SUFFICIENT CONDITIONS FOR THE SOLUTION OF THE SEMIGLOBAL OUTPUT TRACKING PROBLEM USING PRACTICAL INTERNAL MODELS [2]

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Abstract: In a recent work we have introduced the notion of a *practical internal model*, i.e., a compensator yielding suitable observability properties in a nonlinear system, and we have shown that its existence implies the existence of an output feedback controller achieving arbitrarily small asymptotic tracking error. In this paper we show that a sufficient condition for a *practical internal model* to exist is that the plant is differentially flat with respect to the measurable output. We apply this technique to the output regulation problem and we provide sufficient conditions for its semiglobal solution when no disturbances act on the plant. *Copyright © 2001 IFAC*

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1. INTRODUCTION

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurable output, and the vector fields f and h are assumed to be sufficiently smooth in their arguments. Our objective is to solve the following problems.

Problem 1 (Output Feedback Asymptotic Tracking): Given the dynamical system (1) and a sufficiently smooth reference trajectory $r(t) = [r_1(t), \ldots, r_p(t)]^\top$, design a dynamic output feedback controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r) \\ u &= h_c(x_c, y) \end{aligned} \tag{2}$$

where f_c and h_c are sufficiently smooth, such that the closed-loop system (1)-(2) has the property that $e(t) = y(t) - r(t) \to 0$ as $t \to \infty$, and the internal states x and x_c are bounded for all $t \geq 0$, and for all initial conditions $[x(0)^\top, x_c(0)^\top]^\top \in \mathcal{A}$, for some closed set $\mathcal{A}$.

Problem 2 (Output Feedback Practical Tracking): Given the dynamical system (1), a sufficiently smooth reference trajectory $r(t) = [r_1(t), \ldots, r_p(t)]^\top$, and a small scalar $e_0 > 0$, design a dynamic output feedback controller of the form (2) such that the closed-loop system (1)-(2) has the property that there exists a $T > 0$ such that $\|e(t)\| \leq e_0$ for all $t \geq T$, and such that the internal states x and x_c are bounded for all $t \geq 0$, and for all initial conditions $[x(0)^\top, x_c(0)^\top]^\top \in \mathcal{A}$, for some closed set $\mathcal{A}$.

In (Maggiore and Passino 2001), we have showed that if there exists a practical internal model then Problem 2 has a solution. The aim of this paper is to find sufficient conditions for the existence of the practical internal model and conditions for the

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solvability of Problem 1. For convenience of the reader, in what follows we briefly summarize the results in (Maggiore and Passino 2001), together with the definition of a practical internal model. First we need to introduce some basic assumptions.

Assumption A1 (Stable Inverse): For the reference trajectory $r(t)$, there exist sufficiently smooth and bounded functions $x_r(t)$ and $c_r(t)$ such that

$$\begin{aligned} \dot{x_r} &= f(x_r, c_r) \\ r(t) &= h(x_r, c_r) \end{aligned} \tag{3}$$

for some initial condition $x_r(0), c_r(0)$, and for all $t \geq 0$.

Next, consider the change of coordinates $\tilde{x} = x - x_r$, rewrite (1) in new coordinates as

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, u), \tag{4}$$

and notice that the asymptotic stability of the origin of (4) is equivalent to the attractivity of the tracking manifold of (1).

Assumption A2 (Stabilizability of the Tracking Manifold): There exists a smooth function $\bar{u}(\tilde{x}, c_r)$ such that $\bar{u}(0, c_r) = c_r$ and the origin is a uniformly asymptotically stable equilibrium point of $\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, \bar{u}(\tilde{x}, c_r))$, with domain of attraction a closed set $\tilde{\mathcal{D}} \subset \mathbb{R}^n$, i.e., there exists (see (Kurzweil 1956)) a function $V(\tilde{x}, t)$, defined for $\tilde{x} \in \tilde{\mathcal{D}}$, which is continuous with continuous partial derivatives, and continuous positive definite functions $\alpha_1(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}_\infty$, $\alpha_2(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}$, and $\alpha_3(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}$ such that

