Necessary and Sufficient Graphical Conditions for Formation Control of Unicycles

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Abstract–The feasibility problem is studied of achieving a specified formation among a group of autonomous unicycles by local distributed control. The directed graph defined by the information flow plays a key role. It is proved that formation stabilization to a point is feasible if and only if the sensor digraph has a globally reachable node. A similar result is given for formation stabilization to a line and to more general geometric arrangements.

Index Terms–Multi-agent system, distributed control, nonholonomic mobile robots.

I. Introduction

The problem of coordinated control of a group of autonomous wheeled vehicles is of recent interest in control and robotics. Over the past decade, many researchers have worked on formation control problems with differences regarding the types of agent dynamics, the varieties of the control strategies, and the types of tasks demanded. In 1990, Sugihara and Suzuki [1] proposed a simple algorithm for a group of point-mass type robots to form approximations to circles and simple polygons. And in the years following, distributed algorithms were presented in [2]–[4] with the objective of getting a group of such robots to congregate at a common location: This is termed an *agreement problem* [3] or *rendezvous problem* [4]. In [5], Jadbabaie et al. studied a different agreement problem: getting autonomous agents in the plane to move in a common direction. In addition to the references mentioned so far, mathematical analysis and control synthesis for formation control of point-mass type robots were developed in [6]–[8] by different approaches. With regard to a group of wheeled vehicles with nonholonomic constraints, the formation control problem with different objectives was investigated in [9]–[13]. Other relevant recent references are [14]–[21].

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Published in *IEEE Transactions on Automatic Control*, vol. 50, no. 1, 2005, pp. 121–127

As a natural extension of our previous work [7], [11] and motivated by the proposed strategy in [12], [13], in this paper the feasibility problem is studied of achieving a specified formation among a group of unicycles by distributed control. Each unicycle relies only on locally available information, namely, the relative displacements to certain neighbors; in particular, we do not assume that the unicycles possess a common reference frame.

Central to a discussion of formation control is the nature of the information flow throughout the formation. This information flow can be modeled by a directed graph (digraph for short), where a link from node i to node j indicates that vehicle i has access in some way to the position of vehicle j—but only with respect to the local coordinate frame of vehicle i. Such a digraph is assumed in this paper to be static—the dynamic case, where ad hoc links can be established or dropped, is a future topic. We emphasize that modeling the information flow with a static digraph may not accurately model realistic situations whereby sensors have a limited field of view. At the same time, investigating feasibility of formations with static digraphs is a necessary step towards the more realistic dynamic setting. With this in mind, in this paper we use the term *sensor digraph* to denote the digraph defined above. Our analysis relies on several tools from algebraic graph theory, non-negative matrix theory, and averaging theory. We introduce a new concept for our analysis: $H(\alpha, m)$ stability of the Laplacian of the digraph.

Our first main result is that formation stabilization to a common point is feasible if and only if the sensor digraph has a globally reachable node (a node to which there is a directed path from every other node). That is, there exists at least one unicycle that is viewable, perhaps indirectly by hopping from one unicycle to another, by all other unicycles. This is precisely the degree of connectedness required and is much weaker than strong connectedness of the sensor digraph (as in cyclic pursuit [11], for example). Our proof of sufficiency is constructive: We present an explicit smooth periodic feedback controller, and prove convergence using averaging theory.

Our second main result concerns formation stabilization to a line. This turns out to be feasible if and only if there are at most two disjoint closed sets of nodes in the sensor digraph. In addition, we introduce a special sensor digraph which guarantees that all vehicles converge to a line segment, equally spaced. This is an extension to unicycles of a line-formation scheme of Wagner and Bruckstein [22].

Finally, we show how formation stabilization to a common point can be adapted to any geometric pattern if a group of unicycles have a common sense of direction.

II. Problem Statement and Main Results

Before treating unicycles, it is perhaps illuminating to give a result for the much simpler case of point masses. Consider n "point-mass robots" whose positions are modeled by complex numbers, $z_1, \ldots, z_n$, in the plane. Assume a kinematic model of velocity control: $\dot{z}_i = u_i$. Assume each robot obtains the relative positions of a subgroup, N_i, of the other robots. Let y_i denote the vector whose components are the relative positions $z_m - z_i$, as m ranges over N_i. Thus y_i, a vector of dimension the cardinality of N_i, represents the information available to u_i. We allow controllers of the form $u_i = F_i y_i$, or $u_i = 0$ if N_i is empty. Thus, $y_i = 0 \Longrightarrow u_i = 0 \Longrightarrow \dot{z}_i = 0$; that is, robot i does not move if all robots it senses are collocated with it (or if there is no information available to it). The problem of convergence to a common point is this:

Problem 0: Find, if possible, $F_1, \ldots, F_n$ such that

$$(\forall\ z_i(0), i = 1, \ldots, n)(\exists\ z_{ss})(\forall\ i)\ \lim_{t\to\infty} z_i(t) = z_{ss}.$$

Now define the sensor digraph $\mathcal{G}$ for this setup: There is a directed edge from node i to node m if and only if $m \in N_i$.

