Practical Internal Models for Output Feedback Tracking in Nonlinear Systems[1]

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Abstract

We study the solution of the output feedback tracking problem for incompletely observable nonlinear systems. We show that if there exists a *practical internal model*, i.e., a compensator yielding suitable observability properties in the closed-loop system, one can find an output feedback controller achieving arbitrarily small asymptotic tracking error.

1 Introduction

The asymptotic tracking problem entails making a nonlinear system asymptotically track a desired reference trajectory while guaranteeing boundedness of all the internal variables. In its general formulation, even when the state of the plant is known, the tracking problem is considerably more difficult than that of stabilization. For systems in *output feedback form* the global solution of this problem via output feedback is well-known (see [1]) while, for nonminimum-phase systems, achieving asymptotic tracking is considered to be a challenging problem even in the state feedback case. This is due to the fact that input-output linearization techniques do not guarantee that the internal dynamics are stable and hence control techniques based on the Hirshcorn left-inverse of the system cannot be employed. One approach to overcome this problem, introduced in [2] by Devasia *et al.*, involves calculating a stable inverse of the plant, i.e., bounded state and input trajectories yielding the desired output trajectory. Such an inverse is found as the solution of an integral equation which can be calculated iteratively by means of a Picard-like iteration. The stable inverse so obtained is employed as a feedforward term in a regulation scheme and, being in general non-causal, it may require pre-actuation. The other major approach to solving tracking problems is the theory of output regulation (also referred to as the servomechanism theory, see [3]), originally introduced by Davison, Francis, and Wonham in [3, 4] for linear systems, and extended to nonlinear systems by Isidori and Byrnes in [5]. The error feedback output regulation problem entails finding a dynamic controller that makes the output track a reference trajectory and reject a time-varying disturbance, both of which are generated by a neutrally stable exosystem, by using only the feedback given by the tracking error. While the stable inversion approach requires perfect knowledge of the entire reference trajectory as well as the model of the plant, and hence it is inherently non-robust, the output regulation approach is robust but it restricts the class of reference trajectories to be tracked.

This paper aims at providing a unifying framework for output tracking by output feedback. In particular, we seek a solution of the output feedback tracking problem when the reference trajectory is not necessarily generated by an exosystem, and no disturbance acts on the plant. The main contribution here is to replace the internal model (which, in this general setting, is not available) by a *practical internal model*, i.e., a compensator yielding suitable observability properties, and using two observers to estimate the feedforward term and the state of the system on-line. These estimates are then employed by the controller to perform tracking. Hence, similar to the output regulation theory, and unlike the stable inversion approach, the knowledge of the feedforward term and the state of the plant is not needed. Moreover, the choice of the practical internal model is not dictated by the nature of the exosystem, but it is rather a consequence of the observability properties of the plant.

The key result is given in one theorem. It is shown (Theorem 3) that the proposed scheme solves the output feedback practical tracking problem (i.e., tracking with arbitrarily small error). Because of space limitations all the proofs are omitted.

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2 Problem Formulation and Assumptions

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurable output, and the vector fields f and h are assumed to be sufficiently smooth in their arguments. Our objective is to solve the following problem.

Problem 1 (Output Feedback Practical Tracking): *Given the dynamical system (1), a sufficiently smooth reference trajectory $r(t) = [r_1(t), \ldots, r_p(t)]^\top$, and a small scalar $e_0 > 0$, design a dynamic output feedback controller*

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r) \\ u &= h_c(x_c, y) \end{aligned} \tag{2}$$

such that the closed-loop system (1)-(2) has the property that there exists a $T > 0$ such that $\|e(t)\| \leq e_0$ for all $t \geq T$, and such that the internal states x and x_c are bounded for all $t \geq 0$, and for all initial conditions $[x(0)^\top, x_c(0)^\top]^\top \in \mathcal{A}$, for some closed set $\mathcal{A}$.

Assumption A1 (Stable Inverse): For the reference trajectory $r(t)$, there exist sufficiently smooth and bounded functions $x_r(t)$ and $c_r(t)$ such that

$$\begin{aligned} \dot{x_r} &= f(x_r, c_r) \\ r(t) &= h(x_r, c_r) \end{aligned} \tag{3}$$

for some initial condition $x_r(0), c_r(0)$, and for all $t \geq 0$.

Assumption A1 requires the existence of an input-state trajectory $(c_r(t), x_r(t))$ (a stable inverse of the plant in the terminology of [2]) which reproduces the desired output trajectory $r(t)$ or, in other words, the existence of a tracking manifold $\{x \in \mathbb{R}^n \,|\, x = x_r(t)\}$ of (1) which is made invariant under the action of $c_r(t)$. Within an output regulation framework, A1 is equivalent to requiring the existence of a solution to the regulator equations.