$$(i) \quad \alpha_1(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \leq V(\tilde{x}, t) \leq \alpha_2(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \tag{5}$$

$$(ii) \quad \frac{\partial V}{\partial \tilde{x}} \tilde{f}(t, \tilde{x}, \bar{u}(\tilde{x}, c_r)) + \frac{\partial V}{\partial t} \leq -\alpha_3(\|\tilde{x}\|_{\tilde{\mathcal{D}}}), \tag{6}$$

for $\tilde{x} \in \tilde{\mathcal{D}}$, $\tilde{x} \neq 0$, and all $t \geq 0$, where $\|\tilde{x}\|_{\tilde{\mathcal{D}}} \triangleq \max\left\{\|\tilde{x}\|, \frac{1}{\rho(\tilde{x}, \tilde{\mathcal{D}}^o)} - \frac{2}{\rho(0, \tilde{\mathcal{D}}^o)}\right\}$, $\tilde{\mathcal{D}}^o$ is the complement of $\tilde{\mathcal{D}}$ in $\mathbb{R}^n$, and $\rho(\tilde{x}, \tilde{\mathcal{D}}^o)$ denotes the distance of $\tilde{x}$ from the set $\tilde{\mathcal{D}}^o$ (i.e., $\rho(\tilde{x}, \tilde{\mathcal{D}}^o) = \inf_{z \in \tilde{\mathcal{D}}^o} \|\tilde{x} - z\|$).

We now assume x in (1) to be observable from the output y. In order to characterize the observability properties of (1), consider the observability mapping $y_x \triangleq [y_1, \ldots, y_1^{(k_1-1)}, \ldots, y_p, \ldots, y_p^{(k_p-1)}]^\top \triangleq \mathcal{H}_x(x, z)$, where $z \triangleq [u_1, \ldots, u_1^{(n_1-1)}, \ldots, u_m, \ldots, u_m^{(n_m-1)}]^\top \in \mathbb{R}^{n_u}$, $\sum_{i=1}^p k_i = n$, $n_u \triangleq n_1 + \ldots + n_m$, $0 \leq n_i \leq \max\{k_1, \ldots, k_p\}$ (when $\mathcal{H}_x$ does not depend on u_i, then we set $n_i = 0$). Note that the vector z contains only the derivatives of u that end up appearing in the mapping $\mathcal{H}_x$ for the application at hand.

Assumption A3 (Observability): System (1) is observable over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, i.e., there exists a set of indices $\{k_1, \ldots, k_p\}$ such that the mapping $y_e = \mathcal{H}_x(x, z)$ is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $z \in \mathcal{U}$.

Notice that A3 does not require (1) to be uniformly completely observable (UCO), i.e., $\mathcal{X} \times \mathcal{U} = \mathbb{R}^n \times \mathbb{R}^{n_u}$, and thus it relaxes analogous conditions commonly found in the literature (see, e.g., (Tornambè 1992, Teel and Praly 1995)).

Next, we introduce a condition to estimate the functions $x_r(t)$ and $c_r(t)$ on-line. Before stating the assumption, note that it is useful to think of (3) as a copy of the plant with unknown state x_r, unknown input c_r, but a known output which is the reference trajectory $r(t)$. Consider a compensator of the type

$$\begin{aligned} \dot{\zeta}_r &= a(\zeta_r, x_r, v_r) \\ c_r &= b(\zeta_r, x_r), \end{aligned} \tag{7}$$

where $\zeta_r \in \mathbb{R}^q$ ($q \geq m$), $v_r \in \mathbb{R}^m$, a and b are sufficiently smooth, and v_r is the new input of the composite system (3)-(7). Define the observability mapping associated with x_r and ζ_r in the composite system (3)-(7) as $y_{x_r,\zeta_r} \triangleq [y_1, \ldots, y_1^{(\bar{k}_1-1)}, \ldots, y_p, \ldots, y_p^{(\bar{k}_p-1)}]^\top \triangleq \mathcal{H}_{x,\zeta}\left(x_r, \zeta_r, v_r, \ldots, v_r{}^{(\bar{n}_u-1)}\right)$, where $\sum_{i=1}^p \bar{k}_i = n + q$, $0 \leq \bar{n}_u \leq \max\{\bar{k}_1, \ldots, \bar{k}_p\} - 1$.