Before giving our results, we review some notions in graph theory. For a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{1, \ldots, n\}$, if there is a path in $\mathcal{G}$ from one node i to another node j, then j is said to be *reachable* from i, written $i \rightarrow j$. If not, then j is said to be not reachable from i, written $i \nrightarrow j$. If a node i is reachable from every other node in $\mathcal{G}$, then we say it is *globally reachable*. If $\mathcal{U}$ is a nonempty subset of $\mathcal{V}$ and $i \nrightarrow j$ for all $i \in \mathcal{U}$ and $j \in \mathcal{V} - \mathcal{U}$, then $\mathcal{U}$ is said to be *closed*. More information can be found in [23], [24].

Theorem 0: Problem 0 is solvable if and only if $\mathcal{G}$ has a globally reachable node. Moreover, when Problem 0 is solvable, one solution is $F_i = [1\ \cdots\ 1]$.

The easy proof is omitted.

Now we turn to the main topic of unicycles. We can identify the real plane, $\mathbb{R}^2$, and the complex plane, $\mathbb{C}$, by identifying a column vector, z_i, and a complex number, $\mathbf{z}_i$. Now consider a wheeled vehicle with coordinates (x_i, y_i, θ_i) with respect to a global frame ${}^g\Sigma$ (see Fig. 1). The location of the vehicle in the plane is $z_i = [x_i\ y_i]^T$ or $\mathbf{z}_i = x_i + jy_i$. The vehicle has the nonholonomic constraint of pure rolling and non-slipping and is described kinematically as

$$\left\{\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i), \\ \dot{y}_i &= v_i \sin(\theta_i), \\ \dot{\theta}_i &= \omega_i, \end{aligned}\right. \qquad \text{or} \qquad \left\{\begin{aligned} \dot{\mathbf{z}}_i &= v_i e^{j\theta_i}, \\ \dot{\theta}_i &= \omega_i. \end{aligned}\right.$$

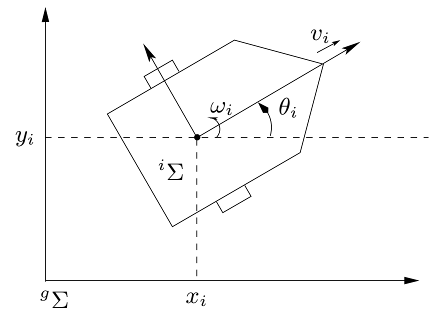

Fig. 1. Wheeled vehicle.

Following [10], we construct a moving frame ${}^i\Sigma$, the Frenet-Serret frame, that is fixed on the vehicle (see Fig. 2). Let r_i be the unit vector tangent to the trajectory at the current location of the vehicle (r_i is the normalized velocity vector) and let s_i be r_i rotated by $\pi/2$. Thus, $\dot{z}_i = v_i r_i$ since the vehicle is moving at speed v_i.

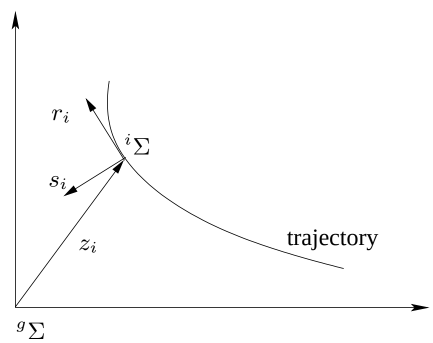

Fig. 2. Frenet-Serret frame.

Now consider n wheeled vehicles, indexed by i. We refer to the individual vehicles as nodes and the information flows as links. Although the vehicles in the group are dynamically decoupled, meaning the motion of one vehicle does not directly affect any of the other vehicles, they are coupled through the information flow. Let N_i denote the set of labels of those vehicles accessible by vehicle i and define the sensor digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: There is a directed edge from node i to node m if and only if $m \in N_i$. We refer to this as the *sensor digraph*.

In this paper, we assume N_i is time-invariant, meaning the information flow topology is static. In the control law that we study, no vehicle can access the absolute positions of other vehicles or its own. Specifically, vehicle i can only get the relative positions of a subgroup of vehicles with respect to its

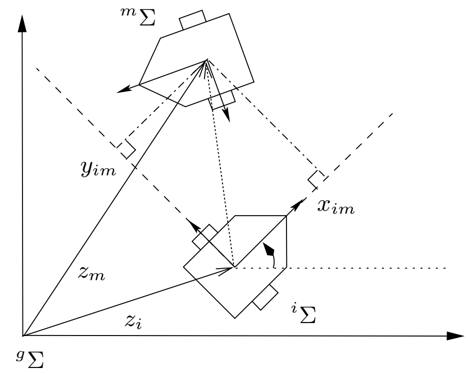

Fig. 3. Local information.

own Frenet-Serret frame (see Fig. 3),

$$\begin{cases} x_{im} = (z_m - z_i) \cdot r_i, \\ y_{im} = (z_m - z_i) \cdot s_i, \end{cases} \quad m \in N_i,$$

where dot denotes dot product. This leads to the following definition.

Definition 1: A controller (v_i, w_i), $i = 1, \ldots n$, is said to be a *local information controller* if

$$\begin{cases} v_i & = g_i(t, x_{im}, y_{im})|_{m \in N_i}, \\ \omega_i & = h_i(t, x_{im}, y_{im})|_{m \in N_i} \end{cases} \quad i = 1, \ldots, n$$

where g_i is such that $\{(\forall\ m \in N_i)\ z_m = z_i\} \Rightarrow \{v_i = 0\}$.