Next, consider the change of coordinates $\tilde{x} = x - x_r$, rewrite (1) in new coordinates as

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, u), \tag{4}$$

and notice that the asymptotic stability of the origin of (4) is equivalent to the attractivity of the tracking manifold of (1).

Assumption A2 (Stabilizability of the Tracking Manifold): There exists a smooth function $\bar{u}(\tilde{x}, c_r)$ such that $\bar{u}(0, c_r) = c_r$ and the origin is a uniformly asymptotically stable equilibrium point of $\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, \bar{u}(\tilde{x}, c_r))$, with domain of attraction a closed set $\tilde{\mathcal{D}} \subset \mathbb{R}^n$, i.e., there exists (see [6]) a function $V(\tilde{x}, t)$, defined for $\tilde{x} \in \tilde{\mathcal{D}}$, which is continuous with continuous partial derivatives, and continuous positive definite functions $\alpha_1(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}_\infty$, $\alpha_2(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}$, and $\alpha_3(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}$ such that

$$(i) \quad \alpha_1(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \leq V(\tilde{x}, t) \leq \alpha_2(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \tag{5}$$

$$(ii) \quad \frac{\partial V}{\partial \tilde{x}} \tilde{f}(t, \tilde{x}, \bar{u}(\tilde{x}, c_r)) + \frac{\partial V}{\partial t} \leq -\alpha_3(\|\tilde{x}\|_{\tilde{\mathcal{D}}}), \tag{6}$$

for $\tilde{x} \in \tilde{\mathcal{D}}$, $\tilde{x} \neq 0$, and all $t \geq 0$, where $\|\tilde{x}\|_{\tilde{\mathcal{D}}} \triangleq \max\left\{\|\tilde{x}\|, \frac{1}{\rho(\tilde{x}, \tilde{\mathcal{D}}^o)} - \frac{2}{\rho(0, \tilde{\mathcal{D}}^o)}\right\}$, $\tilde{\mathcal{D}}^o$ is the complement of $\tilde{\mathcal{D}}$ in $\mathbb{R}^n$, and $\rho(\tilde{x}, \tilde{\mathcal{D}}^o)$ denotes the distance of $\tilde{x}$ from the set $\tilde{\mathcal{D}}^o$ (i.e., $\rho(\tilde{x}, \tilde{\mathcal{D}}^o) = \inf_{z \in \tilde{\mathcal{D}}^o} \|\tilde{x} - z\|$).

Note that $\|\tilde{x}\|_{\tilde{\mathcal{D}}} \to \infty$ on the boundary of $\tilde{\mathcal{D}}$. This, together with the fact that $\alpha_1 \in \mathcal{K}_\infty$, implies that $V(\tilde{x}, t)$ is proper on $\tilde{\mathcal{D}}$. Next, let $\mathcal{D} \triangleq \{x \in \mathbb{R}^n \,|\, \tilde{x} \in \mathcal{D}, \text{ for all } t \geq 0\}$ represent the domain of attraction for the tracking manifold in x coordinates. By the boundedness of $x_r(t)$ we have that if $\tilde{\mathcal{D}}$ is compact then $\mathcal{D}$ is also compact.

Assumption 2 states that when $x(t)$, $x_r(t)$, and $c_r(t)$ are available for feedback, the tracking manifold can be made attractive by setting $u = \bar{u}(\tilde{x}, c_r)$. Unfortunately, the problem of calculating $x_r(t)$ and $c_r(t)$ satisfying (3) may be, in general, intractable. Hence, in the following we will seek a way to estimate these functions on-line.

We now assume x in (1) to be observable from the output y. In order to characterize the observability properties of (1), consider the observability mapping $y_x \triangleq [y_1, \ldots, y_1^{(k_1-1)}, \ldots, y_p, \ldots, y_p^{(k_p-1)}]^\top \triangleq \mathcal{H}_x(x, z)$, where $z \triangleq [u_1, \ldots, u_1^{(n_1-1)}, \ldots, u_m, \ldots, u_m^{(n_m-1)}]^\top \in \mathbb{R}^{n_u}$, $\sum_{i=1}^p k_i = n$, $n_u \triangleq n_1 + \ldots + n_m$, $0 \leq n_i \leq \max\{k_1, \ldots, k_p\}$ (when $\mathcal{H}_x$ does not depend on u_i, then we set $n_i = 0$). Note that the vector z contains only the derivatives of u that end up appearing in the mapping $\mathcal{H}_x$ for the application at hand.

Assumption A3 (Observability): System (1) is observable over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, i.e., there exists a set of indices $\{k_1, \ldots, k_p\}$ such that the mapping $y_e = \mathcal{H}_x(x, z)$ is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $z \in \mathcal{U}$.

Notice that A3 does not require (1) to be uniformly completely observable (UCO), i.e., $\mathcal{X} \times \mathcal{U} = \mathbb{R}^n \times \mathbb{R}^{n_u}$, and thus it relaxes analogous conditions commonly found in the literature (see, e.g., [7, 8]).