Assumption A4 (Practical Internal Model): There exists a compensator of the form (7), which we call a *practical internal model*, which is regular (i.e., for each $x(0)$ and $u(t)$ there exist $\zeta(0)$ and $v(t)$ such that $b(\zeta, x) = u$, for all $t \geq 0$) and such that the following two properties hold for the composite system (3)-(7).
(i) $\mathcal{H}_{x,\zeta}$ does not depend on v_r and its derivatives, i.e., $\mathcal{H}_{x,\zeta} = \mathcal{H}_{x,\zeta}(x_r, \zeta_r)$.
(ii) There exists a set of indices $\{\bar{k}_1, \ldots, \bar{k}_p\}$ such that the mapping $y_{x_r,\zeta_r} = \mathcal{H}_{x,\zeta}(x_r, \zeta_r)$ is invertible with respect to x_r and ζ_r, and its inverse is sufficiently smooth, for all $[x_r{}^\top, \zeta_r{}^\top]^\top \in \mathcal{X}_a$.

Next, we need to guarantee that the reference trajectory is contained in within an observable region.

Assumption A5 (Reference Trajectory): The reference trajectory $r(t)$ is such that, for all $t \geq 0$, $\left[r_1(t), \ldots, r_1^{(\bar{k}_1-1)}(t), \ldots, r_p(t), \ldots, r_p^{(\bar{k}_p-1)}(t)\right]^\top \in C_r \subset \mathcal{H}_{x,\zeta}(\mathcal{X}_a)$, for some convex compact C_r.

Finally, we need to make sure that the state and input trajectories of the closed-loop system travel within the observable domain of the plant (at least in the ideal case when the state feedback controller is employed). To this end, in the following assumption we characterize a subset of the domain of attraction $\mathcal{D}$ which is contained within an observable region of (1). Given any scalar $c > 0$ let $\Omega_c \triangleq \{x \in \mathbb{R}^n \,|\, V(x - x_r, t) \leq c, \text{ for all } t \geq 0\}$, where $V(\tilde{x}, t)$ is defined in A2, and note that for all $c > 0$ we have that, by the properness of V and the definition of $\mathcal{D}$, $\Omega_c \subset \mathcal{D}$. Let Ω^z be the compact set which is invariant with respect to the z trajectories (its existence follows from the smoothness of $\bar{u}$ and the boundedness of $x(t)$, $x_r(t)$, and $c_r(t)$), and consider the mapping $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_u}$, $\mathcal{F}(x, z) \triangleq [\mathcal{H}_x(x, z)^\top, z^\top]^\top$ which, clearly, is a diffeomorphism on $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, and assume that the set $\mathcal{X} \times \mathcal{U}$ satisfies the following.

Assumption A6 (Topology of $\mathcal{X} \times \mathcal{U}$): There exists a positive scalar $\bar{c}$ such that $\mathcal{F}(\Omega_{\bar{c}}, \Omega^z) \subset C_\xi \subset \mathcal{F}(\mathcal{X}, \mathcal{U})$, for some convex compact C_ξ.

Figure 1 shows the structure of the output feedback controller solving Problem 2. The scheme employs two observers, the first one is used to estimate the functions x_r and ζ_r (and hence also c_r), while the second one estimates the state of system (1). Projection algorithms are employed to keep the observer estimates in within the observable regions while preserving their convergence properties. Next, the projected estimates are employed by the stabilizer $\bar{u}$ to drive the closed-loop trajectories inside an arbitrarily small neighborhood of the tracking manifold. Refer to (Maggiore and Passino 2001) for more details on the structure of the output tracking controller.

Theorem 1. (*Theorem 3 in (Maggiore and Passino June 2000)*). Suppose that A1-A6 hold. Then, the controller structure depicted in Figure 1 solves Problem 2 on a set $\mathcal{A}$ whose size depends on the size of the sets $\mathcal{X} \times \mathcal{U}$, $\mathcal{X}_a$, and $\mathcal{D}$. If A2, A3, and A4 hold globally (i.e., C_ξ and C_r can be chosen arbitrarily large), then the solution of Problem 2 is semiglobal and $\mathcal{A}$ can be chosen to be an arbitrarily large compact set.