Notice that in our definition a vehicle does not translate (but it can rotate) when either it cannot obtain local information from any other vehicle or its neighbors have all converged to its position.

In what follows, we present the two main problems investigated in this paper, together with necessary and sufficient conditions for their solutions.

Problem 1: (Formation Stabilization to a Point) Find, if possible, a local information controller such that for all $(x_i(t_0), y_i(t_0), \theta_i(t_0)) \in \mathbb{R}^3$, $i = 1, \ldots n$, and all $t_0 \in \mathbb{R}$,

$$(\exists\ z_{ss} \in \mathbb{R}^2)\ (\forall\ i)\ \lim_{t \to \infty} z_i(t) = z_{ss}.$$

Theorem 1: (Section III) Problem 1 is solvable if and only if the sensor digraph has a globally reachable node.

Problem 2: (Formation Stabilization to a Line) Find, if possible, a local information controller such that for all $(x_i(t_0), y_i(t_0), \theta_i(t_0)) \in \mathbb{R}^3$, $i = 1, \ldots n$, and all $t_0 \in \mathbb{R}$, all vehicles converge to form a line.

Theorem 2: (Section IV) Problem 2 is solvable if and only if there are at most two disjoint closed sets of nodes in the sensor digraph.

In Section IV we also introduce a special sensor digraph which guarantees that all vehicles converge to a line segment, equally spaced. In Section V we show how our solution to Problem 1 can be employed to achieve formation stabilization to any geometric pattern.

III. Formation Stabilization to a Point

In this section we prove Theorem 1. The proof requires the following lemmas.

Lemma 1: [1] A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| \geq 2$ has no globally reachable node if and only if it has two disjoint closed subsets of $\mathcal{V}$.

Proof: (*Sufficiency*) Sufficiency follows directly.

(*Necessity*) We prove necessity by means of a constructive algorithm. Firstly, select any node, say v_{i_1}, in $\mathcal{V}$ and partition $\mathcal{V}$ as $\mathcal{V} = \{v_{i_1}\} \cup \mathcal{V}_1 \cup \mathcal{V}_1'$, where every node in $\mathcal{V}_1$ can reach v_{i_1} and no node in $\mathcal{V}_1'$ can reach v_{i_1}. Then $\mathcal{V}_1'$ is closed. Also, $\mathcal{V}_1' \neq \phi$, since v_{i_1} is not globally reachable.

Secondly, select any node, say v_{i_2} in $\mathcal{V}_1'$. Since $\mathcal{V}_1'$ is not empty, we can always find one. Check if the node v_{i_2} is globally reachable in the induced subgraph $\mathcal{G}(V_1')$.

If so, then partition $\mathcal{V}$ as $\mathcal{V} = \mathcal{W}_1 \cup \mathcal{W}_1' \cup \mathcal{V}_1'$, where every node in $\mathcal{W}_1$ can reach (some node in) $\mathcal{V}_1'$ and no node in $\mathcal{W}_1'$ can reach $\mathcal{V}_1'$. Then $\mathcal{W}_1'$ is closed. Also, $\mathcal{W}_1' \neq \phi$, since v_{i_2} is not globally reachable. Thus $\mathcal{V}_1'$ and $\mathcal{W}_1'$ are two disjoint closed subsets for the digraph $\mathcal{G}$.

If instead the condition above is false, partition $\mathcal{V}_1'$ as $\mathcal{V}_1' = \{v_{i_2}\} \cup \mathcal{V}_2 \cup \mathcal{V}_2'$, where every node in $\mathcal{V}_2$ can reach v_{i_2} and no node in $\mathcal{V}_2'$ can reach v_{i_2}. Then $\mathcal{V}_2'$ is closed and nonempty. Next, select any node, say v_{i_3}, in $\mathcal{V}_2'$ and check if v_{i_3} is globally reachable in the induced subgraph $\mathcal{G}(V_2')$. If so, then the conclusion follows by the same argument as above. If it does not, repeat this procedure again until this condition holds. Since the digraph has a finite number of nodes and $\mathcal{V}_k'$ is getting smaller each step, eventually the condition must hold and two disjoint closed subsets will have been constructed. ■

The following is a useful algebraic characterization of the property that a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has a globally reachable node, expressed in terms of the Laplacian L. The proof is omitted (see [26]).

Lemma 2: The digraph has a globally reachable node if and only if 0 is a simple eigenvalue of L.

The information flow produces a kind of symmetry in the system equations with respect to the x and y coordinates. For this reason, the Laplacian L leads to the matrix $L_{(2)} = L \otimes I_2$ (Kronecker product),

[1]This lemma is logically equivalent to Theorem 5 of [16]; we became aware of this reference after the first submission of our paper.

which we now study. A definition from [25] will be modified in order to better suit our application. Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be a partition of $\{1, 2, \ldots, n\}$. A block diagonal matrix with diagonal blocks indexed by $\alpha_1, \alpha_2, \ldots, \alpha_p$ is said to be α-*diagonal*.

