When is a Lagrangian Control System With Virtual Holonomic Constraints Lagrangian?

Alireza Mohammadi *,[1], **Manfredi Maggiore** *,[1], **Luca Consolini** **,

* *Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Road, Toronto, ON, M5S 3G4, Canada* (alireza.mohammadi@mail.utoronto.ca, maggiore@control.utoronto.ca)
** *Dipartimento di Ingegneria dell'Informazione, Via Usberti 181/a, 43124 Parma, Italy* (lucac@ce.unipr.it)

Abstract: This paper investigates a class of Lagrangian control systems with n degrees-of-freedom (DOF) and $n-1$ actuators, assuming that $n-1$ virtual holonomic constraints have been enforced via feedback, and a basic regularity condition holds. The reduced dynamics of such systems are described by a second-order unforced differential equation. We present necessary and sufficient conditions under which the reduced dynamics are those of a mechanical system with one DOF and, more generally, under which they have a Lagrangian structure.

1. INTRODUCTION

A virtual holonomic constraint (VHC) is a relation of the form $h(q) = 0$ that can be made invariant via feedback control. The early idea of VHCs appeared in Nakanishi et al. [2000] where the authors enforced the angle of a virtual pendulum on the configuration of a brachiating robot in order to follow the target dynamics of a harmonic oscillator and to imitate the pendulum-like motion of an ape's brachiation. In the past decade, VHCs have emerged as a useful tool for motion control in biped robots (see, e.g., Plestan et al. [2003], Westervelt et al. [2003], Chevallereau et al. [2008]), and as an approach to motion planning for general robotic systems (e.g., Shiriaev et al. [2005, 2006, 2010], Freidovich et al. [2008]). In the context of motion control of biped robots, researchers encode a walking gait by imposing, through feedback control, relations between the joint angles of the robot, and they show that when the relations are satisfied, the reduced motion arising is a stable limit cycle corresponding to a periodic walking motion (see, e.g., Plestan et al. [2003], Westervelt et al. [2003]). In the context of motion planning, researchers use VHCs to aid the selection of closed orbits corresponding to desired repetitive behaviors, which can then be stabilized in a variety of ways (as in, e.g., Shiriaev et al. [2005, 2006]).

A fundamental question that naturally arises in the context of virtual holonomic constraints is whether the reduced dynamics induced by a given VHC are Lagrangian. There are several appealing reasons for posing such a question. First, if the reduced dynamics are Lagrangian, then periodic motion planning and control of oscillations are feasible (see Shiriaev et al. [2005]). Second, one can leverage the structural properties of Lagrangian systems to understand the qualitative properties of the reduced dynamics. Finally, this question has an intrinsic theoretical value as it fits within the framework of a classical problem of mathematical physics known as the *inverse problem of Lagrangian mechanics*, posed first by Helmholtz [1887]. This problem asks if a given set of differential equations represents a Lagrangian system. Comprehensive historical surveys regarding this problem can be found in Santilli [1978], Tonti [1985], Krupková and Prince [2008]. See Saunders [2010] for a survey on recent findings. The focus of the literature in this area is on finding conditions for the existence of a local Lagrangian function for differential equations of arbitrary even order. In the context of virtual holonomic constraints, the inverse problem of Lagrangian mechanics takes on a special significance, since it was shown in Maggiore and Consolini [2013], Consolini and Maggiore [2012] that virtual holonomic constraints may induce reduced dynamics that *fail* to be Lagrangian. Specifically, when enforcing $n-1$ virtual holonomic constraints on a Lagrangian control system with n DOF and $n-1$ actuators, the reduced dynamics are described by a second-order unforced differential equation whose state space is either a plane or a cylinder. In the former case, the reduced dynamics are *always* Lagrangian, while in the latter case they may fail to be such (see Maggiore and Consolini [2013], Consolini and Maggiore [2012]). The work in Maggiore and Consolini [2013], Jankuloski et al. [2012] provided sufficient conditions for the existence of a Lagrangian structure, but a number of open problems remain. To begin with, the work in Maggiore and Consolini [2013], Jankuloski et al. [2012] investigated the existence of a Lagrangian function in the special form of kinetic minus potential energy. The existence of more general Lagrangian structures was not investigated. Further, necessary and sufficient conditions for the existence of a Lagrangian structure are not available to date. Finally, while the work in Maggiore and Consolini [2013], Consolini and Maggiore

[1] A. Mohammadi was supported in part by the Ontario Graduate Scholarship (OGS) and by the Natural Sciences and Engineering Research Council of Canada (NSERC). M. Maggiore was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

[2012] presented counter-examples of non-Lagrangian reduced dynamics, it is not clear whether the existence of a Lagrangian structure is "typical" or "exceptional" for the reduced dynamics induced by a VHC.

Contributions of the paper. In this paper we consider Lagrangian control systems with n DOF and $n-1$ actuators. As an application example of this class of systems, one can mention the single support phase of walking in biped robot locomotion. We assume that a regular VHC, $h(q) = 0$, of order $n-1$ (the definition will be given in Section 2) has been enforced via feedback control, and we investigate the resulting reduced dynamics. These are given by a second-order unforced differential equation of the form

$$\ddot{s} = \Psi_1(s) + \Psi_2(s)\dot{s}^2, \tag{1}$$

where the state space is either $\mathbb{R}^2$ or the cylinder $\mathbb{S}^1 \times \mathbb{R}$. In Section 3, we present necessary and sufficient conditions under which (1) admits a global mechanical structure, i.e., it results from the Euler-Lagrange equation with a Lagrangian function of the form $L(s, \dot{s}) = (1/2)M(s)\dot{s}^2 - V(s)$, with $M \neq 0$. We then present necessary and sufficient conditions under which (1) admits *any* global Lagrangian structure, i.e., all solutions of (1) satisfy the Euler-Lagrange equation for an appropriate Lagrangian function $L(s, \dot{s})$. An outcome of our results is that the existence of a global Lagrangian structure for systems of the form (1) is *exceptional* when the state space is $\mathbb{S}^1 \times \mathbb{R}$. We remark that while the literature on the inverse problem of Lagrangian mechanics focuses on the existence of a local Lagrangian, the problem investigated in this paper is of a global nature, but it only concerns the second-order system (1).