The main obstacle to the solution of Problem 1 is the fact that v_r in (7) is unknown. This represents a severe limitation because the tracking manifold is *not* an equilibrium point of the closed-loop system in Theorem 1, and hence cannot be asymptotically stabilized.

2. A SUFFICIENT CONDITION FOR THE SOLUTION OF PROBLEM 2

A class of systems which has drawn increasing attention in the past decade is the class of differentially flat systems. In what follows we will briefly recall the properties of differentially flat systems and relate them to the theory introduced in this paper. Keeping in mind that (3) and (1) share identical properties (provided A1 holds), for notational convenience we apply the following definition to the *copy of the plant* (3) assuming, for the moment, that A1 does not hold and thus $r(t)$ is *not* the prespecified reference trajectory, but rather the generic output of the copy of the plant. We will later show that differential flatness of the plant implies that A1 holds and thus $r(t)$ actually *is* the prespecified reference trajectory and that all the assumptions required by Theorem 1 are indeed satisfied.

System (3) is differentially flat (see (Fliess *et al.* 1995)) with respect to the flat output r (or, equivalently, the plant (1) is differentially flat with respect to the flat output y) if there exists a compensator

$$\begin{aligned} \dot{\bar{\zeta}}_r &= \bar{a}(\bar{\zeta}_r, x_r, v_r) \\ c_r &= \bar{b}(\bar{\zeta}_r, x_r, v_r) \end{aligned} \tag{8}$$

such that the augmented system (3)-(8) is feedback linearizable with respect to the output function r and there exists a function γ such that $\bar{\zeta}_r(t) = \gamma(x_r, c_r, \dot{c}_r, \ldots, c_r{}^{(n_\zeta)})$ for some positive n_ζ. This means that, if $p = m$ and (3) is globally differentially flat with flat output r, the change of coordinates $\xi_r = \mathcal{H}(x_r, \bar{\zeta}_r)$, where

$$\xi_r = [r_1, \ldots, r_1^{(l_1-1)}, \ldots, r_p, \ldots, r_m^{(l_m-1)}]^\top$$
$$\mathcal{H}(x_r, \bar{\zeta}_r) = \left[h_1, \ldots, \bar{\varphi}_1^{(l_1-1)}, \ldots, h_m, \ldots, \bar{\varphi}_m^{(l_m-1)}\right]^\top,$$

for some set of indices $\{l_1, \ldots, l_m\}$, is a diffeomorphism independent of v_r and, in new coordinates, (3)-(8) reads as

$$\begin{aligned} \dot{\xi}_r &= A_c \xi_r + B_c\left[\alpha(\xi_r) + \beta(\xi_r)\, v_r\right] \\ r &= C_c \xi_r \end{aligned} \tag{9}$$

where (A_c, B_c, C_c) is in controllable/observable canonical form and β is an invertible $m \times m$ matrix. Note that (8) differs from the practical internal model in that the function $\bar{b}$ is allowed to depend on v_r, and hence estimating $x_r(t)$ and $\zeta_r(t)$ is not sufficient to calculate $c_r(t)$. This obstacle is removed by using the dynamic feedback transformation

$$\begin{aligned} v_r &= -\beta^{-1}(\xi_r)(\alpha(\xi_r) - \zeta_r{}') \\ \dot{\zeta}_r{}' &= v_r{}', \end{aligned} \tag{10}$$

