathbb{R}$ such that $\sigma(\mathbb{D}) = \{(y_1, y_2) : \gamma(y_1, y_2) = 0\}$. The lift of this path is a generalized cylinder given by $\Gamma = \{x : \gamma(x_1, x_2) = 0\}$. By checking the conditions of Corollary 1 using the standard definition of relative degree we get

$$L_g\gamma = 0$$
$$L_g L_f\gamma = v\cos x_3 \frac{\partial\gamma}{\partial x_2} - v\sin x_3 \frac{\partial\gamma}{\partial x_1}.$$

We thus fail to achieve the desired relative degree of 2 at any $x \in \mathbb{R}^3$ such that

$$x_3 = \arctan\left(\frac{\partial\gamma}{\partial x_2} \Big/ \frac{\partial\gamma}{\partial x_1}\right), \tag{7}$$

which cannot occur $\forall x \in \Gamma^*$ (see Figure 2). ■

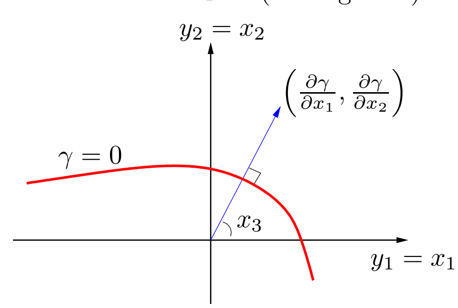


Fig. 2. Geometric interpretation of condition (7).

Figure 3 depicts a few phase curves of the unicycle approaching and following a unit circle. The controller is designed using $\alpha = x_1^2 + x_2^2 - 1$.

### *5.2 Rear-wheel driving car-like robot*

Consider the kinematic model of a rear-wheel drive car-like robot with fixed translational speed $v \neq 0$ and steering angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$\dot{x} = \begin{bmatrix} v\cos x_3 \\ v\sin x_3 \\ \dfrac{v}{\ell}\tan x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{8}$$
$$y = \text{col}\,(x_1, x_2)\,.$$

[2] Actually, Theorem 2 *partially* recovers this well-known property of unicycles in that it requires that the curve $\sigma$ has no self-intersections (see Assumption 1(iii)).

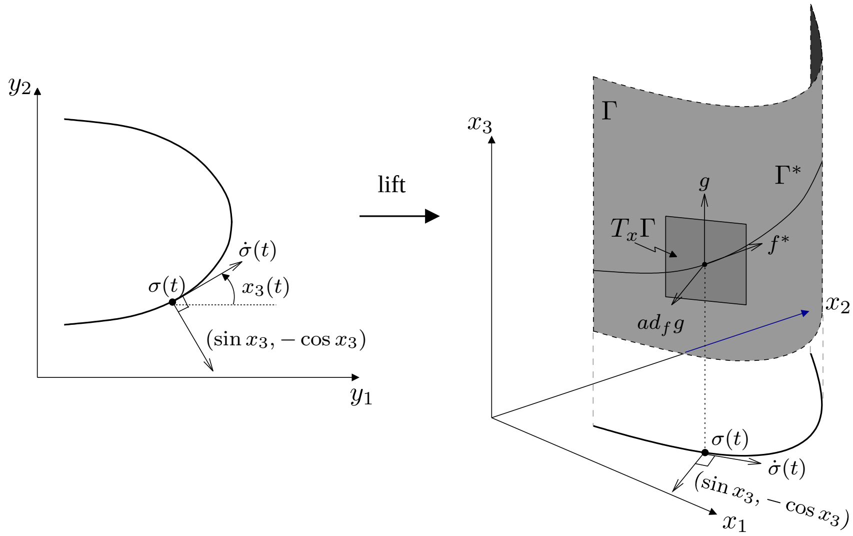


Fig. 1. Maneuver regulation for the unicycle with forward velocity $v = 1$.

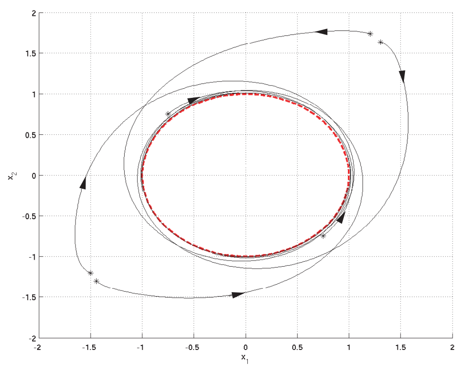

Fig. 3. Unicycle approaching and following the unit circle.