Notation. We let $\mathbf{n} := \{1, \ldots, n\}$, and given $x \in \mathbb{R}^n$, we denote $\|x\| := (x^\top x)^{1/2}$. Given $x \in \mathbb{R}$ and $T > 0$, then $[x]_T := x$ modulo T. The set of real numbers modulo T is denoted by $[\mathbb{R}]_T$. Therefore, $[\mathbb{R}]_T = \{[x]_T : x \in \mathbb{R}\}$. The set $[\mathbb{R}]_T$ can be given the structure of a smooth manifold diffeomorphic to the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ through the map $[x]_T \mapsto \exp(i2(\pi/T)[x]_T)$. Given a function $h : \mathcal{Q} \to \mathbb{R}^k$, we define $h^{-1}(0) := \{q \in \mathcal{Q} : h(q) = 0\}$. Given a smooth manifold $\mathcal{Q}$, we denote by $T\mathcal{Q}$ its tangent bundle, $T\mathcal{Q} := \{(p, v_p) : p \in \mathcal{Q}, v_p \in T_p\mathcal{Q}\}$. If $h : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a smooth map between manifolds, and $p \in \mathcal{Q}_1$, $dh_p : T_p\mathcal{Q}_1 \to T_{h(p)}\mathcal{Q}_2$ denotes the differential of h at p, while $dh : T\mathcal{Q}_1 \to T\mathcal{Q}_2$ denotes the global differential of h, defined as $dh : (p, v_p) \mapsto (h(p), dh_p(v_p))$. If $h : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a diffeomorphism, then we say that $\mathcal{Q}_1, \mathcal{Q}_2$ are diffeomorphic, and we write $\mathcal{Q}_1 \simeq \mathcal{Q}_2$. In this case, the global differential $dh : T\mathcal{Q}_1 \to T\mathcal{Q}_2$ is a diffeomorphism as well (see [Lee, 2013, Corollary 3.22]).

2. PRELIMINARIES ON VIRTUAL HOLONOMIC CONSTRAINTS

In this section we review basic material taken from Maggiore and Consolini [2013]. Consider a Lagrangian control system with n DOF and $n-1$ actuators modelled as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = B(q)\tau.$$

In the above, $q = (q_1, \ldots, q_n) \in \mathcal{Q}$ is the configuration vector. We assume that each component q_i, $i \in \mathbf{n}$, is either a linear displacement in $\mathbb{R}$, or an angular displacement in $[\mathbb{R}]_{T_i}$, for some $T_i > 0$ (typically, $T_i = 2\pi$). With this assumption, the configuration manifold $\mathcal{Q}$ is a generalized cylinder, and $T\mathcal{Q}$ is the Cartesian product $T\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^n$. The term $B(q)\tau$ represents external forces produced by the control vector $\tau \in \mathbb{R}^{n-1}$. We assume that $B : \mathcal{Q} \to \mathbb{R}^{n\times(n-1)}$ is smooth and $\operatorname{rank} B(q) = n-1$ for all $q \in \mathcal{Q}$. Further, the function $\mathcal{L} : T\mathcal{Q} \to \mathbb{R}$ is assumed to be smooth and to have the special form $\mathcal{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$, where $D(q)$, the generalized mass matrix, is symmetric and positive definite for all $q \in \mathcal{Q}$. We will assume that there exists a left annihilator of B on Q. That is to say, there exists a smooth function $B^\perp : \mathcal{Q} \to \mathbb{R}^{1\times n}$ which does not vanish and is such that $B^\perp(q)B(q) = 0$ on $\mathcal{Q}$. With the above mentioned assumptions, the Lagrangian control system takes on the following standard form

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla P(q) = B(q)\tau. \tag{2}$$

Definition 2.1. (Maggiore and Consolini [2013]) A **virtual holonomic constraint (VHC)** of order $n-1$ for system (2) is a relation $h(q) = 0$, where $h : \mathcal{Q} \to \mathbb{R}^{n-1}$ is a smooth function which has a regular value at 0, i.e., $\operatorname{rank}(dh_q) = n-1$ for all $q \in h^{-1}(0)$, and is such that the set

$$\Gamma = \{(q, \dot{q}) : h(q) = 0, \, dh_q\dot{q} = 0\} \tag{3}$$

is controlled invariant. That is to say, there exists a smooth feedback $\tau : \Gamma \to \mathbb{R}^{n-1}$ such that Γ is positively invariant for the closed-loop system. The set Γ is called the **constraint manifold** associated with $h(q) = 0$. A VHC is said to be **stabilizable** if there exists a smooth feedback $\tau(q, \dot{q})$ that asymptotically stabilizes Γ. Such a stabilizing feedback is said to **enforce the VHC** $h(q) = 0$. △

Definition 2.2. (Maggiore and Consolini [2013]) A relation $h(q) = 0$, where $h : \mathcal{Q} \to \mathbb{R}^{n-1}$ is a smooth function, is a **regular VHC** of order $n-1$ for (2) if system (2) with output function $e = h(q)$ has well-defined vector relative degree $\{2, \ldots, 2\}$ everywhere on the constraint manifold given in (3). △

A regular VHC is a VHC. Indeed, the condition that the output function $e = h(q)$ has vector relative degree $\{2, \ldots, 2\}$ implies (see Isidori [1995]) that $\operatorname{rank}(dh_q) = n-1$ for all $q \in h^{-1}(0)$. Moreover, the zero dynamics manifold exists and it coincides with Γ, implying that Γ is controlled invariant. Regular VHCs enjoy two important properties. First, under mild assumptions (see Maggiore and Consolini [2013]), regular VHCs are stabilizable by input-output feedback linearizing feedback. The second useful property of regular VHCs is that they induce well-defined reduced dynamics. Specifically, the dynamics on Γ (i.e., the zero dynamics associated with the output $e = h(q)$) are given by a second-order unforced system. In order to find the reduced dynamics, we follow a procedure presented in Jankuloski et al. [2012]. We first pick a regular parametrization $\sigma : \Theta \to \mathcal{Q}$ of the curve $h^{-1}(0)$, where $\Theta = \mathbb{R}$ if $h^{-1}(0) \simeq \mathbb{R}$, while $\Theta = [\mathbb{R}]_T$, $T > 0$, if $h^{-1}(0) \simeq \mathbb{S}^1$. The map $\sigma : \Theta \to \sigma(\Theta) = h^{-1}(0)$ is a diffeomorphism. Therefore, the global differential $d\sigma : T\Theta \to Th^{-1}(0)$, $(s, \dot{s}) \mapsto (\sigma(s), \sigma'(s)\dot{s})$ is a diffeomorphism as well. Since, $Th^{-1}(0) = \Gamma$, we conclude that $T\Theta \simeq \Gamma$. Next, multiplying (2) on the left by $B^\perp(q)$ we obtain $B^\perp D\ddot{q} + B^\perp(C\dot{q} + \nabla P) = 0$. The dynamics on Γ are found by restricting the equation above on Γ. To this end, we use the fact that $d\sigma : T\Theta \to \Gamma$ is a diffeomorphism, and we let $q = \sigma(s)$, $\dot{q} = \sigma'(s)\dot{s}$, and $\ddot{q} = \sigma'(s)\ddot{s} + \sigma''(s)\dot{s}^2$. By so doing, we obtain