Definition 2: Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be a partition of $\{1, 2, \ldots, n\}$ and $m \geq 0$ an integer. An $n \times n$ matrix A is said to be $H(\alpha, m)$-stable if

(a) 0 is an eigenvalue of A of algebraic and geometric multiplicity m, while all other eigenvalues have negative real part,

(b) for every α-diagonal positive definite symmetric matrix R, 0 is an eigenvalue of RA of algebraic and geometric multiplicity m, while all other eigenvalues have negative real part.

Lemma 3: Let $\alpha = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$. If the digraph $\mathcal{G}$ is strongly connected, then $-L_{(2)}$ is $H(\alpha, 2)$ stable.

The proof is omitted due to space limitation (see [26]).

Lemma 4: Let $\alpha = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$. The matrix $-L_{(2)}$ is $H(\alpha, 2)$ stable if and only if the digraph $\mathcal{G}$ has a globally reachable node.

Proof: (*Sufficiency*) Let $\mathcal{V}'$ be the set of all the globally reachable nodes. It is not empty but contains either all n nodes or r $(1 \leq r < n)$ nodes. In the former case, the digraph $\mathcal{G}$ is strongly connected and therefore $-L_{(2)}$ is $H(\alpha, 2)$ stable by Lemma 3. In the latter case, we can express L, without loss of generality, by $L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}$, where the associated digraph $\mathcal{G}(L_1)$ is strongly connected and L_3 is nonsingular. Hence, by Lemma 3, $-L_{1_{(2)}}$ is $H(\alpha, 2)$ stable. Furthermore, one can easily verify that L_3 is a nonsingular M-matrix. Then it follows that there exists a positive diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_{(n-r)})$ such that $Q = L_3^T P + P L_3$ is positive definite. Thus for every α-diagonal positive definite symmetric matrix R_2, let $\bar{P} = P_{(2)} R_2^{-1} = R_2^{-1} P_{(2)}$, which is positive definite. Applying properties of the Kronecker product yields $(-R_2 L_{3_{(2)}})^T \bar{P} + \bar{P}(-R_2 L_{3_{(2)}}) = -Q_{(2)}$, which is negative definite. Hence $-L_{3_{(2)}}$ is $H(\alpha, 0)$ stable and therefore $-L_{(2)}$ is $H(\alpha, 2)$ stable.

(*Necessity*) Since $-L_{(2)} = -L \otimes I_2$ is $H(\alpha, 2)$ stable, it follows that $-L_{(2)}$ has a 0 eigenvalue of algebraic multiplicity 2 and then, by a property of the Kronecker product, $-L$ has a simple eigenvalue at 0. Using Lemma 2, the digraph has a globally reachable node. ■

Proof of Theorem 1: (*Sufficiency*) Define the time-varying feedback controller for each vehicle i, $(i = 1, \ldots, n)$

$$\begin{cases} v_i(t) = k \sum\limits_{m \in N_i} x_{im}(t) = k \sum\limits_{m \in N_i} (z_m(t) - z_i(t)) \cdot r_i(t), \\ \omega_i(t) = \cos(t), \end{cases} \tag{1}$$

where $k > 0$.

We begin by noticing that $\{(\forall\ m \in N_i)\ z_m = z_i\} \Rightarrow \{v_i = 0\}$. Further, for any $t_0 \in \mathbb{R}$, $\theta_i(t) = \theta_i(t_0) + \sin(t),\ i = 1, \ldots, n$, which is periodic with period 2π. Next, using the identity $(z \cdot r)r = (rr^T)z$, we get $\dot{z}_i = v_i r_i = k \sum_{m \in N_i} [(z_m - z_i) \cdot r_i]\, r_i = k\ r_i r_i^T \sum_{m \in N_i} (z_m - z_i)$. Define $M(\theta_i(t)) := r_i r_i^T$ and $H(\theta(t)) :=$ diag$(M(\theta_1(t)) \cdots M(\theta_n(t)))$. Thus, the overall position dynamics become

$$\dot{z} = -kH(\theta(t))L_{(2)}z, \tag{2}$$

where $z \in \mathbb{R}^{2n}$ is the position vector $z = [z_1^T\ \cdots\ z_n^T]^T$ and $L_{(2)} = L \otimes I_2$ (L is the Laplacian of the sensor digraph). And the corresponding averaged system of (2) is

$$\dot{z} = -kH_{av}L_{(2)}z, \tag{3}$$

where $H_{av} = diag(\bar{M}_1, \ldots, \bar{M}_n)$, $\bar{M}_i := \begin{bmatrix} m_i^1 & m_i^2 \\ m_i^2 & m_i^3 \end{bmatrix}$, and

$$\begin{aligned} m_i^1 &= \tfrac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta_i(\tau))d\tau, \\ m_i^2 &= \tfrac{1}{2\pi} \int_0^{2\pi} \cos(\theta_i(\tau))\sin(\theta_i(\tau))d\tau, \\ m_i^3 &= \tfrac{1}{2\pi} \int_0^{2\pi} \sin^2(\theta_i(\tau))d\tau. \end{aligned}$$