and letting $v_r{}'$ be the control input of the new compensator (8)-(10). By setting $\zeta_r = [\bar{\zeta}_r{}^\top, \zeta_r{}'^\top]^\top$, $a(\zeta_r, x_r, v_r{}') = [\bar{a}^\top(\bar{\zeta}_r, x_r, -\beta^{-1}(\xi_r)(\alpha(\xi_r) - \zeta_r{}')), v_r{}'^\top]^\top$, $b(\zeta_r, x_r) = \bar{b}(\bar{\zeta}_r, x_r, \zeta_r{}')$, we get a compensator of the type (7). The observability mapping of the composite system (3)-(8)-(10) is given by $[r_1, \ldots, r_1^{(l_1)}, \ldots, r_m, \ldots, r_m^{(l_m)}]^\top = \mathcal{H}_{x,\zeta}(x_r, \zeta_r) = [h_1, \ldots, \bar{\varphi}_1^{(l_1-1)}, \zeta_r{}'_1, \ldots, h_m, \ldots, \bar{\varphi}_m^{(l_m-1)}, \zeta_r{}'_m]^\top$. Clearly, the properties of the mapping $\mathcal{H}$, namely its invertibility and independence

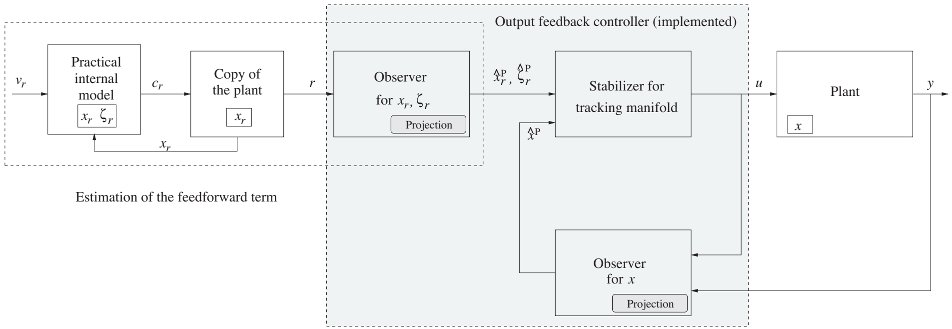

Fig. 1. Block diagram of the output feedback controller solving Problem 2.

of v_r, are still valid for $\mathcal{H}_{x,\zeta}$, and therefore A4 is satisfied on $\mathbb{R}^n \times \mathbb{R}^q$ with $\bar{k}_i = l_i + 1$, $i = 1,\ldots,m$. Furthermore, the observability of the augmented system, together with the fact that $\bar{\zeta}_r(t) = \gamma(x_r, c_r, \dot{c}_r, \ldots, c_r{}^{(n_\zeta)})$, implies that (1) is also observable with respect to y and hence A3 holds on $\mathbb{R}^n \times \mathbb{R}^{n_u}$. Next, given a smooth reference trajectory $r(t)$, the tracking manifold for the composite system is given by

$$\left[x_r{}^\top(t), \zeta_r{}^\top(t)\right]^\top = \mathcal{H}^{-1}_{x,\zeta}\left(\left[r_1(t),\ldots,r_1^{(l_1)}(t),\ldots,\right.\right.$$
$$\left.\left.\ldots, r_m(t),\ldots,r_m^{(l_m)}(t)\right]^\top\right),$$
$$v_r{}' = \left[r_1^{(l_1+1)},\ldots,r_m^{(l_m+1)}\right],$$

from which one obtains $c_r(t) = b(\zeta_r(t), x_r(t))$, and thus A1 is satisfied. As for Assumption A2, from (9) we have that a copy of the compensator (8) can be also employed to (globally) dynamically stabilize the tracking manifold as follows

$$\begin{aligned}
\dot{\bar{\zeta}} &= \bar{a}(\bar{\zeta}, x, v)\\
v &= v_r + K\left(\mathcal{H}(x,\bar{\zeta}) - \mathcal{H}(x_r,\bar{\zeta}_r)\right),\\
u &= \bar{b}(\bar{\zeta}, x, v).
\end{aligned}$$

Clearly, if the tracking manifold of (1) is stabilizable by static feedback then there is no need to include (8) in the state feedback tracking controller. Finally, we conclude this discussion by noting that since A1-A4 hold globally, Problem 2 is semiglobally solvable. The previous considerations are summarized in the following proposition.

Proposition 1: If system (1) is globally differentially flat with respect to the flat output y, then the output feedback practical tracking problem is solvable semiglobally.