Next, we introduce a condition to estimate the functions $x_r(t)$ and $c_r(t)$ on-line. Before stating the assumption, note that it is useful to think of (3) as a copy of the plant with unknown state x_r, unknown input c_r, but a known output which is the reference trajectory $r(t)$. Consider a compensator of the type

$$\begin{aligned} \dot{\zeta}_r &= a(\zeta_r, x_r, v_r) \\ c_r &= b(\zeta_r, x_r), \end{aligned} \tag{7}$$

where $\zeta_r \in \mathbb{R}^q$, $v_r \in \mathbb{R}^m$, a and b are sufficiently smooth, and v_r is the new input of the composite system (3)-(7). The order q of the compensator is constrained to be greater than or equal to m, and the compensator is required to be *regular*, i.e., its output is able to reproduce any given function $c_r(t)$, provided $\zeta_r(t)$ and $v_r(t)$ are appropriately chosen. Thus, the regularity of (7) is related to its invertibility with respect to c_r. Define the observability mapping associated with x_r and ζ_r in the composite system (3)-(7) as $y_{x_r,\zeta_r} \triangleq [y_1, \dots, y_1^{(\bar{k}_1-1)}, \dots, y_p, \dots, y_p^{(\bar{k}_p-1)}]^\top \triangleq \mathcal{H}_{x,\zeta}\left(x_r, \zeta_r, v_r, \dots, {v_r}^{(\bar{n}_u-1)}\right)$, where $\sum_{i=1}^p \bar{k}_i = n + q$, $0 \leq \bar{n}_u \leq \max\{\bar{k}_1, \dots, \bar{k}_p\} - 1$. Assume that the compensator (7) is such that the composite system (3)-(7) is observable. Since the observability properties of the plant, characterized by A3, hold for all $[x^\top, z^\top]^\top \in \mathcal{X} \times \mathcal{U}$ which, in general, is a subset of $\mathbb{R}^n \times \mathbb{R}^{n_u}$, there exist trajectories $(x(t), z(t))$ for which the mapping $\mathcal{H}_x$ is not invertible and hence x is not observable. Analogously, since the observability properties stated in A3 clearly hold for the copy of the plant (3), we have that the compensator (7) may generate input trajectories $c_r(t)$ driving (3) in unobservable regions. As a result, the observability properties of the composite system (3)-(7) (and specifically the invertibility of $\mathcal{H}_{x,\zeta}$) will not hold globally, so we will assume the composite system (3)-(7) to be observable on some suitable set $\mathcal{X}_a$ which we will specify in the assumption to follow. Furthermore, we assume that the compensator (7) is such that the mapping $\mathcal{H}_{x,\zeta}$ does not depend on $v_r, \dots, {v_r}^{(\bar{n}_u-1)}$, so that the knowledge of y_{x_r,ζ_r} (the time derivatives of r) is sufficient to calculate x_r and ζ_r (by inverting $\mathcal{H}_{x,\zeta}$) and consequently c_r (by letting $c_r = b(x_r, \zeta_r)$), even without knowing the input v_r. The following assumption states the above requirements in a precise way.

Assumption A4 (Practical Internal Model): There exists a compensator of the form (7), which we call a *practical internal model*, which is regular (i.e., for each $x(0)$ and $u(t)$ there exist $\zeta(0)$ and $v(t)$ such that $b(\zeta, x) = u$, for all $t \geq 0$) and such that the following two properties hold for the composite system (3)-(7).
(i) $\mathcal{H}_{x,\zeta}$ does not depend on v_r and its derivatives, i.e., $\mathcal{H}_{x,\zeta} = \mathcal{H}_{x,\zeta}(x_r, \zeta_r)$.
(ii) There exists a set of indices $\{\bar{k}_1, \dots, \bar{k}_p\}$ such that the mapping $y_{x_r,\zeta_r} = \mathcal{H}_{x,\zeta}(x_r, \zeta_r)$ is invertible with respect to x_r and ζ_r, and its inverse is sufficiently smooth, for all $[{x_r}^\top, {\zeta_r}^\top]^\top \in \mathcal{X}_a$.

If the observability mapping $\mathcal{H}_x$ of the plant does not depend on u then A4 is automatically satisfied and the practical internal model is simply given by m integrators, $\dot{z}_i = v_i,\ u_i = \zeta_i,\ \ i = 1, \dots, m$. In the general case when $\mathcal{H}_x$ depends on u and its time derivatives, one can show that a sufficient condition for A4 to be satisfied is that the plant (1) is differentially flat with respect to the output y (see [9]). Notice also that there is a relationship between the observability domain $\mathcal{X} \times \mathcal{U}$ for the plant and that of the composite system, $\mathcal{X}_a$. In particular, by denoting $\mathcal{P}_x(\mathcal{X}_a)$ the projection of $\mathcal{X}_a$ onto the x subspace and by $b(\mathcal{X}_a)$ the image of $\mathcal{X}_a$ under the mapping b one has that $\mathcal{P}_x(\mathcal{X}_a) \times b(\mathcal{X}_a) \supset \mathcal{X} \times \mathcal{U}$. Next, we need to guarantee that the reference trajectory is contained in within an observable region.