By the Cauchy-Schwarz inequality, $m_i^1 m_i^3 \geq (m_i^2)^2$. Since $\theta_i(t)$ is not constant, the inequality holds strictly. So $\bar{M}_i$ is positive definite and therefore H_{av} is positive definite. More exactly, H_{av} is α-diagonal positive definite with $\alpha = \{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$. By the condition that the sensor digraph has a globally reachable node, it follows from Lemma 4 that $-L_{(2)}$ is $H(\alpha, 2)$ stable. So there is a similarity transformation F such that $-F^{-1}L_{(2)}F =$ diag$(-L_{11},\ 0_{2\times 2})$, where $-L_{11}$ is Hurwitz and the last two column vectors of F are in the null space of $L_{(2)}$. Without loss of generality, we can choose the last two column vectors of F to be $\mathbf{1} \otimes I_{2\times 2}$. Applying the transformation $e = [e_1^T\ e_2^T]^T = F^{-1}z$ to the system (2), where $e_1 \in \mathbb{R}^{2n-2}, e_2 \in \mathbb{R}^2$, we have

$$\dot{e} = -kF^{-1}H(\theta(t))L_{(2)}Fe = k \begin{bmatrix} A_{11}(t) & 0 \\ A_{12}(t) & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

And correspondingly, for the averaged system (3), we have

$$\dot{e} = -kF^{-1}H_{av}L_{(2)}Fe = k \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{12} & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Now, the reduced averaged system $\dot{e}_1 = k\bar{A}_{11}e_1$ is exponentially stable since $-L_{(2)}$ is $H(\alpha, 2)$ stable. Then, by Theorem 8.3 in [27], there exists a positive constant k^* such that, for all $0 < k < k^*$, global exponential stability of the reduced original system $\dot{e}_1 = kA_{11}(t)e_1$ is established. Also, since $\dot{e}_2 = kA_{12}(t)e_1$ and $A_{12}(t)$ is uniformly bounded, it follows that $\dot{e}_2 \to 0$ exponentially when $t \to \infty$. This implies that e_2 tends to some finite constant vector, say $z_{ss} = [x_{ss}\ y_{ss}]^T$. In conclusion, $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} Fe(t) = \mathbf{1} \otimes z_{ss}$.

(*Necessity*) Assume formation stabilization to a point by local information controller is feasible. By way of contradiction, suppose the sensor digraph has no globally reachable node. Then it follows from Lemma 1 that there are two disjoint closed sets of nodes in the sensor digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, say $\mathcal{V}_1$ and $\mathcal{V}_2$. Given the initial conditions satisfying $z_i(0) = z_{ss_1},\ i \in \mathcal{V}_1$ and $z_j(0) = z_{ss_2},\ j \in \mathcal{V}_2$, then for each vehicle i in $\mathcal{V}_1$, $(\forall\ m \in N_i)\ z_m = z_i$ and so $\dot{z}_i = 0$. Meanwhile, for each vehicle j in $\mathcal{V}_2$, $(\forall\ m \in N_j)\ z_m = z_j$ and so $\dot{z}_j = 0$. Hence, if $z_{ss_1} \neq z_{ss_2}$, they can not gather at the same point, a contradiction. ■

IV. Formation Stabilization to a Line

We begin this section with a proof of Theorem 2.

Proof of Theorem 2: (*Sufficiency*) By the condition that there are at most two disjoint closed sets of nodes in the sensor digraph, by Lemma 1, either the sensor digraph has a globally reachable node, or there are exactly two disjoint closed sets of nodes in it. In the first case, by Theorem 1, formation stabilization to a point is feasible, which is a special instance of line formation. In the second case, we let $\mathcal{V}_1$, $\mathcal{V}_2$ be the two disjoint closed sets of nodes in the sensor digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and let $\mathcal{V}_3 = \mathcal{V} - \mathcal{V}_1 - \mathcal{V}_2$. Thus, the induced subgraphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E} \cap (\mathcal{V}_1 \times \mathcal{V}_1))$, $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E} \cap (\mathcal{V}_2 \times \mathcal{V}_2))$ both have a globally reachable node. (To see this point, suppose one of these two induced subgraphs has no globally reachable set. Then by Lemma 1 there are two disjoint closed sets in it and therefore there are three disjoint closed sets in $\mathcal{G}$, a contradiction.) If $\mathcal{V}_3$ is empty, then by Theorem 1 each group of vehicles whose indices are in $\mathcal{V}_i,\ i = 1, 2,$ can converge to a point and therefore the whole group of wheeled vehicles form a line. On the other hand, if $\mathcal{V}_3$ is not empty, then every node in $\mathcal{V}_3$ can reach either $\mathcal{V}_1$ or $\mathcal{V}_2$. Without loss of generality, we assume the graph Laplacian has the form $L = \begin{bmatrix} L_{11} & 0 & 0 \\ 0 & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$, where L_{11} and L_{22} are the Laplacian matrices corresponding to the induced subgraph $\mathcal{G}_1$ and $\mathcal{G}_2$. One can easily verify that L_{33} is a nonsingular M-matrix.