$$\ddot{s} = \Psi_1(s) + \Psi_2(s)\dot{s}^2, \tag{4}$$

where $\Psi_1(s) = -\left.\frac{B^\perp \nabla P}{B^\perp D\sigma'}\right|_{q=\sigma(s)}$,

$$\Psi_2(s) = -\left.\frac{B^\perp D\sigma'' + \sum_{i=1}^n B_i^\perp \sigma'^\top Q_i \sigma'}{B^\perp D\sigma'}\right|_{q=\sigma(s)},$$

and where $B_i^\perp$ is the i^{th} component of $B^\perp$ and $(Q_i)_{jk} = 1/2(\partial_{q_k} D_{ij} + \partial_{q_j} D_{ik} - \partial_{q_i} D_{kj})$. The unforced autonomous system (4) represents the reduced dynamics of system (2) when the regular VHC of order $n-1$, $h(q) = 0$, is enforced. The state space of (4) is $T\Theta = \Theta \times \mathbb{R}$ which, as we have seen, is diffeomorphic to Γ. The set $T\Theta$ is a plane if $h^{-1}(0) \simeq \mathbb{R}$, and a cylinder if $h^{-1}(0)$ is a Jordan curve.

3. MAIN RESULTS

In this section we formulate and solve the main problem investigated in this paper for a two-dimensional system of the form (4), with state space $\mathcal{X} = T\Theta$, with $\Theta = \mathbb{R}$ or $[\mathbb{R}]_T$, $T > 0$. The functions $\Psi_i : \Theta \to \mathbb{R}$, $i = 1, 2$, are assumed to be smooth. We begin by defining precisely the Lagrangian structures under consideration.

Definition 3.1. System (4) is said to be:

(a) **Euler-Lagrange (EL) with Lagrangian L** if there exists a smooth Lagrangian function $L : \mathcal{X} \to \mathbb{R}$ such that the following two properties hold:
  (i) The Lagrangian L is nondegenerate, i.e., $\partial^2 L/\partial \dot{s}^2 > 0$ for all $(s, \dot{s}) \in \mathcal{X}$.
  (ii) All solutions $(s(t), \dot{s}(t))$ of (4) satisfy the Euler-Lagrange equation
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}}(s(t), \dot{s}(t)) - \frac{\partial L}{\partial s}(s(t), \dot{s}(t)) = 0 \tag{5}$$
  for all t in their maximal interval of definition.

(b) **Mechanical** if it is EL with Lagrangian $L(s, \dot{s}) = (1/2)M(s)\dot{s}^2 - V(s)$, where $M : \Theta \to (0, \infty)$, $V : \Theta \to \mathbb{R}$ are smooth.

(c) **Singular Euler-Lagrange (SEL) with Lagrangian L** if there exists a smooth Lagrangian function $L : \mathcal{X} \to \mathbb{R}$ such that $\partial^2 L/\partial \dot{s}^2$ is not identically zero and property (ii) of part (a) holds. Moreover, if L is any function satisfying the latter two properties, then
  (i)′ L is degenerate, i.e., $\partial^2 L/\partial \dot{s}^2$ has zeros.

△

Remark 3.2. EL systems with Lagrangian L are Hamiltonian with Hamiltonian function given by the Legendre transform of L (see, e.g., Arnold [1989]). On the other hand, while SEL systems have a Lagrangian structure, they are generally not Hamiltonian because the Legendre transform of L may not be well-defined. Moreover, SEL systems are not mechanical since, by definition, $\partial^2 L/\partial \dot{s}^2 = M(s) > 0$ for a mechanical system. If L is the Lagrangian of an EL system of the form (4), the Euler-Lagrange equation (5) defines a smooth vector field on $\mathcal{X}$ which coincides with (4). Indeed, requirement (i) in Definition 3.1(a) ensures that the coefficient of $\ddot{s}$ in (5) is not zero, and therefore (5) defines a smooth vector field on $\mathcal{X}$. Moreover, by uniqueness of solutions of (4) and requirement (ii) in Definition 3.1(a), the local phase flow of this vector field must coincide with the local phase flow of (4). Hence, the vector field arising from (5) must coincide with (4). On the other hand, for a SEL system (see Remark 5.4), the Euler-Lagrange equation (5) gives rise to the equation $\alpha(s, \dot{s})\left[\ddot{s} - \Psi_1(s) - \Psi_2(s)\dot{s}^2\right] = 0$, where α is a smooth function with zeros. It follows from this identity that the Euler-Lagrange equation does not give rise to a well-defined vector field on $\mathcal{X}$, and the collection of its solutions $(s(t), \dot{s}(t))$ contains, but is not equal to the collection of solutions of (4). We will illustrate this fact with an example in Section 6. Finally, we remark that the requirement, in Definition 3.1(c), that $\partial^2 L/\partial \dot{s}^2$ is not identically zero guarantees that the Euler-Lagrange equation (5) gives rise to a second-order differential equation. △

Inverse Lagrangian Problem (ILP). *Find necessary and sufficient conditions under which system* (4) *is, respectively, EL, mechanical, or SEL.*

In order to present the solution of ILP, we let $\tilde{\Psi}_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, be defined as $\tilde{\Psi}_i(x) := \Psi_i([x]_T)$, and we define the **virtual mass** $\tilde{M} : \mathbb{R} \to (0, \infty)$ and **virtual potential** $\tilde{V} : \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} \tilde{M}(x) &= \exp\Big(-2\int_0^x \tilde{\Psi}_2(\tau)\,d\tau\Big), \\ \tilde{V}(x) &= -\int_0^x \tilde{\Psi}_1(\tau)\tilde{M}(\tau)\,d\tau. \end{aligned} \tag{6}$$

We now present the main results of this paper.

Theorem 3.3. (**Solution to ILP - Part 1).** If $\Theta = \mathbb{R}$, then system (4) with state space $\mathcal{X} = T\Theta$ is mechanical, with $M = \tilde{M}$ and $V = \tilde{V}$, where $\tilde{M}, \tilde{V}$ are defined in (6) with $\tilde{\Psi}_i(x) = \Psi_i(x)$.