Assumption A5 (Reference Trajectory): The reference trajectory $r(t)$ is such that, for all $t \geq 0$, $\left[r_1(t), \dots, r_1^{(\bar{k}_1-1)}(t), \dots, r_p(t), \dots, r_p^{(\bar{k}_p-1)}(t)\right]^\top \in C_r \subset \mathcal{H}_{x,\zeta}(\mathcal{X}_a)$, for some convex compact C_r.

This condition may be relaxed by requiring it to be satisfied asymptotically, rather than for all $t \geq 0$. In this case, slight modifications in the proofs show that the results of this paper still hold. Finally, we need to make sure that the state and input trajectories of the closed-loop system travel within the observable domain of the plant (at least in the ideal case when the state feedback controller is employed). To this end, in the following assumption we characterize a subset of the domain of attraction $\mathcal{D}$ which is contained within an observable region of (1). Given any scalar $c > 0$ let $\Omega_c \triangleq \{x \in \mathbb{R}^n \,|\, V(x - x_r, t) \leq c, \text{ for all } t \geq 0\}$, where $V(\tilde{x}, t)$ is defined in A2, and note that for all $c > 0$ we have that, by the properness of V and the definition of $\mathcal{D}$, $\Omega_c \subset \mathcal{D}$. Let Ω^z be the compact set which is invariant with respect to the z trajectories (its existence follows from the smoothness of $\bar{u}$ and the boundedness of $x(t), x_r(t)$, and $c_r(t)$), and consider the mapping $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_u}$, $\mathcal{F}(x, z) \triangleq [\mathcal{H}_x(x, z)^\top, z^\top]^\top$ which, clearly, is a diffeomorphism on $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, and assume that the set $\mathcal{X} \times \mathcal{U}$ satisfies the following.

Assumption A6 (Topology of $\mathcal{X} \times \mathcal{U}$): There exists a positive scalar $\bar{c}$ such that $\mathcal{F}(\Omega_{\bar{c}}, \Omega^z) \subset C_\xi \subset \mathcal{F}(\mathcal{X}, \mathcal{U})$, for some convex compact C_ξ.

This assumption, taken from [10], expresses the requirement that there exist invariant sets $\Omega_{\bar{c}}$ and Ω^z for x and z, respectively, which can be inscribed in an observable domain whose image under $\mathcal{F}$ (the set C_ξ) is convex. Note that A6 implies that, when $x = x_r$ and $u = c_r$ (i.e., when tracking is achieved), the state of the plant is observable. This is a trivial necessary condition for the solvability of the tracking problem by output feedback.

3 Solution of Problem 1 Using a Separation Principle

In this section we will develop dynamic output feedback controllers solving Problem 1 using a separation principle. In order to do that we will formulate the tracking problem in a way that will allow us to use the methodology developed in [11, 10] for output feedback stabilization. Our scheme employs two observers, the first one is used to estimate the functions x_r and ζ_r (and hence also c_r), while the second one estimates the state of system (1). Projection algorithms are employed to keep the observer estimates in within the observable regions while preserving their convergence properties. Next, the projected estimates are employed by the stabilizer $\bar{u}$ to drive the closed-loop trajectories inside an arbitrarily small neighborhood of the tracking manifold.

3.1 Estimation of $x_r(t), c_r(t)$

By using A1 and the regularity of compensator (7) we have that there exists sufficiently smooth and bounded functions $\zeta_r(t)$ and $v_r(t)$ such that

$$\begin{aligned} \dot{x_r} &= f(x_r, b(\zeta_r, x_r)) \\ \dot{\zeta}_r &= a(\zeta_r, x_r, v_r) \\ r(t) &= h(x_r, b(\zeta_r, x_r)) \end{aligned} \tag{8}$$

where $c_r = b(\zeta_r, x_r)$. Rewrite (8) as

$$\begin{aligned} \dot{x_a} &= f^a(x_a, v_r) \\ r(t) &= h(x_a) \end{aligned} \tag{9}$$