Consider again the problem of maneuvering the output of the car to an arbitrary curve $\sigma(\mathbb{D})$ satisfying Assumption 1 with $r \geq 3$. Following the same reasoning used in Section 5.1, it is easy to verify that Assumptions 2 and 3 are satisfied and

$$\Gamma^* = \Big\{ x \in \mathbb{R}^4 : x = (\sigma(s), \varphi(s), \arctan\left(\frac{\ell}{v}\dot\varphi(s)\right), s \in \mathbb{D} \Big\},$$

where $\varphi(s) = \text{arctan2}\,(\dot\sigma(s))$ is the angle of $\dot\sigma(s)$ with respect to the positive $y_1$ axis.

In checking the conditions of Theorem 2, we require that

$$T_x\Gamma^* + \text{span}\,\{g, ad_f g, ad_f^2 g\} = \mathbb{R}^4,$$

where

$$ad_f g = -\frac{v}{\ell}\left(\sec^2 x_4\right)\frac{\partial}{\partial x_3}$$

$$ad_f^2 g = -\frac{v^2}{\ell}\sin x_3 \left(\sec^2 x_4\right)\frac{\partial}{\partial x_1} + \frac{v^2}{\ell}\cos x_3 \left(\sec^2 x_4\right)\frac{\partial}{\partial x_2}.$$

The condition above is satisfied provided $x_4 \neq \pm\frac{\pi}{2} + 2k\pi$ which does not occur on $\Gamma^*$. Finally, since $[g, ad_f g] \in \text{span}\,\{g, ad_f g\}$, the conditions of Theorem 2 are satisfied and therefore we conclude that Problem 1 has a solution. As in the unicycle example, we now show that the path constraint defining $\Gamma$ can always be used to define the function $\alpha$ in Theorem 1.

*Lemma 4.* If the curve $\sigma$ in (8) satisfies Assumption 1 with $r \geq 3$ then Corollary 1 applies to the car system (8).

*Proof :* Similar to the proof of Lemma 3.

The points at which we fail to achieve the required relative degree of $n - n^* = 3$ have the same geometric interpretation as in the unicycle example (i.e., a singularity occurs when the car is perpendicular to the curve). Let us now apply this result to a more specific and useful application example. We attempt to make the car follow an approximation of the Toronto Indy race track, generated by a $4^{th}$ order spline. Splines are useful in path generation since they can be utilized to model arbitrary paths to any degree of accuracy. An $n^{th}$ order spline is class $C^{n-1}$. Here, we use $4^{th}$ order splines to obtain $C^3$ curves. Let $K$ represent an ordered collection of knots and let $I$ represent an index set of all piecewise polynomials. Introduce the following notation to represent the spline we wish to follow, which may have an arbitrary number of knots

$$\sigma : \lambda \in \mathbb{R} \mapsto \Bigg(\lambda, \sum_{i\in I}\sum_{j=0}^{4}(a_j^i(\lambda - K_i)^j) (1(\lambda - K_i) - 1(\lambda - K_{i+1}))\Bigg) \tag{9}$$

where $1(t)$ is the unit step function. We enforce that the spline be the graph of a function in $\mathbb{R}^2$. This guarantees injectivity and properness and therefore this class of curves satisfy Assumption 1.

Let $\sigma_i$ represent a single segment of the spline. Then, Assumption 2 is also satisfied since

$$\sigma(\mathbb{D}) = \Big\{ y \in \mathbb{R}^2 : y_2 - \sum_{i \in I} \sum_{j=0}^{4} (a_j^i (y_1 - K_i)^j) \\ (1(y_1 - K_i) - 1(y_1 - K_{i+1})) = 0 \Big\}.$$

By Lemma 4, in each knot interval we can use the sole constraint defining $\Gamma$, $\alpha_i(x) = x_2 - \sum_{j=1}^{4} a_j^i (x_1 - K_i)^j$, and apply Corollary 1 Define a coordinate transformation in each knot interval $(K_i, K_{i+1})$,

$$T_i : x \mapsto \mathrm{col}(\varphi_i(x), \alpha_i(x), L_f \alpha_i(x), L_f^2 \alpha_i(x))$$

yielding the normal form (4). The smooth linearizing feedback solving the maneuver regulation problem is

$$u_i = \frac{-L_f^3 \alpha_i - k_1 \alpha_i - k_2 L_f \alpha_i - k_3 L_f^2 \alpha_i}{L_g L_f \alpha_i}$$

with $k_1, k_2, k_3 > 0$. In order to traverse the entire path, the controller must switch from $u_i$ to $u_{i+1}$ when $x_1$ switches from knot interval $(K_i, K_{i+1})$ to $(K_{i+1}, K_{i+2})$. The controller $u_i$ is a function of the path constraint and its first 3 derivatives. Thus, in order for the *switching* to be bumpless, we require a $4^{th}$ order or higher spline. Figure 4 shows the car following a $4^{th}$ order spline consisting of 38 knots for various initial conditions.