By using the controller (1), the overall position dynamics are given by

$$\begin{cases} \dot{z}^1 = -kH(\theta^1(t))L_{11_{(2)}} z^1, \\ \dot{z}^2 = -kH(\theta^2(t))L_{22_{(2)}} z^2, \\ \dot{z}^3 = -kH(\theta^3(t)) \left(L_{33_{(2)}} z^3 + L_{31_{(2)}} z^1 + L_{32_{(2)}} z^2 \right), \end{cases}$$

where $z^i, \theta^i,\ i = 1, 2, 3$ are the corresponding position vector and orientation vector respectively. From the proof of Theorem 1, we have that there exist positive constants k_1^* and k_2^* such that, for all $0 < k < k_1^*$, $\lim_{t\to\infty} z^1(t) = \mathbf{1} \otimes z_{ss_1}$, where $z_{ss_1} = [x_{ss_1}\ y_{ss_1}]^T$, and for all $0 < k < k_2^*$, $\lim_{t\to\infty} z^2(t) = \mathbf{1} \otimes z_{ss_2}$, where $z_{ss_2} = [x_{ss_2}\ y_{ss_2}]^T$. The change of variables $\varsigma = L_{33_{(2)}} z^3 + L_{31_{(2)}} z^1 + L_{32_{(2)}} z^2$ yields

$$\dot{\varsigma} = -kL_{33_{(2)}} H(\theta^3(t))\varsigma - kL_{31_{(2)}} H(\theta^1(t))L_{11_{(2)}} z^1 - kL_{32_{(2)}} H(\theta^2(t))L_{22_{(2)}} z^2. \tag{4}$$

Since L_{33} is a nonsingular M-matrix, by the same argument as in the proof of Lemma 6, $-L_{33_{(2)}}$ and $-L_{33_{(2)}}^T$ are both $H(\alpha, 0)$ stable. By Theorem 8.3 in [27], there exists a positive constant k_3^* such that, for all $0 < k < k_3^*$, the origin of the nominal system $\dot{\varsigma} = -kL_{33_{(2)}} H(\theta^3(t))\varsigma$ is globally exponentially stable. Furthermore, notice that the other two terms in (4) both exponentially converge to zero. Hence (4) can be viewed as an exponentially stable system with an exponentially vanishing input, and thus its origin is exponentially stable. Let $k^* = \min\{k_1^*, k_2^*, k_3^*\}$. Hence, for all $0 < k < k^*$,

$$\begin{aligned} \lim_{t\to\infty} z^3(t) &= -\left(L_{33_{(2)}}\right)^{-1} L_{31_{(2)}} \lim_{t\to\infty} z^1(t) - \left(L_{33_{(2)}}\right)^{-1} L_{32_{(2)}} \lim_{t\to\infty} z^2(t) \\ &= -\left(L_{33}^{-1} \cdot L_{31} \cdot \mathbf{1}\right) \otimes z_{ss_1} - \left(L_{33}^{-1} \cdot L_{32} \cdot \mathbf{1}\right) \otimes z_{ss_2}. \end{aligned}$$

Notice that $[L_{31}\ L_{32}\ L_{33}] \cdot \mathbf{1} = 0$ and so $-\left(L_{33}^{-1} \cdot L_{31} \cdot \mathbf{1}\right) - \left(L_{33}^{-1} \cdot L_{32} \cdot \mathbf{1}\right) = \mathbf{1}$. Hence, all $(z_i(t))_{i\in\mathcal{V}_3}$ approach a convex combination of z_{ss_1} and z_{ss_2}, which means the wheeled vehicles with indices in $\mathcal{V}_3$ eventually move to the line formed by two points z_{ss_1} and z_{ss_2} in the plane.

(*Necessity*) Suppose by way of contradiction that there are three disjoint closed sets of nodes in the sensor digraph, say $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$. Then let the initial conditions of the vehicles in $\mathcal{V}_j,\ j = 1, 2, 3$ be chosen such that $z_i(0) = z_{ss_j},\ i \in \mathcal{V}_j$. Hence, for each vehicle i in $\mathcal{V}_j$, $(\forall\ m \in N_i) z_m = z_i$ and so $\dot{z}_i = 0$. Then three groups of vehicles form a geometric pattern specified by three points z_{ss_1}, z_{ss_2} and z_{ss_3}. These three points can be arbitrarily set that may not form a line, a contradiction. ■

Theorem 2 has an interesting special case when the two disjoint closed sets of nodes in the sensor digraph both have only one member, say nodes 1 and n. Vehicles 1 and n are called *edge leaders*. The edge leaders here are not necessarily wheeled vehicles—they can be virtual beacons or landmarks. But the vehicles respond to these edge leaders much like they respond to real neighbor vehicles. The purpose of the edge leaders is to introduce the mission: to direct the vehicle group behavior. We emphasize that the edge leaders are not central coordinators. They do not broadcast instructions. They only play the role of individual vehicles, but cannot sense other vehicles or communicate with them. As for the remaining vehicles, $i,\ i = 2, \ldots, n-1$, we assume that each agent can sense or communicate with agents $i-1$ and $i+1$. This gives the sensor digraph in Fig. 4. It is readily seen that the digraph in Fig. 4 has exactly 2 disjoint closed sets of nodes. We now show that in this special case all vehicles converge to a uniform distribution on the line segment specified by the two edge leaders.

Fig. 4. The sensor digraph for n agents with two edge leaders.