where $x_a = [{x_r}^\top, {\zeta_r}^\top]^\top$, and notice that A4 implies that x_a is observable with respect to $r(t)$, viewed as an output of system (9). The input of system (9), the function v_r, is in general not known but, by A4, point (i), we have that the observability mapping $\mathcal{H}_{x,\zeta}$ associated with (9) does not depend on it and, by A5, the identity $[{x_r}^\top, {\zeta_r}^\top]^\top = \mathcal{H}_{x,\zeta}^{-1}\left([r_1, \dots, r_1^{(\bar{k}_1-1)}, \dots, r_p, \dots, r_p^{(\bar{k}_p-1)}]\right)$ is well-defined for all $t \geq 0$. Hence, in theory, the problem of estimating x_r, c_r could be solved by using the approach of Teel and Praly in [12], i.e., by estimating the derivatives of the reference trajectory $r(t)$, inverting the nonlinear map $\mathcal{H}_{x,\zeta}$, and setting $c_r = b(\zeta_r, x_r)$. Assuming the analytical knowledge of $\mathcal{H}_{x,\zeta}^{-1}$ is, however, too restrictive and hence, instead of estimating the derivatives of $r(t)$, we will rely on the nonlinear observer introduced in [11] to estimate x_r and ζ_r, directly. The observer for (9) has the form

$$\begin{aligned} \dot{\hat{x}}_a &= f^a(\hat{x}_a, 0) + \left[\frac{\partial \mathcal{H}_{x,\zeta}(\hat{x}_a)}{\partial \hat{x}_a}\right]^{-1} (\mathcal{E}^a)^{-1} L^a (r - \hat{r}) \\ \hat{r} &= h(\hat{x}_a) \end{aligned} \tag{10}$$

where $\hat{x}_a = [{\hat{x}_r}^\top, {\hat{\zeta}_r}^\top]^\top$, $L^a = \textit{block-diag}[L_1^a, \dots, L_p^a]$, L_i^a are Hurwitz vectors of dimension $\bar{k}_i \times 1$, for $i = 1, \dots, p$, and $\mathcal{E}^a = \textit{block-diag}[\mathcal{E}_1^a, \dots, \mathcal{E}_p^a]$, where $\mathcal{E}_i^a = \textit{diag}[\varepsilon, \varepsilon^2, \dots, \varepsilon^{\bar{k}_i}]$, and $\varepsilon \in \mathbb{R}$. Notice that the unknown function v_r is replaced by 0 in (10). This is due to the fact that here we estimate the state of the dynamical system (9) in the presence of an unknown input. In contrast to the result in [13] where a similar observer is employed, the presence of v_r here does not allow us to guarantee asymptotic convergence of the estimation error. The following weaker convergence result, however, will be sufficient for the purpose of solving Problem 1.

Theorem 1 *Consider system (9) and suppose A4 is satisfied with $\mathcal{X}_a = \mathbb{R}^n \times \mathbb{R}^q$. Then, the observer (10) guarantees that $\hat{x}_a(t)$ is bounded for all $t \geq 0$ and, for all $\delta, T > 0$ there exists $\bar{\varepsilon}$, $0 < \bar{\varepsilon} \leq 1$ such that $\|\hat{x}_a - x_a\| \leq \delta$ for all $t \geq T$, whenever $\varepsilon \in (0, \bar{\varepsilon})$.*

3.2 Estimation of $x(t)$

Next, we turn our attention to the problem of estimating the state x in (1). To this end, notice that the mapping $\mathcal{H}_x$ depends on $z = [u_1, \dots, u_1^{(n_1-1)}, \dots, u_m, \dots, u_m^{(n_m-1)}]^\top$. The standard approach to estimate x entails adding m chains of integrators at the input side of (4):

$$\begin{aligned} &\dot{z}_{i,j} = z_{i,j+1}, \quad j = 1, \dots, n_i - 1 \\ &\dot{z}_{i,n_i} = u_i', \qquad i = 1, \dots, m \\ &u_i = z_{i,1} \end{aligned} \tag{11}$$

and redesigning an appropriate control law for the augmented system. In this way, the states of the i-th chain of integrators represent the derivatives of u_i. Next, one can get x by estimating the vector y_x containing the time derivatives of y and setting $\hat{x} = \mathcal{H}_x^{-1}(\hat{y}_x, z)$. This idea, originally introduced by Tornambé in [7], has later been employed throughout the output feedback control literature (see, e.g., [12]). The additional design complication of finding a control law for the augmented system, however, is a shortcoming one would like to avoid. In the following, we will adopt one of the two observer designs introduced in [10] which does not require the analytical calculation of the inverse of $\mathcal{H}_x$ and, moreover, avoids the design complication mentioned above by estimating the vector z. The reader may refer to [10] for the details; here we will briefly illustrate the proposed estimator and provide the convergence result without proof. Consider the following nonlinear observer,