Proof. By inspection, the Euler-Lagrange equation with Lagrangian $L(s, \dot{s}) = (1/2)M(s)\dot{s}^2 - V(s)$ produces equation (4).

Remark 3.4. When system (4) is mechanical and $\Theta = \mathbb{R}$, the total energy $E_0(s, \dot{s}) = (1/2)\tilde{M}(s)\dot{s}^2 + \tilde{V}(s)$ is a first integral of the system, i.e., it is constant along all solutions. Shiriaev et al. [2005, 2006] present an "integral of motion" for (4) which does not coincide with E_0, but it relies on the functions $\tilde{M}(s)$ and $\tilde{V}(s)$ in (6). The "integral of motion" in Shiriaev et al. [2005, 2006] is not a first integral because it is only constant along one solution of the system. △

Theorem 3.5. (**Solution to ILP - Part 2).** If $\Theta = [\mathbb{R}]_T$, then the following statements about system (4) with state space $\mathcal{X} = T\Theta$ are equivalent:
 (i) System (4) is EL.
 (ii) System (4) is mechanical.
 (iii) The functions $\tilde{M}$ and $\tilde{V}$ in (6) are T-periodic.

Moreover, if (4) is EL, then the Lagrangian function $L : T[\mathbb{R}]_T \to \mathbb{R}$ is given by $L(s, \dot{s}) = (1/2)M(s)\dot{s}^2 - V(s)$, where $M : [\mathbb{R}]_T \to (0, \infty)$ and $V : [\mathbb{R}]_T \to \mathbb{R}$ are the unique smooth functions such that $\tilde{M} = M \circ \pi$ and $\tilde{V} = V \circ \pi$, where $\pi : \mathbb{R} \to [\mathbb{R}]_T$ is defined as $\pi(x) = [x]_T$.

Remark 3.6. The sufficiency part of the theorem was proved in Maggiore and Consolini [2013], Jankuloski et al. [2012], but we present it in Section 5 for completeness. △

Theorem 3.7. (**Solution to ILP - Part 3).** If $\Theta = [\mathbb{R}]_T$, then the following statements about system (4) with state space $\mathcal{X} = T\Theta$ are equivalent:
 (i) System (4) is SEL.
 (ii) The function $\tilde{M}$ is T-periodic, while $\tilde{V}$ is not T-periodic.

Moreover, if (4) is SEL, then the Lagrangian function $L : T[\mathbb{R}]_T \to \mathbb{R}$ is the unique smooth function such that $L(\pi(x), \dot{x}) = \tilde{L}(x, \dot{x})$ for all $(x, \dot{x}) \in \mathbb{R} \times \mathbb{R}$, where π is defined in Theorem 3.5,

$$\tilde{L}(x,\dot{x}) = -\sin(2\pi f_0 \tilde{E}_0(x,\dot{x})) + \sqrt{2 f_0 \tilde{M}(x)}\,\pi\dot{x}\times \Bigg[\cos(2\pi f_0 \tilde{V}(x))\,\mathbf{C}\Big(\sqrt{2f_0\tilde{M}(x)}\,\dot{x}\Big) - \sin(2\pi f_0 \tilde{V}(x))\,\mathbf{S}\Big(\sqrt{2f_0\tilde{M}(x)}\,\dot{x}\Big)\Bigg], \tag{7}$$

where $f_0 = 1/\tilde{V}(T)$, $\tilde{E}_0(x,\dot{x}) = (1/2)\tilde{M}(x)\dot{x}^2 + \tilde{V}(x)$, and $\mathbf{C}(\cdot)$, $\mathbf{S}(\cdot)$ are the Fresnel cosine and sine integrals, defined as $\mathbf{C}(x) = \int_0^x \cos(\pi t^2/2)dt$, $\mathbf{S}(x) = \int_0^x \sin(\pi t^2/2)dt$.

Remark 3.8. Theorems 3.5 and 3.7 show that, when $\Theta = [\mathbb{R}]_T$ (which, in the setup presented in Section 2, corresponds to the situation when the VHC $h(q) = 0$ is a Jordan curve) the property of (4) being either EL or SEL is *exceptional,* in that it is not satisfied by a generic system of the form (7) with state space $T\Theta$. Indeed, in order for (4) to be EL or SEL it is required at a minimum that $\tilde{M}(x)$ be T-periodic, which corresponds to requiring that the T-periodic function $\tilde{\Psi}_2 : \mathbb{R} \to \mathbb{R}$ has zero average. In other words, the set $\{\tilde{\Psi}_2 : \mathbb{R} \to \mathbb{R} | \int_0^T \tilde{\Psi}_2(\tau)d\tau = 0\}$ has "measure zero" in the set of all smooth T-periodic and real-valued functions defined on the real line. △

In the next two sections we prove Theorems 3.5 and 3.7 assuming that $\Theta = [\mathbb{R}]_T$. We now provide an outline of the arguments that follow.

Outline of proofs of Theorems 3.5 and 3.7.

Step 1. In Section 4, we define a lifted system, $\ddot{x} = \tilde{\Psi}_1(x) + \tilde{\Psi}_2(x)\dot{x}^2$, with state space $\mathbb{R}^2$. In Lemma 4.1, we show that trajectories of the lifted system are related to trajectories of system (4) through the map $d\pi$, where $\pi(x) = [x]_T$.

Step 2. In Lemma 4.2, we show that solutions of the Euler-Lagrange equation (5) are related through the map $d\pi$ to solutions of the Euler-Lagrange equation with Lagrangian $\tilde{L} = L \circ d\pi$.

Step 3. Leveraging Lemmas 4.1 and 4.2, in Proposition 4.3 we show that (4) is EL or SEL if and only if the lifted system is EL or SEL with a Lagrangian $\tilde{L}(x,\dot{x})$ which is T-periodic with respect to x.

Step 4. In Section 5, we find necessary and sufficient conditions for the existence of a Lagrangian $\tilde{L}$ for the lifted system which enjoys the periodicity property of Proposition 4.3. In Proposition 5.1 we show that in order for a function $\tilde{L}(x,\dot{x})$ which is nondegenerate and T-periodic with respect to x to be a Lagrangian for the lifted system, it is necessary and sufficient that $\tilde{M}$ and $\tilde{V}$ in (6) are T-periodic. This result proves Theorem 3.5.

Step 5. In Lemma 5.2, we find expressions for $\tilde{M}(x+nT)$, $\tilde{V}(x+nT)$, $n \in \mathbb{Z}$.