3. SOLUTION OF PROBLEM 1 USING THE INTERNAL MODEL PRINCIPLE

In this section we will attempt to build a bridge between some well-known results in nonlinear regulator theory and the methodology adopted here: this will help us in solving Problem 1. Consider the system

$$\begin{aligned}
\dot{x} &= f(x,u,w)\\
e &= h(x) - q(w)
\end{aligned} \tag{11}$$

where $x \in X \subset \mathbb{R}^n$, $u \in \mathbb{R}^m$, $e \in \mathbb{R}^m$, and $w \in W \subset \mathbb{R}^r$. The variable e represents the measurable error between the output $h(x)$ and the reference $q(w)$. Assume that w is an unknown disturbance input generated by the exosystem

$$\dot{w} = s(w), \tag{12}$$

that $f(x,u,w)$, $h(x)$, $q(w)$, and $s(w)$ are sufficiently smooth functions and, moreover, that $f(0,0,0) = 0$, $h(0) = 0$, $q(0) = 0$, and $s(0) = 0$.

Problem 3 (Semiglobal Output Feedback Output Regulation): Given a system of the form (11) with an exosystem of the type (12), and arbitrary compact sets $\mathcal{K}_x \subset \mathbb{R}^n$ and $\mathcal{K}_w \subset \mathbb{R}^r$, find a controller of the type (2) (where $r = q(w)$) and a compact set $\mathcal{K}_{x_c} \subset \mathbb{R}^{n_c}$, such that the trajectory $(x(t), x_c(t))$ of the closed-loop system

$$\begin{aligned}
\dot{w} = s(w), \quad \dot{x} &= f(x, h_c(x_c,y), w),\\
\dot{x}_c &= f_c(x_c, h(x), q(w))
\end{aligned} \tag{13}$$

with initial conditions $(x(0), x_c(0), w(0)) \in \mathcal{K}_x \times \mathcal{K}_{x_c} \times \mathcal{K}_w$ is well-defined for all $t \geq 0$, bounded and such that $e(t) = h(x) - q(w) \to 0$ as $t \to \infty$.

The formulation of Problem 3 is slightly more general than the standard formulation of the semiglobal output regulation problem, in that the origin of the unforced system is not required to be exponentially stable, and the feedback variables of the controller are given by the system output $h(x)$ and the reference trajectory $q(w)$, and thus the controller is not restricted to use the error feedback e. The solution to this problem presently not known. In recent works researchers have concentrated in devising classes of nonlinear systems for which the global robust output regulation problem (see (Serrani and Isidori 2000)) or its semiglobal version (see (Mahmoud and Khalil 1997), (Isidori 1997),(Khalil 2000), (Serrani *et al.* 2000)) may be solved. In most of the works (11) is required to have a well-defined relative degree and to be minimum phase with input to state stable zero dynamics (see (Mahmoud and

Khalil 1997), (Isidori 1997),(Khalil 2000)), or simply minimum phase (see (Serrani *et al.* 2000)). Return now to the output feedback tracking problem and notice that in Problem 1 f does not depend on the disturbance input w and $r(t)$ is not restricted to be the output $q(w(t))$ of the exosystem (12). Furthermore, the controller (2) uses input $h(x,u)$ and $r(t)$ separately, rather than their difference $e(t)$. In order to establish a foundation for a comparison between Problem 1 and Problem 3 assume that $r(t)$ is generated by the exosystem (12), i.e., $r = q(w)$, and notice that imposing the existence of a tracking manifold in A1 is equivalent to the existence of a solution to the regulator equations

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w), \; 0 = h(\pi(w)) - q(w) \tag{14}$$

with $x_r(t) = \pi(w(t))$, and $c_r(t) = c(w(t))$. In this framework Assumption A4, regarding the existence of an appropriate compensator, can be replaced by the internal model principle, where the compensator (7) takes the form of an internal model

$$\dot{\zeta}_r = a(\zeta_r), \qquad c_r = b(\zeta_r). \tag{15}$$

Noting that the internal model (15) is an unforced system, the obstacle to asymptotic tracking caused by the presence of the unknown input v_r in (7) is removed. In the general case when an internal model is not available (or does not exist) a practical internal model (7) may exist and could be employed to solve the practical tracking problem (see Theorem 1). Notice that the requirement, in A4, that the augmented system (1)-(15) be observable on $\mathcal{X}_a$, together with the stabilizability assumption A2, forms a set of conditions conceptually similar to stabilizability/detectability conditions on the plant augmented with the internal model used, e.g., in (Isidori 1995).