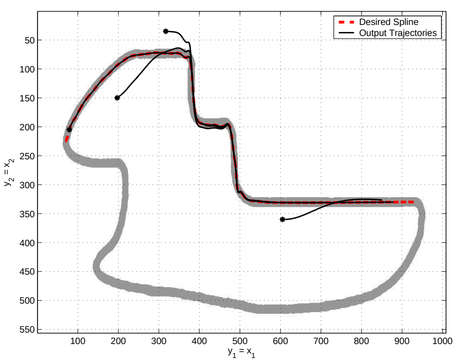


Fig. 4. The car following the Toronto Indy race track.

### 5.3 1-trailer Systems

Consider the kinematic model of a 1-trailer system with a unicycle as the tractor or lead vehicle, see Figure 5.

$$\dot{x} = \begin{bmatrix} v \cos x_3 \\ v \sin x_3 \\ 0 \\ \dfrac{v}{L} \sin(x_3 - x_4) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u \qquad (10)$$
$$y = \mathrm{col}\,(x_1 - L \cos x_4, x_2 - L \sin x_4)\,.$$

This model is a slight variation of the standard trailer model (see Fliess et al. (1995); Samson

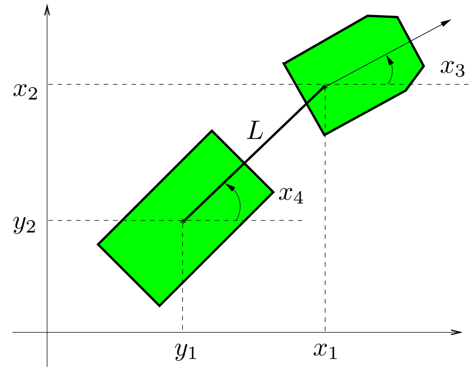


Fig. 5. Kinematic model of the Standard 1-trailer.

(1995)). The subsequent discussion applies to that model as well. Our goal is to force the output of system (10) (i.e. the position of the last trailer) to follow a sinusoid $\sigma : \lambda \mapsto \mathrm{col}\left(\lambda, 100 \cos\left(\frac{\lambda}{100}\right)\right)$. The curve $\sigma$ satisfies Assumption 1 and the first part of Assumption 2 with $\sigma(\mathbb{D}) = \Big\{ y \in \mathbb{R}^2 : y_2 - 100 \cos\left(\frac{y_1}{100}\right) = 0 \Big\}$. The lift

$$\Gamma = (\gamma \circ h)^{-1}(0) = \Big\{ x \in \mathbb{R}^4 : x_2 - L \sin x_4 - \\ 100 \cos\left(\frac{1}{100}(x_1 - L \cos x_4)\right) = 0 \Big\} \qquad (11)$$

is an embedded submanifold and thus Assumption 2 is satisfied. We require various Lie brackets:

$$ad_f g = (v \sin x_3) \frac{\partial}{\partial x_1} + (-v \cos x_3) \frac{\partial}{\partial x_2} + \\ \left(-\frac{v}{L} \cos(x_3 - x_4)\right) \frac{\partial}{\partial x_4} \qquad (12)$$

$$ad_f^2 g = -\frac{v^2}{L^2} \frac{\partial}{\partial x_4}. \qquad (13)$$

To check if Assumption 3 is satisfied, we must first characterize $\Gamma^*$. It is well-known that the standard n-trailer system is flat Fliess et al. (1995): specifying a path for the final vehicle completely determines the state of the system and the control input. This leads to the strong suspicion that $n^* = 1$ and thus we expect to be able to characterize $\Gamma^*$ as the zero level set of $n - n^* = 3$ functions. We characterize $\Gamma^*$ using the zero dynamics algorithm of Isidori (1995); Isidori and Moog (1988). We initialize the algorithm at any $x^* \in \Gamma^*$. Let $u_0 \in \mathbb{R}$ be such that $f(x^*) + g(x^*) u_0 \in T_{x^*} \Gamma^*$. At the first step of the algorithm we have

$$M_0 = \Gamma = (\gamma \circ h)^{-1}(0) = M_0^c, \;\; H_0(x) := \gamma \circ h(x),$$

where $M_0^c$ denotes the connected component of $M_0$ containing $x^*$. The next step of the algorithm yields

$$M_1 = \Big\{ x \in M_0^c : f(x) \in \mathrm{span}\,\{g\}(x) + T_x M_0^c \Big\}$$

n where $T_x M_0^c = \ker(dH_0)$. Equivalently,

$$M_1 = \Big\{ x \in M_0^c : \langle dH_0, f(x) \rangle + \langle dH_0, g(x) \rangle u = 0 \\ \text{is solvable for } u \Big\}.$$