Step 6. Using Lemma 5.2, in Proposition 5.3, we prove that the lifted system is SEL with a Lagrangian $\tilde{L}(x,\dot{x})$ which is T-periodic with respect to x if and only if $\tilde{M}$ in (6) is T-periodic, while $\tilde{V}$ isn't. In light of Proposition 4.3, this proves Theorem 3.7.

4. LIFT OF ILP TO $\mathbb{R}^2$

Let $\pi : \mathbb{R} \to [\mathbb{R}]_T$ be defined as $\pi(x) = [x]_T$, and let $\bar{\pi} : T\mathbb{R} \to T[\mathbb{R}]_T$ denote the global differential of π, $\bar{\pi} := d\pi$, so that $\bar{\pi}(x,\dot{x}) = ([x]_T, d\pi_x\dot{x}) = ([x]_T,\dot{x})$. Given two functions $f : [\mathbb{R}]_T \to \mathbb{R}$ and $F : T[\mathbb{R}]_T \to \mathbb{R}$, we define their **lifts** to be functions $\tilde{f} := f \circ \pi : \mathbb{R} \to \mathbb{R}$, and $\tilde{F} := F \circ \bar{\pi} : T\mathbb{R} \to \mathbb{R}$. If $\tilde{L} : T\mathbb{R} \to \mathbb{R}$ is a smooth function, its associated Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}} - \frac{\partial \tilde{L}}{\partial x} = 0. \tag{8}$$

Finally, we define the lift of system (4) as

$$\ddot{x} = \tilde{\Psi}_1(x) + \tilde{\Psi}_2(x)\dot{x}^2, \tag{9}$$

where $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are the lifts of Ψ_1 and Ψ_2, namely, $\tilde{\Psi}_i(x) := \Psi_i([x]_T)$. The state space of the above differential equation is $\tilde{\mathcal{X}} = T\mathbb{R}$. We will apply to system (9) the terminology of Definition 3.1, whereby L will be replaced by $\tilde{L}$. The proofs of technical results are omitted due to space limitations.

Lemma 4.1. The vector field of equation (4) is $\bar{\pi}$-related to the vector field of (9). Therefore, pair $(s(t),\dot{s}(t))$ is a solution of (4) if and only if there exists a solution $(x(t),\dot{x}(t))$ of (9) such that $(s(t),\dot{s}(t)) = \bar{\pi}(x(t),\dot{x}(t))$.

Lemma 4.2. Let $I \subset \mathbb{R}$ be an open interval, and $s : I \to [\mathbb{R}]_T$, $x : I \to \mathbb{R}$ be C^1 signals such that $(s(t),\dot{s}(t)) = \bar{\pi}(x(t),\dot{x}(t))$ for all $t \in I$. Then, the pair $(s(t),\dot{s}(t))$ satisfies the Euler-Lagrange equation (5) with smooth Lagrangian $L : T[\mathbb{R}]_T \to \mathbb{R}$ if and only if the pair $(x(t),\dot{x}(t))$ satisfies the lifted Euler-Lagrange equation (8) with smooth Lagrangian $\tilde{L} = L \circ \bar{\pi}$.

Proposition 4.3. The following statements are equivalent

- (i) System (4) with state space $\mathcal{X} = T[\mathbb{R}]_T$ is EL (resp., SEL) with Lagrangian L.
- (ii) System (9) with state space $\tilde{\mathcal{X}} = T\mathbb{R}$ is EL (resp., SEL) with Lagrangian $\tilde{L} = L \circ \bar{\pi}$.

Proof. Let $\tilde{L} = L \circ \bar{\pi}$. Then, it is easy to see that $(\partial^2\tilde{L}/\partial\dot{x}^2)(x,\dot{x}) = (\partial^2 L/\partial\dot{s}^2)(\bar{\pi}(x,\dot{x}))$. Therefore, L is nondegenerate (respectively, degenerate) if and only if $\tilde{L}$ is nondegenerate (respectively, degenerate). Now, suppose that system (4) is EL (respectively, SEL) with Lagrangian L. Consider an arbitrary solution of (9), namely, $(x(t),\dot{x}(t))$, where $x : I \to \mathbb{R}$ is C^1 and $I \subset \mathbb{R}$ is an open interval. By Lemma 4.1, $(s(t),\dot{s}(t)) := \bar{\pi}(x(t),\dot{x}(t))$ is a solution of (4), and thus satisfies the Euler-Lagrange equation (5). By Lemma 4.2, $(x(t),\dot{x}(t))$ satisfies the Euler-Lagrange equation with Lagrangian $\tilde{L} = L \circ \bar{\pi}$. Since $(x(t),\dot{x}(t))$ is an arbitrary solution of (9), and since $\bar{\pi} : T\mathbb{R} \to T[\mathbb{R}]_T$ is onto, system (9) is EL (respectively, SEL) with Lagrangian $\tilde{L} = L \circ \bar{\pi}$. The proof that if (9) is EL (respectively, SEL) with Lagrangian $\tilde{L} = L \circ \bar{\pi}$, then (4) is EL (respectively, SEL) with Lagrangian L is analogous. We consider an arbitrary solution $(s(t),\dot{s}(t))$ of (4), and we let $(x(t),\dot{x}(t))$ be a solution of (9) such that $(s(t),\dot{s}(t)) = \bar{\pi}(x(t),\dot{x}(t))$. Such a solution exists by Lemma 4.1 and the fact that $\bar{\pi}$ is onto. Thus, $(x(t),\dot{x}(t))$ is a solution of the Euler-Lagrange equation (8) with Lagrangian $\tilde{L} = L \circ \bar{\pi}$. By Lemma 4.2, $(s(t),\dot{s}(t))$ is a solution of the Euler-Lagrange equation (5) with Lagrangian L. Since $(s(t),\dot{s}(t))$ is an arbitrary solution of (4), we conclude that (4) is EL (respectively, SEL).

5. PROOFS OF MAIN RESULTS

By virtue of Proposition 4.3, solving ILP and finding a Lagrangian L for system (4) is equivalent to solving ILP and finding a Lagrangian $\tilde{L}$ for the lifted system (9) such that $\tilde{L} = L \circ \bar{\pi}$, for some smooth $L : T[\mathbb{R}]_T \to \mathbb{R}$. Given a smooth function $\tilde{L} : T\mathbb{R} \to \mathbb{R}$, there exists a smooth function $L : T[\mathbb{R}]_T \to \mathbb{R}$ satisfying $\tilde{L} = L \circ \bar{\pi}$ if and only if $\tilde{L}$ is T-periodic with respect to its first argument, i.e., $\tilde{L}(x+T, \dot{x}) = \tilde{L}(x, \dot{x})$ for all $(x, \dot{x}) \in T\mathbb{R}$. In this section, we leverage this fact to prove Theorems 3.5 and 3.7.