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, \hat{u}) + \left[\frac{\partial \mathcal{H}_x(\hat{x}, \hat{z})}{\partial \hat{x}}\right]^{-1} \Big[(\mathcal{E}^x)^{-1} L\,(y - \hat{y}) \\ &\qquad - \frac{\partial \mathcal{H}_x(\hat{x}, \hat{z})}{\partial \hat{z}} (\mathcal{E}^z)^{-1} K C (u - \hat{u})\Big] \\ \hat{y} &= h(\hat{x}, \hat{u}) \end{aligned} \tag{12}$$

$$
\begin{aligned}
\dot{\hat{z}} &= A\hat{z} + (\mathcal{E}^z)^{-1}K(u-\hat{u}) \\
\hat{u} &= C\hat{z}
\end{aligned} \tag{13}
$$

where $L = block\text{-}diag[L^1,\dots,L^p], K = block\text{-}diag[K^1,\dots,K^m]$, and L^i, K^j are Hurwitz vectors of dimension $k_i \times 1$ and $n_j \times 1$, respectively, for $i = 1,\dots,p$, $j = 1,\dots,m$. Analogously, we let $A = block\text{-}diag[A^1,\dots,A^m]$, $B = block\text{-}diag[B^1,\dots,B^m]$, $C = block\text{-}diag[C^1,\dots,C^m]$, where A^i, B^i,and C^i are in controllable/observable canonical form and have dimensions $n_i \times n_i$, $n_i \times 1$, and $1 \times n_i$, respectively. Finally, $\mathcal{E}^x = block\text{-}diag[\mathcal{E}^x_1,\dots,\mathcal{E}^x_p]$, where $\mathcal{E}^x_i = diag[\rho,\rho^2,\dots,\rho^{k_i}]$ and $\rho \in \mathbb{R}$, and $\mathcal{E}^z = block\text{-}diag[\mathcal{E}^z_1,\dots,\mathcal{E}^z_p]$, where $\mathcal{E}^z_j = diag[\epsilon,\epsilon^2,\dots,\epsilon^{n_j}]$ and $\epsilon \in \mathbb{R}$. Let $U \triangleq [u_1^{(n_1)},\dots,u_m^{(n_m)}]^\top$, then the vector $\dot{z}$ can be expressed as $\dot{z} = Az + BU$, and $u = Cz$.

Theorem 2 *[10] Consider system (1) and suppose A3 is satisfied for $\mathcal{X} = \mathbb{R}^n, \mathcal{U} = \mathbb{R}^{n_u}$, x and z are confined to within a compact invariant set Ω, and U is bounded for all $t \geq 0$. Then, the cascaded observers (12)-(13) guarantee that $\hat{x}$ and $\hat{z}$ are bounded for all $t \geq 0$, and the estimation error converges arbitrarily fast to an arbitrarily small neighborhood of the origin, i.e., for all $\delta, T > 0$, there exist $\bar{\rho}, \bar{\epsilon}$, $0 < \bar{\rho}, \bar{\epsilon} \leq 1$, such that $\|\hat{x} - x\| \leq \delta$, $\|\hat{z} - z\| \leq \delta$, for all $t \geq T$, whenever $\rho \in (0,\bar{\rho}), \epsilon \in (0,\bar{\epsilon})$.*

3.3 Observer Estimates Projection

The estimates $\hat{x}_a = [\hat{x}_r{}^\top(t), \hat{\zeta}_r{}^\top(t)]^\top$, $\hat{x}(t)$, and $\hat{z}(t)$ generated by the observers (10) and (12)-(13) cannot be directly used in the closed-loop design for two reasons. First, the observer estimates are well-defined only when A3 and A4 hold globally. Furthermore, the observers (10)-(12) and (13) exhibit "peaking" which may destroy the closed-loop stability. In order to eliminate the peaking phenomenon and, at the same time, guarantee that the observer estimates $\hat{x}_a(t)$, $\hat{x}(t)$, and $\hat{z}(t)$ are confined to within observable regions, we apply the projection idea of [11, 10]. In what follows we define the projection of $\hat{x}_a(t)$ on the compact set $\mathcal{H}^{-1}_{x,\zeta}(C_r)$, where C_r is defined in A5. Let $\xi_r = \mathcal{H}_{x,\zeta}(x_r,\zeta_r)$, $\hat{\xi}_r = \mathcal{H}_{x,\zeta}(\hat{x}_r,\hat{\zeta}_r) = \mathcal{H}_{x,\zeta}(\hat{x}_a)$, and note that $\dot{\hat{\xi}}_r = (\partial\mathcal{H}_{x,\zeta}(\hat{x}_a)/\partial\hat{x}_a)\,\dot{\hat{x}}_a$, which is well-defined when $\hat{x}_a \in \mathcal{X}_a$. The projection is defined by

$$
\dot{\hat{x}}_a^P = \left[\frac{\partial\mathcal{H}_{x,\zeta}}{\partial\hat{x}_a}\right]^{-1}\left\{\mathcal{P}_a(\hat{x}_a, \dot{\hat{x}}_a)\right\} \tag{14}
$$