At this step, $L_g H_0 \equiv 0$ and $L_f H_0 \neq 0$, so letting $H_1 = \mathrm{col}(H_0, L_f H_0)$ we have

$$M_1 = M_1^c = \Big\{ x \in \mathbb{R}^4 : H_1(x) = 0 \Big\}.$$

The next iteration yields

$$M_2 = \Big\{ x \in M_1^c : \langle dH_1, f(x) \rangle + \langle dH_1, g(x) \rangle u = 0 \\ \text{is solvable for } u \Big\}, \tag{14}$$

where $L_g H_1 = \mathrm{col}(0, L_g L_f H_0)$. Due to the complexity of the expression, it is not immediately clear whether $L_g H_1 = 0$ on $M_1^c$. If $L_g L_f H_0$ is nonzero on $M_1^c$ then it would follow that (14) can be solved for $u$ leading to the conclusion that $M_2 = M_1^c$ and the algorithm terminates yielding $n^* = 2$. Otherwise, the algorithm continues. Suspecting that $n^* = 1$, we continue the algorithm under the assumption that $L_g L_f H_0(x) = 0$ on $M_1^c$, later, we verify that this assumption is valid. Let $H_2 = \mathrm{col}(H_1, L_f H_1) = \mathrm{col}(H_1, L_f^2 H_0)$. The next iteration in the algorithm yields

$$M_3 = \Big\{ x \in M_2^c : \langle dH_2, f(x) \rangle + \langle dH_2, g(x) \rangle u = 0 \Big\}.$$

We are now set to show, numerically, the following two facts

(1) $(\forall\, x \in M_1^c \cap U)\ L_g L_f H_0(x) = 0$
(2) $(\forall\, x \in M_2^c \cap U)\ L_g L_f^2 H_0 \neq 0$

(where $U$ is some open set containing $x^*$) which allow us to conclude that the assumption above is in fact correct and that the algorithm terminates with $n^* = 1$ and $\Gamma^* \cap U = H_2^{-1}(0) \cap U$. We do that by numerically generating a uniform orthogonal grid of $M_1^c$ (a two dimensional manifold) and $M_2^c$ (a one dimensional manifold) and checking whether properties (1) and (2) hold at the resulting grid points. To this end, given an $n$-dimensional submanifold $M$ in $\mathbb{R}^m$, expressed as $M = H^{-1}(0)$, where $H = \mathrm{col}(H_1, \ldots, H_{m-n}) : \mathbb{R}^m \to \mathbb{R}^{m-n}$, we introduce the following

Procedure uniform grid

1. Find a basis $\{v_1, \ldots, v_n\}$ of $T_x M = \ker(dH_x)$
2. Apply Gram-Schmidt orthonormalization to get $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$
3. Let $G = \sum_{i=1}^{m-n} H_i^2$ and let

$$\hat{v}_i = \begin{cases} \tilde{v}_i - \mu \dfrac{\nabla G}{\|\nabla G\|} & \text{if } \|\nabla G\| \geq \epsilon \\ \tilde{v}_i - \mu \dfrac{\nabla G}{\epsilon} & \text{if } \|\nabla G\| < \epsilon. \end{cases}$$

   where $\mu,\ \epsilon > 0$.
4. Choose $x^0$ near $M$ and numerically integrate the $\hat{v}_i$ to generate a grid.

The procedure above uses a continuous approximation to the gradient vector field $\nabla G$ to make $M$ attractive. The parameter $\mu$ controls the speed of convergence, here we set $\mu = 1$. The value of $\epsilon$ should be significantly larger than the integration tolerance used in step 4. We use a tolerance of $10^{-12}$ and $\epsilon = 10^{-3}$. Figure 6 illustrates the concept of gridding for the case $\dim M = 2$. The

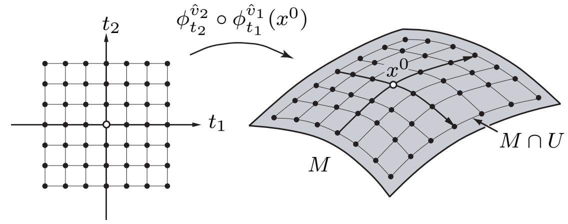


Fig. 6. Gridding a two dimensional manifold.