Proposition 5.1. The lifted system (9) is EL with a smooth Lagrangian $\tilde{L} : T\mathbb{R} \to \mathbb{R}$ such that $\tilde{L}(x+T, \dot{x}) = \tilde{L}(x, \dot{x})$ for all $(x, \dot{x}) \in T\mathbb{R}$, if and only if the virtual mass $\tilde{M}$ and virtual potential $\tilde{V}$ in (9) are T-periodic. If this is the case, then system (4) is mechanical with Lagrangian $L = (1/2)M(s)\dot{s}^2 - V(s)$, where M and V are defined through $\tilde{M} = M \circ \pi$, $\tilde{V} = V \circ \pi$.

Proof. ($\Leftarrow$) If $\tilde{M}$, $\tilde{V}$ are T-periodic, then $\tilde{L}(x, \dot{x}) = (1/2)\tilde{M}(x)\dot{x}^2 - \tilde{V}(x)$ is T-periodic with respect to x, and $\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}} - \frac{\partial \tilde{L}}{\partial x} = \tilde{M}(x)\big(\ddot{x} - \tilde{\Psi}_1(x) - \tilde{\Psi}_2(x)\dot{x}^2\big)$. Since $\tilde{M} > 0$, the lifted system is mechanical with Lagrangian $\tilde{L}$.
($\Rightarrow$) Assume that system (9) is EL with smooth Lagrangian $\tilde{L} : T\mathbb{R} \to \mathbb{R}$ such that $\tilde{L}(x+T, \dot{x}) = \tilde{L}(x, \dot{x})$ for all $(x, \dot{x}) \in T\mathbb{R}$. By definition of EL system, $\tilde{L}$ is nondegenerate, i.e., $\partial^2\tilde{L}/\partial\dot{x}^2 \neq 0$. Define a smooth function $\tilde{E} : T\mathbb{R} \to \mathbb{R}$ as $\tilde{E}(x, \dot{x}) := \dot{x}\frac{\partial \tilde{L}}{\partial \dot{x}}(x, \dot{x}) - \tilde{L}(x, \dot{x})$. By differentiating the expression for $\tilde{E}$ above along the vector field of (9), it is readily seen that $\tilde{E}$ is an integral of motion for (9), i.e., $\dot{\tilde{E}} = 0$. Consequently, $\tilde{E}$ must satisfy the first-order linear PDE

$$\frac{\partial \tilde{E}}{\partial x}\dot{x} + \frac{\partial \tilde{E}}{\partial \dot{x}}\left(\tilde{\Psi}_1(x) + \tilde{\Psi}_2(x)\dot{x}^2\right) = 0. \tag{10}$$

Its general solution, obtained via the method of characteristics in Pinchover and Rubinstein [2005], is $\tilde{E}(x, \dot{x}) = F(\tilde{E}_0(x, \dot{x}))$, where F is a smooth function and $\tilde{E}_0(x, \dot{x}) = \frac{1}{2}\tilde{M}(x)\dot{x}^2 + \tilde{V}(x)$. Using the definition of $\tilde{E}$, we have $\frac{\partial \tilde{E}}{\partial \dot{x}} = \dot{x}\frac{\partial^2 \tilde{L}}{\partial \dot{x}^2}$ for all $(x, \dot{x}) \in T\mathbb{R}$. Therefore, $\frac{\partial^2 \tilde{L}}{\partial \dot{x}^2} = \tilde{M}(x)F'(\tilde{E}_0(x, \dot{x}))$. Since $\partial^2\tilde{L}/\partial\dot{x}^2 > 0$ and $\tilde{M} > 0$, it follows that $F'(\tilde{E}_0(x, \dot{x})) > 0$ for all $(x, \dot{x}) \in \mathbb{R}^2$, and thus F is strictly increasing. Furthermore, we know that $\tilde{E}(x+T, \dot{x}) = \tilde{E}(x, \dot{x})$ for all $(x, \dot{x}) \in \mathbb{R}^2$. Therefore, for all $(x, \dot{x}) \in T\mathbb{R}$, we have $F(\tilde{E}_0(x+T, \dot{x})) = F(\tilde{E}_0(x, \dot{x}))$, which implies that $\tilde{E}_0(x+T, \dot{x}) = \tilde{E}_0(x, \dot{x})$. Since $\dot{x}$ is arbitrary, this latter identity implies that $\tilde{M}$ and $\tilde{V}$ are T-periodic. Since $\tilde{M}$ and $\tilde{V}$ are T-periodic, then $(1/2)\tilde{M}(\tilde{x})\dot{\tilde{x}}^2 - \tilde{V}(\tilde{x})$ is a Lagrangian for the lifted system (9). By Proposition 4.3, $L(s, \dot{s}) = (1/2)M(s)\dot{s}^2 - V(s)$ is a Lagrangian for the original system (4).

Lemma 5.2. Consider the virtual mass and virtual potential in (9). For all $n \in \mathbb{Z}$ and all $x \in \mathbb{R}$, the following holds:

$$\tilde{M}(x+nT) = \tilde{M}(T)^n\tilde{M}(x) \tag{11}$$

$$\tilde{V}(x+nT) = \begin{cases} \tilde{M}(T)^n\tilde{V}(x) + \tilde{V}(T)\dfrac{\tilde{M}(T)^n - 1}{\tilde{M}(T) - 1}, \\ \qquad \text{if } \tilde{M}(T) \neq 1, \\ \tilde{V}(x) + n\tilde{V}(T), \\ \qquad \text{if } \tilde{M}(T) = 1. \end{cases} \tag{12}$$

Proposition 5.3. The lifted system (9) is SEL with a smooth Lagrangian $\tilde{L} : T\mathbb{R} \to \mathbb{R}$ such that $\tilde{L}(x+T, \dot{x}) = \tilde{L}(x, \dot{x})$ for all $(x, \dot{x}) \in T\mathbb{R}$, if and only if the virtual mass $\tilde{M}(x)$ in (6) is T-periodic, and the virtual potential $\tilde{V}(x)$ is not T-periodic.