$$
\mathcal{P}_a(\hat{x}_a, \dot{\hat{x}}_a) = \begin{cases} \dot{\hat{\xi}}_r - \Gamma\,\dfrac{N_r(N_r\,\dot{\hat{\xi}}_r)}{N_r^\top\Gamma N_r} & \text{if } N_r^\top\,\dot{\hat{\xi}}_r \geq 0 \\ & \text{and } \hat{\xi}_r \in \partial C_r \\ \dot{\hat{\xi}}_r & \text{otherwise} \end{cases}
$$

where $\Gamma_1 = (S_1\bar{\mathcal{E}}^a)^{-1}(S_1\bar{\mathcal{E}}^a)^{-1}$, $S_1 = S_1^\top$ denotes the matrix square root of P_1, the solution of the Lyapunov equation $(A-LC)^\top P_1 + P_1(A-LC) = -I$, $\bar{\mathcal{E}}^a$ is the scaling introduced in the proof of Theorem 1, and $N_r(\hat{x}_a)$ is the normal vector to ∂C_r, the boundary of the set C_r, at $\hat{x}_a$. A straightforward adaptation of Lemma 1 in [11] to the case under consideration shows that, if A5 holds and (14) is applied to (10), $\hat{x}_a^P(t) = [\hat{x}_r^{P\top}, \hat{\zeta}_r^{P\top}]^\top$ is bounded and contained in the observable region $\mathcal{X}_a$ for all $t \geq 0$. Moreover, the original convergence characteristics of the observer in Theorem 1 are preserved.

Next, we turn our attention to the estimates $\hat{x}(t)$ and $\hat{z}(t)$. In order to guarantee that the observer (12)-(13) is well-defined when the observability assumption A3 does not hold globally, we need to apply the projection to $\hat{x}$ and $\hat{z}$. Let $\xi = \mathcal{H}_x(x,z)$, $\hat{\xi} = \mathcal{H}_x(\hat{x},\hat{z})$, and note that $\dot{\hat{\xi}} = [\partial\mathcal{H}_x(\hat{x},\hat{z})/\partial\hat{x}]\,\dot{\hat{x}} + [\partial\mathcal{H}_x(\hat{x},\hat{z})/\partial\hat{z}]\,\dot{\hat{z}}$, which is well-defined when $\hat{x} \in \mathcal{X}$, $\hat{z} \in \mathcal{U}$. Further, let P_2 be the positive definite solution of the Lyapunov equation $P_2(A-KC) + (A-KC)^\top P_2 = -I$, and $\bar{\mathcal{E}}^x \triangleq block\text{-}diag[\bar{\mathcal{E}}^x_1,\dots,\bar{\mathcal{E}}^x_p]$, $\bar{\mathcal{E}}^z \triangleq block\text{-}diag[\bar{\mathcal{E}}^z_1,\dots,\bar{\mathcal{E}}^z_m]$, where $\bar{\mathcal{E}}^x_i \triangleq diag\left[1/\rho^{k_i-1},\dots,1\right]$, $i = 1,\dots,p$, and $\bar{\mathcal{E}}^z_i \triangleq diag\left[1/\epsilon^{n_i-1},\dots,1\right]$, $i = 1,\dots,m$. We are now ready to define the projections for $\hat{x}(t)$ and $\hat{z}(t)$:

$$
\dot{\hat{x}}^P = \left[\frac{\partial\mathcal{H}_x}{\partial\hat{x}}\right]^{-1}\left\{\mathcal{P}_1\left(\hat{\xi},\dot{\hat{\xi}},\hat{z},\dot{\hat{z}}\right) - \frac{\partial\mathcal{H}_x}{\partial\hat{z}}\dot{\hat{z}}^P\right\}
$$

$$
\mathcal{P}_1\left(\hat{\xi},\dot{\hat{\xi}},\hat{z},\dot{\hat{z}}\right) = \begin{cases} \dot{\hat{\xi}} - \Gamma_1\dfrac{N_\xi\left(N_\xi^\top\dot{\hat{\xi}} + N_z^\top\dot{\hat{z}}\right)}{N_\xi^\top\Gamma_1 N_\xi + N_z^\top\Gamma_2 N_z} \\ \quad \text{if } N_\xi^\top\dot{\hat{\xi}} + N_z^\top\dot{\hat{z}} \geq 0 \\ \quad \text{and } [\hat{\xi}^\top,\hat{z}^\top]^\top \in \partial C_\xi \\ \dot{\hat{\xi}} \quad \text{otherwise} \end{cases} \tag{15}
$$

$$
\dot{\hat{z}}^P = \mathcal{P}_2\left(\hat{\xi},\dot{\hat{\xi}},\hat{z},\dot{\hat{z}}\right)
$$