Theorem 3: Consider a group of n wheeled vehicles with two stationary edge leaders labeled 1 and n. Then, there exists a positive constant k^* such that for all $0 < k < k^*$, the following smooth time-varying feedback control law for each vehicle $i,\ (i = 2, \ldots, n-1)$

$$\begin{cases} v_i(t) = k \sum\limits_{j=N_i} x_{ij}(t), \quad N_i = \{i-1, i+1\} \\ \omega_i(t) = \cos(t) \end{cases}$$

guarantees that all the vehicles converge to a uniform distribution on the line segment specified by the two edge leaders.

Proof: Let L be the Laplacian of the sensor digraph in Fig. 4 and let $z = [z_1^T \ \cdots \ z_n^T]^T$. It follows from Theorem 2 that $L_{(2)} z(\infty) = 0$. Consider the following partition of $\{1, 2, \ldots, n\}$, $\{m_1, m_2, \ldots, m_{2n}\} = \{1, 3, \ldots, 2n-1, 2, 4, \ldots, 2n\}$. Then the associated permutation matrix P has the unit coordinate vectors $\mathbf{e}_{m_1}, \mathbf{e}_{m_2}, \ldots, \mathbf{e}_{m_{2n}}$ for its columns. Now observe that the matrix P performs the transformation $P^T(L \otimes I_2)P = I_2 \otimes L = \text{diag}(L, L)$ and $P^T z = [x\ y]^T$, where $x = [x_1 \ \cdots \ x_n]^T$ and $y = [y_1 \ \cdots \ y_n]^T$. Thus, $Lx(\infty) = 0$ and $Ly(\infty) = 0$. Also note that $\text{Ker}(L) = \text{span}\{\xi_1, \xi_2\}$, where $\xi_1 = [0\ 1\ \cdots\ n-1]^T$, $\xi_2 = [n-1\ \ n-2\ \cdots\ 0]^T$, so $x(\infty) = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $y(\infty) = \beta_1 \xi_1 + \beta_2 \xi_2$. Since $x_1(\infty) = x_1(0)$, $x_n(\infty) = x_n(0)$ and $y_1(\infty) = y_1(0)$, $y_n(\infty) = y_n(0)$, so we solve for $\alpha_1 = \frac{x_n(0)}{n-1}$, $\alpha_2 = \frac{x_1(0)}{n-1}$ and $\beta_1 = \frac{y_n(0)}{n-1}$, $\beta_2 = \frac{y_1(0)}{n-1}$. This shows that all vehicles asymptotically approach a uniform distribution on the line. ∎

V. Formation Stabilization to Any Geometric Pattern

In this section, we turn our attention to the problem of formation stabilization to any geometric pattern. Following [3], we let Π be a predicate describing a geometric pattern, such as a point, a regular polygon, a line segment, etc. Such a predicate specifies a formation up to translation and rotation. By *formation stabilization of a group of n vehicles to* Π, we mean that the vehicles (globally exponentially) converge to a distribution satisfying Π.

We suppose that a group of wheeled vehicles have a *common sense of direction*, represented by the angle ψ in Fig. 5. For instance, each vehicle carries a navigation device such as a compass. Alternatively, all vehicles initially agree on their orientation and use it as the common direction. The common direction may not coincide with the positive x-axis of the global frame. Let $\phi_i = \theta_i - \psi$ (see Fig. 5). We assume that vehicle i can measure its own ϕ_i.

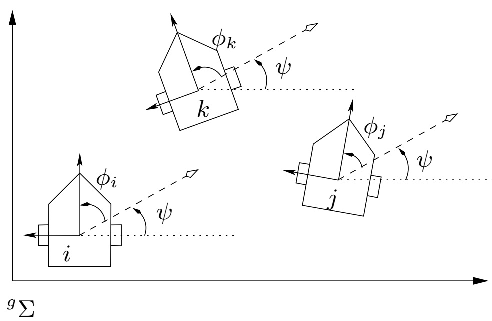

Fig. 5. A group of vehicles have a common sense of direction.

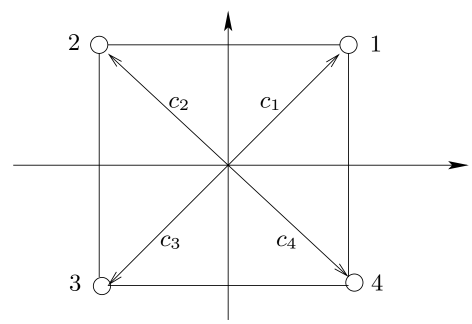

Fig. 6. An example for a square formation.

There are two ways to describe a geometric pattern in the plane. One way is by inter-node distances, d_{ij}, as in the rigid formation framework of [8]. The other way is by specifying the position vector, c_i, of each node with respect to a common coordinate frame. As an example, a square formation described by $c_i,\ i = 1, 2, 3, 4,$ is given in Fig. 6. It is worth noting that, for all $(R, b) \in SE(2)$, the vectors $\hat{c}_i = Rc_i + b$ describe the same geometric formation as the one specified by c_i. So given a desired geometric formation pictured by $c_i,\ i = 1, \dots, n$, our objective is to stabilize the position state z_i of each vehicle to $\hat{c}_i = Rc_i + b,\ i = 1, \dots, n$, for some R and b. To achieve a desired geometric formation characterized by $c = [c_1^T \ \cdots \ c_n^T]^T$, we can simply translate the formation vector c into a control offset $d = L_{(2)}c$ so that the forward control velocity is 0 when the group of vehicles has achieved a formation. We denote the offset for each vehicle by $d_i = [d_{x_i} \ d_{y_i}]^T$ or $\mathbf{d}_i = d_{x_i} + jd_{y_i}$.