Proof. ($\Leftarrow$) Suppose that the virtual mass $\tilde{M}(x)$ is T-periodic and the virtual potential $\tilde{V}(x)$ is not T-periodic, so that $\tilde{V}(T) \neq 0$ and $f_0 = 1/\tilde{V}(T)$ is well-defined. Consider the function $\tilde{L} : T\mathbb{R} \to \mathbb{R}$ defined in (7). With our definition of f_0, $\tilde{L}(x, \dot{x})$ is T-periodic with respect to x. Moreover, by direct computation, we have

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}} - \frac{\partial \tilde{L}}{\partial x} = \tilde{\alpha}(x, \dot{x})\left(\ddot{x} - \tilde{\Psi}_1(x) - \tilde{\Psi}_2(x)\dot{x}^2\right), \tag{13}$$

where $\tilde{\alpha}(x, \dot{x}) = (\partial^2\tilde{L})/(\partial\dot{x}^2) = 2\pi f_0\tilde{M}(x)\cos(2\pi f_0\tilde{E}_0(x, \dot{x}))$. Note first that $\tilde{\alpha}$ is not identically zero because $\tilde{V}$ is not identically zero (if it were, then $\tilde{V}$ would be T-periodic, contradicting our assumption). At the same time, we now show that $\tilde{\alpha}$ has zeros. By assumption, $\tilde{M}(T) = \tilde{M}(0) = 1$ and $\tilde{V}(T) \neq V(0) = 0$. By identity (12) in Lemma 5.2, $\tilde{V}(x) \to \pm\infty$ as $|x| \to \infty$, and the two limits as $x \to \pm\infty$ have opposite sign, which implies that the continuous map $\tilde{V} : \mathbb{R} \to \mathbb{R}$ is onto. Thus, there exists $\bar{x} \in \mathbb{R}$ such that $2\pi f_0\tilde{V}(\bar{x}) = \pi/2$, implying that $\tilde{\alpha}(\bar{x}, 0) = 0$. We have shown that $\tilde{\alpha}$ has zeros, which implies that $\tilde{L}$ is degenerate. By definition, all solutions of the lifted system (9) satisfy the differential equation $\ddot{x} = \tilde{\Psi}_1(x) + \tilde{\Psi}_2(x)\dot{x}^2$. Therefore, by identity (13), any solution of (9) satisfies the Euler-Lagrange equation with a degenerate Lagrangian $\tilde{L}$. In order to complete the proof that system (9) is SEL, we need to show that if $\tilde{L}'$ is any other Lagrangian for system (9), then $\tilde{L}$ is degenerate, i.e., $\partial^2\tilde{L}'/\partial\dot{x}^2$ has zeros. Suppose there exists a nondegenerate Lagrangian $\tilde{L}'$ for system (9). Then, system (9) is EL, which by Proposition 5.1 implies that $\tilde{V}$ is T-periodic, a contradiction.
($\Rightarrow$) Suppose that the lifted system (9) is SEL, and let $\tilde{L}$ be a degenerate Lagrangian such that $\tilde{L}(x, \dot{x})$ is T-periodic with respect to x, and $\partial^2\tilde{L}/\partial\dot{x}^2$ has zeros, but it is not identically zero. We need to show that $\tilde{M}(T) = 1$, so that $\tilde{M}$ in (6) is T-periodic (this fact will imply that $\tilde{V}$ is not T-periodic, because if it were so, then by Proposition 5.1 the system would be EL). As in the proof of Proposition 5.1, let $\tilde{E} = \dot{x}\partial\tilde{L}/\partial\dot{x} - \tilde{L}$. Then, $\tilde{E}$ satisfies the linear PDE (10), whose general solution is $\tilde{E}(x, \dot{x}) = F(\tilde{E}_0(x, \dot{x}))$, with $\tilde{E}_0(x, \dot{x}) = (1/2)\tilde{M}(x)\dot{x}^2 + \tilde{V}(x)$. Since $\tilde{L}$ is T-periodic with respect to x, so is $\tilde{E}$. Therefore, $\tilde{E}(x, \dot{x}) = \tilde{E}(x+nT, \dot{x})$ for all $(x, \dot{x}) \in T\mathbb{R}$ and all $n \in \mathbb{Z}$. Using Lemma 5.2, for all $n \in \mathbb{Z}$ we have $F(E_0(x, \dot{x})) = F\big(\tilde{E}_0(x+nT, \dot{x})\big)$ $= F\left(\tilde{M}(T)^n\tilde{E}_0(x, \dot{x}) + \tilde{V}(T)\frac{\tilde{M}(T)^n - 1}{\tilde{M}(T) - 1}\right)$. We claim that if $\tilde{M}(T) \neq 1$, then F is a constant function. Indeed, for any $p \in \mathrm{Im}(\tilde{E}_0)$ and any $n \in \mathbb{Z}$, we have $F(p) = F\left(\tilde{M}(T)^n p + \tilde{V}(T)\frac{\tilde{M}(T)^n - 1}{\tilde{M}(T) - 1}\right)$. If $\tilde{M}(T) > 1$, taking the limit as $n \to -\infty$ in both sides of the identity above we get $F(p) = F\left(\frac{-\tilde{V}(T)}{\tilde{M}(T) - 1}\right)$. If $\tilde{M}(T) < 1$, the same identity

is obtained by taking the limit for $n \to +\infty$. Since the right-hand side of the identity above does not depend on p, $F : \mathrm{Im}(\tilde{E}_0) \to \mathbb{R}$ is a constant map. Thus, for all $(x, \dot{x}) \in T\mathbb{R}$ we have $\frac{\partial \tilde{E}}{\partial \dot{x}} = \dot{x}\frac{\partial^2 \tilde{L}}{\partial \dot{x}^2} = 0$, and so $\partial^2 \tilde{L}/\partial \dot{x}^2 \equiv 0$, contradicting our hypothesis on $\tilde{L}$.

Remark 5.4. Since the degenerate Lagrangian $\tilde{L}(x, \dot{x})$ in (7) is smooth and T-periodic with respect to x, there exists a smooth function $L : T[\mathbb{R}]_T \to \mathbb{R}$ such that $L \circ \bar{\pi} = \tilde{L}$. By Lemma 4.2, since $\tilde{\alpha}(x, \dot{x})$ is T-periodic with respect to x, (13) implies that L satisfies the identity $\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = \alpha(s, \dot{s})\left(\ddot{s} - \Psi_1(s) - \Psi_2(s)\dot{s}^2\right)$, where α and $\tilde{\alpha}$ are related through $\tilde{\alpha} = \alpha \circ \bar{\pi}$. △

6. EXAMPLES

We now present a number of examples illustrating the results of this paper.