map generated by the gridding algorithm is isometric, i.e., unit time intervals are mapped to unit length intervals in the grid. The vector fields $\hat{v}_i$ are continuous and the manifold $M$ is attractive for each of them. This is useful for numerical stability of the procedure: if the algorithm is initialized at a point $x^0$ which is not exactly on $M$, the procedure generates grid points that get closer and closer to $M$. Finally, the flows associated to the vector fields $\hat{v}_i$ commute, i.e., starting from a point $x \in M$, $\phi_{t_j}^{\hat{v}_j} \circ \phi_{t_i}^{\hat{v}_i}(x) = \phi_{t_i}^{\hat{v}_i} \circ \phi_{t_j}^{\hat{v}_j}(x)$.

We apply the procedure above to $M_1^c$ (choosing $x^0 = x^*$) to determine if $L_g L_f H_0 = 0$ at each point of the grid. The result, shown in Figure 7, is that $|L_g L_f H_0| < 10^{-11}$ and validates our conjecture (1). Next, we apply the gridding algorithm

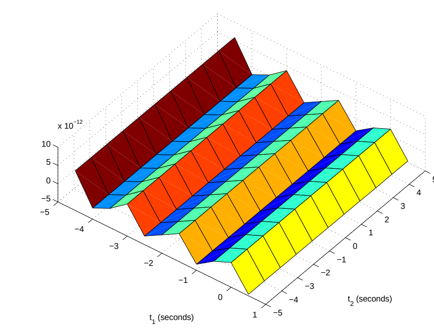


Fig. 7. The value of $L_g L_f H_0$ over a uniform 2-dimensional grid of $M_1^c$.

to $M_2^c$ (choosing $x^0 = x^*$). Figure 8 shows that $L_g L_f^2 H_0 \neq 0$ on $M_2^c$ thus validating conjecture (2). We have thus numerically shown that $n^* = 1$, as expected, and $\Gamma^* = M_2^c$ near $x^*$. In particular, we have seen that $L_g H_0 \equiv 0$, $L_g L_f H_0 = 0$ on $\Gamma^*$, and $L_g L_f^2 H_0 \neq 0$ on $\Gamma^*$. It can be shown that $L_g L_f H_0$ changes sign in *any* neighborhood of $\Gamma^*$, implying that $H_0(x) = \gamma \circ h(x)$ does not yield a well-defined relative degree and thus Corollary 1 does not apply. Actually, using symbolic mathematics software by Kugi et al. (2003)), one finds that $\mathrm{inv}(D) = \mathrm{inv}(\mathrm{span}\{g, ad_f g, ad_f^2 g\}) = \mathbb{R}^4$,

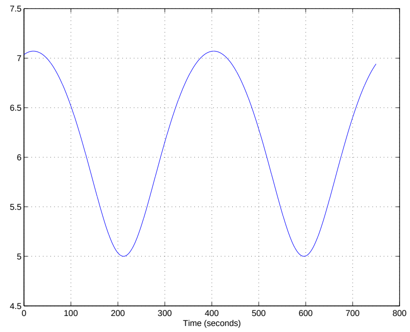


Fig. 8. The value of $L_g L_f^2 h$ on $M_2^c$.

and thus, by Corollary 2, the 1-trailer system is *not* transversely feedback linearizable. However, the discussion above immediately gives that $\gamma \circ h$ yields a robust relative degree 3 on $\Gamma^*$, and thus one can perform approximate input-output linearization on the transverse dynamics of system (10) (Hauser et al. (1992)). We say that the system is approximately transversely feedback linearizable. Figure 9 shows various simulation results for initial conditions on and off the desired sinusoid. The trailer system follows the path for initial conditions sufficiently close to $\Gamma^*$.

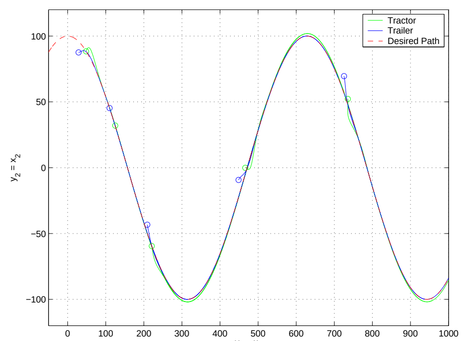


Fig. 9. Approximate transverse feedback linearization of the trailer system

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