$$
\mathcal{P}_2\left(\hat{\xi},\dot{\hat{\xi}},\hat{z},\dot{\hat{z}}\right) = \begin{cases} \dot{\hat{z}} - \Gamma_2\dfrac{N_z\left(N_\xi^\top\dot{\hat{\xi}} + N_z^\top\dot{\hat{z}}\right)}{N_\xi^\top\Gamma_1 N_\xi + N_z^\top\Gamma_2 N_z} \\ \quad \text{if } N_\xi^\top\dot{\hat{\xi}} + N_z^\top\dot{\hat{z}} \geq 0 \\ \quad \text{and } [\hat{\xi}^\top,\hat{z}^\top]^\top \in \partial C_\xi \\ \dot{\hat{\xi}} \quad \text{otherwise} \end{cases} \tag{16}
$$

where $\Gamma_2 = (S^2\bar{\mathcal{E}}^z)^{-1}(S^2\bar{\mathcal{E}}^z)^{-1}$ and $S^2 = S^{2\top}$ denotes the matrix square root of P_2. Finally, N_ξ, N_z are the ξ and z components of the normal vector N to ∂C_ξ, the boundary of C_ξ, at $\hat{\xi}$, $\hat{z}$ i.e., $N(\hat{\xi},\hat{z}) = \left[N_\xi^\top(\hat{\xi},\hat{z}), N_z^\top(\hat{\xi},\hat{z})\right]^\top$. Using projection (15), (16) and A6, it is again possible to prove that $\hat{x}^P(t)$ and $\hat{z}^P(t)$ are contained in the observable region $\mathcal{X} \times \mathcal{U}$ for all $t \geq 0$ and the results of Theorem 2 are preserved.

3.4 Solution of Problem 1

Taking in account the restriction on $\bar{c}$ imposed by A6, choose a scalar $\underline{c}$ such that $0 < \underline{c} < \bar{c}$ (therefore $\Omega_{\underline{c}} \subset \Omega_{\bar{c}} \subset \mathcal{D}$).

Theorem 3 *Suppose that A1-A6 hold. Then, there exist positive scalars $\bar{\varepsilon}, \bar{\rho}, \bar{\epsilon}$ such that the output feedback practical tracking problem is solvable on $\mathcal{A} = \Big\{x \in \mathbb{R}^n, x_c \in \mathbb{R}^{2n+n_u+q} \,|\, x(0) \in \Omega_{\underline{c}}, \hat{x}_a^P(0) \in \mathcal{H}_{x,\zeta}^{-1}(C_r)\,, [\hat{x}^{P\top}(0), \hat{z}^{P\top}(0)]^\top \in \mathcal{F}^{-1}(C_\xi)\Big\}$, for any $0 < \underline{c} < \bar{c}$, by letting $u = \bar{u}(\hat{x}^P - \hat{x}_r^P, b(\hat{\zeta}_r^P, \hat{x}_r^P))$ be the output of the dynamic output feedback controller given by the observers (10), (12)-(13), and the projections (14), (15)-(16), with state $x_c \triangleq [\hat{x}_a^{P\top}, \hat{x}^{P\top}, \hat{z}^{P\top}]^\top = [\hat{x}_r^{P\top}, \hat{\zeta}_r^{P\top}, \hat{x}^{P\top}, \hat{z}^{P\top}]^\top \in \mathbb{R}^{2n+n_u+q}$, and choosing $\varepsilon \in (0, \bar{\varepsilon})$, $\rho \in (0, \bar{\rho})$, $\epsilon \in (0, \bar{\epsilon})$.*

Theorem 3 implies that, if A2, A3, and A4 hold globally, then the solution of Problem 1 is semiglobal, i.e., C_ξ and C_r can be chosen arbitrarily large, and $\mathcal{A}$ can be chosen to be an arbitrarily large compact set in $\mathbb{R}^{2n+n_u+q}$. Recovering the asymptotic stability of the tracking error requires more stringent conditions than those introduced in Section 2.

Conclusions. We have shown that the output feedback (practical) tracking problem can be solved if one can find a compensator (the practical internal model) yielding suitable observability properties in the closed-loop system. It can be shown (see [9]) that, within an output regulation framework, the methodology introduced here can be used to yield semiglobal asymptotic tracking provided the plant equation is not affected by any disturbance. Hence, it must be stressed that, contrary to current results in output regulation theory (see, e.g., [14]), the methodology presented here *does not* handle the presence of uncertainties or disturbances: more research is needed to address this concern. On the other hand, however, the class of systems considered in this paper is not restricted to be in lower triangular form, nor are the reference trajectories restricted to be the outputs of a known exosystem. In this respect, the practical internal model may be viewed as a robust counterpart of the standard internal model, in that it can be used when the information about the exosystem is not accurate, or even when the exosystem is not present at all. Finding necessary and sufficient conditions for the existence of a practical internal model, as well as a constructive methodology to find it, represent open research topics.

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