Example 6.1. Consider the system $\ddot{s} = \frac{1}{2+\cos(s)}[\sin(2s) - \sin(s)\dot{s}^2]$, where $s \in [\mathbb{R}]_{2\pi}$. The virtual mass and potential are given by $\tilde{M}(x) = 9/(\cos x + 2)^2$ and $\tilde{V}(x) = 4 - 18(\cos x + 1)/(\cos x + 2)^2$. Since $\tilde{M}$ and $\tilde{V}$ are 2π-periodic, by Theorem 3.5 the system is EL and mechanical. △

Example 6.2. For the system $\ddot{s} = \cos(s) + 0.5 + \cos(s)\dot{s}^2$, where $s \in [\mathbb{R}]_{2\pi}$, we have $\tilde{M}(x) = \exp\left(-2\int_0^x \tilde{\Psi}_2(\tau)d\tau\right) = \exp\left(-2\int_0^x \cos\tau d\tau\right) = \exp(-2\sin x)$, which is 2π-periodic. On the other hand, one can check that $\tilde{V}(2\pi) = -\int_0^{2\pi}(\cos\tau + 0.5)\exp(-2\sin\tau)d\tau \simeq 7.1615 \neq 0$, so that $\tilde{V}$ is not 2π-periodic. By Theorem 3.7, the system is SEL. △

Example 6.3. For the system $\ddot{s} = \lambda$, with $\lambda \neq 0$ and $s \in [\mathbb{R}]_T$, we have $\tilde{M}(x) = 1$ and $\tilde{V}(x) = -\lambda x$. Since $\tilde{M}$ is T periodic and $\tilde{V}$ isn't, the system is SEL. By Theorem 3.7, the Lagrangian is given by (7). The Euler-Lagrange equation with this Lagrangian reads $\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}} - \frac{\partial \tilde{L}}{\partial x} = \frac{2\pi}{\lambda T}\cos\left(\frac{2\pi}{\lambda T}(\dot{x}^2/2 - \lambda x)\right)(\ddot{x} - \lambda) = 0$. We see that all solutions of the system $\ddot{s} = \lambda$ satisfy the Euler-Lagrange equation, but there are signals $(x(t), \dot{x}(t)) = (T/4 + kT, 0)$, $k \in \mathbb{Z}$ satisfying the Euler-Lagrange equation which do not satisfy the equation $\ddot{s} = \lambda$. Thus, the collection of solutions of a SEL system is contained in, but is not equal to, the collection of solutions of the associated Euler-Lagrange equation. △

Example 6.4. Consider the system $\ddot{s} = -\cos(s) - 2 + (\sin(s) + 2)\dot{s}^2$ with $s \in [\mathbb{R}]_{2\pi}$. We have $\Psi_1(s) = -\cos(s) - 2 < 0$ and $\int_0^{2\pi}\tilde{\Psi}_2(\tau)d\tau = \int_0^{2\pi}(\sin\tau + 2)d\tau = 4\pi > 0$. This latter identity implies that $\tilde{M}(2\pi) \neq 0$, so that $\tilde{M}$ is not 2π-periodic, and the system is neither EL nor SEL. Moreover, one can show, see Consolini and Maggiore [2012], that this system has an exponentially stable limit cycle with domain of attraction including $\mathcal{D} = \{(s, \dot{s}) \in T[\mathbb{R}]_{2\pi} : \dot{s} \leq 0\}$. △

REFERENCES

V. I. Arnold. *Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics, Vol. 60).* Springer, 1989.

C. Chevallereau, J.W. Grizzle, and C.L. Shih. Asymptotically stable walking of a five-link underactuated 3D bipedal robot. *IEEE Transactions on Robotics*, 25(1): 37–50, 2008.

L. Consolini and M. Maggiore. Control of a bicycle using virtual holonomic constraints. *Automatica*, 2012. accepted for publication.

G. Darboux. *Leçons sur la Théorie Générale des Surfaces.* Gauthier-Villars, Paris, 1894.

L. Freidovich, A. Robertsson, A. Shiriaev, and R. Johansson. Periodic motions of the pendubot via virtual holonomic constraints: Theory and experiments. *Automatica*, 44(3):785–791, 2008.

H. Helmholtz. Über die physikalische Bedeutung des Princips der kleinsten Wirkung. *J. Reine Angew. Math.*, 100:137–166, 1887.

A. Isidori. *Nonlinear Control Systems.* Springer, New York, third edition, 1995.

D. Jankuloski, M. Maggiore, and L. Consolini. Further results on virtual holonomic constraints. In *the 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, Bertinoro, Italy, 2012.

O. Krupková and G. E. Prince. Second order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations. In D. Krupka and D. J. Saunders, editors, *Handbook of Global Analysis*, pages 837–904. Elsevier, 2008.

J. M. Lee. *Introduction to Smooth Manifolds.* Springer, second edition, 2013.

M. Maggiore and L. Consolini. Virtual holonomic constraints for Euler-Lagrange systems. *IEEE Trans. Automat. Contr.*, 58(4):1001–1008, 2013.

J. Nakanishi, T. Fukuda, and D.E. Koditschek. A brachiating robot controller. *IEEE Transactions on Robotics and Automation*, 16(2):109–123, 2000.

Y. Pinchover and J. Rubinstein. *An Introduction to Partial Differential Equations.* Cambridge University Press, 2005.

F. Plestan, J.W. Grizzle, E.R. Westervelt, and G. Abba. Stable walking of a 7-DOF biped robot. *IEEE Transactions on Robotics and Automation*, 19(4):653–668, 2003.

R. M. Santilli. *Foundations of Theoretical Mechanics I, The Inverse Problem in Newtonian Mechanics.* Springer-Verlag, New York, 1978.

D. J. Saunders. Thirty years of the inverse problem of calculus of variations. *Reports on Mathematical Physics*, 66(1):43–53, Aug. 2010.

A. Shiriaev, J.W. Perram, and C. Canudas-de-Wit. Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Transactions on Automatic Control.*, 50(8):1164–1176, August 2005.

A. Shiriaev, A. Robertsson, J. Perram, and A. Sandberg. Periodic motion planning for virtually constrained Euler-Lagrange systems. *Systems & Control Letters*, 55: 900–907, 2006.

A.S. Shiriaev, L.B. Freidovich, and S.V. Gusev. Transverse linearization for controlled mechanical systems with several passive degrees of freedom. *IEEE Transactions on Automatic Control*, 55(4):893–906, 2010.

E. Tonti. Inverse problem: Its general solution. In G. M. Rassias and T. M. Rassias, editors, *Differential Geometry, Calculus of Variations, and Their Applications*, pages 497–510. Marcel Dekker, Inc., 1985.

E.R. Westervelt, J.W. Grizzle, and D.E. Koditschek. Hybrid zero dynamics of planar biped robots. *IEEE Transactions on Automatic Control*, 48(1):42–56, 2003.