In conclusion, our framework can be employed for the solution of Problem 1 when an exogenous system generates the reference trajectories. In this case, Problem 1 can be interpreted as an output regulation problem for systems of the form (1), where the state equation does not depend on the exogenous disturbance input $w(t)$. To this end, we need to modify the theory developed in (Maggiore and Passino June 2000). Consider the dynamic extension

$$\begin{aligned} \dot{z}_{i,j} &= z_{i,j+1}, \; j = 1, \ldots, n_i - 1 \\ \dot{z}_{i,n_i} &= u_i', \quad\; i = 1, \ldots, m \\ u_i &= z_{i,1}, \end{aligned} \tag{16}$$

and redesign a state feedback stabilizing control law $\bar{u}'$ for the augmented system (4)-(16), so that z is available for feedback and does not need to be estimated. Assume x, x_r and ζ_r are available for feedback and define the change of coordinates $z_\Delta = z - z_u(\tilde{x}, \zeta_r)$ where $z_u(\tilde{x}, \zeta_r) = [\bar{u}_1, \ldots, \bar{u}_1^{(n_1-1)}, \ldots, \bar{u}_m, \ldots, \; \bar{u}_m^{(n_m-1)}]^\top$, and let $U'(\tilde{x}, \zeta_r) \triangleq [\bar{u}_1^{(n_1)}, \ldots, \; \bar{u}_m^{(n_m)}]^\top$. Note that, since $c_r = b(\zeta_r)$, in the definition of z_u above the time derivatives of c_r can be expressed as functions of ζ_r. In $(\tilde{x}, z_\Delta)$-coordinates the extended system reads as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, C z_\Delta + \bar{u}) \\ \dot{z}_\Delta &= A z_\Delta + B[u' - U'(\tilde{x}, \zeta_r)] \end{aligned} \tag{17}$$

where (A, B, C) is in controllable/observable canonical form and u' is the new control input. By construction, when $u' - U'(\tilde{x}, \zeta_r)$ is zero, the origin is an equilibrium point of (17) and, when $z_\Delta = 0$, we have that, by A2, the $\tilde{x}$ subsystem is asymptotically stable with domain of attraction $\mathcal{D}$ (in x coordinates). A slight variation of Lemma 9.2.1 in (Isidori 1995) proves that there exists a smooth function $\bar{u}'(t, \tilde{x}, z, \zeta_r)$ which makes (17) asymptotically stable with domain of attraction $\tilde{\mathcal{D}}' = \mathcal{D} \times \mathbb{R}^{n_u}$, and a corresponding Lyapunov function $V'(\tilde{x}, z, t)$ enjoying properties similar to (i) and (ii) in A2. Next, given any scalar $c > 0$, let $\Omega_c' \triangleq \{x \in \mathbb{R}^n, z \in \mathbb{R}^{n_u} \,|\, V'(x - x_r, z, t) \leq c \text{ for all } \; t \geq 0\}$. Recalling that now z is available for feedback, we can employ this knowledge to define an asymptotic observer for x

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, Cz) + \left[\frac{\partial \mathcal{H}_x(\hat{x}, z)}{\partial \hat{x}}\right]^{-1} (\mathcal{E}^x)^{-1} L\,(y - \hat{y}) \\ \hat{y} &= h(\hat{x}, Cz), \end{aligned} \tag{18}$$

which, by Theorem 1 in (Maggiore and Passino June 2000), is such that the equilibrium point $x - \hat{x} = 0$ is exponentially stable and asymptotically stable with basin of attraction a suitable region. Moreover, the rate of convergence of $\hat{x}$ to x can be made arbitrarily fast. Next, since now z is the state of the compensator (16) and is available for feedback, A6 has to be modified as follows.