Next, we show that the time-varying control law for each vehicle $i,\ (i = 1, \dots, n)$

$$\begin{cases} v_i(t) = k\left\{[1 \ 0]\, R(-\phi_i(t))d_i + \sum_{m \in N_i} x_{im}(t)\right\} \\ \omega_i(t) = \cos(t) \end{cases} \tag{5}$$

achieves formation stabilization to Π, where R is a rotation matrix defined by $R(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.

Theorem 4: Let Π be a desired geometric formation described by $c = [c_1 \ \cdots \ c_n]^T$. Suppose a group of n wheeled vehicles have a common sense of direction and formation stabilization to a point is feasible. Then there exists a positive constant k^* such that for all $0 < k < k^*$, the smooth time-varying feedback control law (5) with $d = L_{(2)}c$ guarantees global exponential formation stabilization to Π.

Proof: Using the control law (5), one obtains the following closed-loop system

$$\dot{z}_i = kM(\theta_i(t)) \left\{ R(\psi)d_i + \sum_{m \in N_i} (z_m - z_i) \right\},$$

or in vector form, $\dot{z} = kH(\theta(t)) \left\{ -L_{(2)}z + (I_n \otimes R(\psi))\, d \right\}$. By a property of the Kronecker product, $(I_n \otimes R(\psi))(L \otimes I_2) = L \otimes R(\psi) = (L \otimes I_2)(I_n \otimes R(\psi))$. Furthermore, since $d = L_{(2)}c$, we obtain $\dot{z} = -kH(\theta(t))L_{(2)} \left\{ z - (I_n \otimes R(\psi))c \right\}$. Under the coordinate transformation $\varsigma = z - (I_n \otimes R(\psi))c$, we get $\dot{\varsigma} = -kH(\theta(t))L_{(2)}\varsigma$. By the proof of Theorem 1, $\lim_{t \to \infty} \varsigma(t) = \mathbf{1} \otimes z_{ss}$ for some constant position $z_{ss} = [x_{ss} \ y_{ss}]^T$. Hence, $\lim_{t \to \infty} z_i(t) = R(\psi)c_i + z_{ss}, \ i = 1, \ldots, n$, which means that the group of vehicles form a geometric formation specified by c. ■

Remark 1: Notice that $\theta_i(t) = \theta_i(t_0) + \sin(t)$, so if a group of n wheeled vehicles achieve an agreement on their initial orientation $\theta_i(t_0)$ and choose it as their common direction, the control law (5) becomes

$$\begin{cases} v_i(t) = k \left\{ \text{Re}\left(e^{-j\sin(t)}\right) \mathbf{d}_i + \sum_{m \in N_i} x_{im}(t) \right\} \\ \omega_i(t) = \cos(t) \end{cases}$$

The agreement on their orientation can be implemented by an alignment strategy as shown in [5].

Fig. 7 shows the simulation for a circle formation of ten wheeled vehicles with the sensor digraph in Fig. 8. The circle formation is described by $\mathbf{c}_i = 75e^{(j\frac{2(i-1)\pi}{10})}, i = 1, \ldots, 10$.

VI. Conclusions

In this paper, the feasibility problem of achieving a specified geometric formation of a group of unicycles was investigated. Necessary and sufficient graphical conditions for the existence of local information controller to assure the asymptotic convergence of the closed system were derived. The sufficiency proof also presented a constructive method for control law synthesis.

Further research issues include: developing a better behaved controller to solve this problem which does not keep the unicycles wiggling and developing more general results for the dynamic sensor graph case, where ad hoc links can be established and dropped.

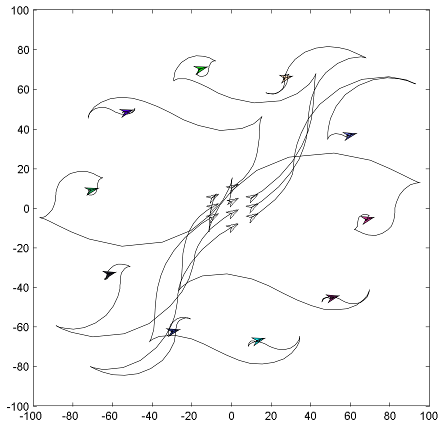

Fig. 7. Ten wheeled vehicles form a circle formation.

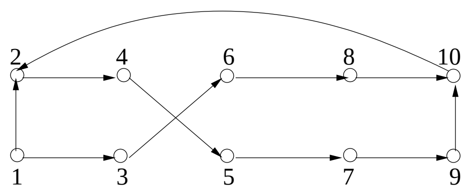

Fig. 8. The sensor digraph of a group of ten wheeled vehicles.

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