Assumption A7 (Topology of $\mathcal{X} \times \mathcal{U}$): Assume there exists a positive scalar $\bar{c}$ such that $\mathcal{F}(\Omega_{\bar{c}}') \subset C_\xi \subset \mathcal{F}(\mathcal{X}, \mathcal{U})$, for some convex compact C_ξ.

Then, the projection for $\hat{x}$ introduced in (Maggiore and Passino June 2000) becomes

$$\begin{aligned} \dot{\hat{x}}^P &= \left[\frac{\partial \mathcal{H}_x}{\partial \hat{x}}\right]^{-1} \left\{ \mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, z, \dot{z}\right) - \frac{\partial \mathcal{H}_x}{\partial z} \dot{z} \right\} \\ \mathcal{P}_1 &= \begin{cases} \dot{\hat{\xi}} - \Gamma_1 \dfrac{N_\xi \left(N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{z}\right)}{N_\xi^\top \Gamma_1 N_\xi + N_z^\top \Gamma_2 N_z} \\ \text{ if } N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{z} \geq 0 \text{ and } [\hat{\xi}^\top, z^\top]^\top \in \partial C_\xi \\ \dot{\hat{\xi}} \text{ otherwise} \end{cases} \end{aligned} \tag{19}$$

Choose a scalar $\underline{c}$ such that $0 < \underline{c} < \bar{c}$. The solution of Problem 1 if given by the following.

Theorem 2. Consider system (1) with the exosystem (12) generating the reference trajectory $r(t) = q(w)$. Suppose A1-A5, and A7 are satisfied and the compensator (7) in A4 is replaced by the internal model

$$\dot{\zeta}_r = a(\zeta_r), \qquad c_r = b(\zeta_r). \tag{20}$$

Then, the output feedback tracking problem is solvable on $\mathcal{A}' = \{x \in \mathbb{R}^n, x_c \in \mathbb{R}^{2n+n_u+q} \,|\, [x^\top(0), z^\top(0)]^\top \in \Omega'_{\underline{c}}, \hat{x}_a^P(0) \in \mathcal{H}_{x,\zeta}^{-1}(C_r), \hat{x}^{P\top}(0) \in \mathcal{H}_x^{-1}(C_\xi)\}$, for any $0 < \underline{c} < \bar{c}$, by letting

$$u' = \bar{u}'(t, \hat{x}^P - \hat{x}_r^P, z, \hat{\zeta}_r^P) \tag{21}$$

(refer to (Maggiore and Passino June 2000) or see Figure 1 for the meaning of the variables $\hat{x}_r^P$ and $\hat{\zeta}_r^P$) be the output of the dynamic output feedback controller with state $x_c \triangleq [z^\top, \hat{x}_a^{P\top}, \hat{x}^{P\top}]^\top \in \mathbb{R}^{2n+n_u+q}$, and choosing large enough gains in the observers.

Theorem 2, besides providing a set of sufficient conditions for the solvability of Problem 1, allows us to apply the methodology introduced in (Maggiore and Passino June 2000) to the output regulation problem.

4. CONCLUSIONS

We have shown that the output feedback (practical) tracking problem can be solved if one can find a compensator (the practical internal model) yielding suitable observability properties in the closed-loop system. We have established a connection between different existing techniques (differential flatness, stable inversion, and output regulation theory) and the approach followed in this paper. It must be stressed that, contrary to current results in output regulation theory (see, e.g., (Isidori 1997), (Mahmoud and Khalil 1997), (Serrani and Isidori 2000)), the methodology presented here *does not* handle the presence of uncertainties or disturbances: more research is needed to address this concern. On the other hand, however, the class of systems considered in this paper is not restricted to be in lower triangular form, nor are the reference trajectories restricted to be the outputs of a known exosystem. In this respect, the practical internal model may be viewed as a robust counterpart of the standard internal model, in that it can be used when the information about the exosystem is not accurate, or even when the exosystem is not present at all. Finding necessary and sufficient conditions for the existence of a practical internal model, as well as a constructive methodology to find it, represent open research topics.

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