Reduction Principles and the Stabilization of Closed Sets for Passive Systems

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Abstract

In this paper we explore the stabilization of closed invariant sets for passive systems, and present conditions under which a passivity-based feedback asymptotically stabilizes the goal set. Our results rely on novel reduction principles allowing one to extrapolate the properties of stability, attractivity, and asymptotic stability of a dynamical system from analogous properties of the system on an invariant subset of the state space.

I. Introduction

The notion of passivity for state space representations of nonlinear systems, pioneered by Willems in the early 1970's, [1], [2], was instrumental for much research on nonlinear equilibrium stabilization. Key contributions in this area were made in the early 1980's by Hill and Moylan in [3], [4], [5], [6], and later by Byrnes, Isidori, and Willems, in their landmark paper [7]. More recently, in a number of papers [8], [9], [10], Shiriaev and Fradkov addressed the problem of stabilizing compact invariant sets for passive nonlinear systems. Their work is a direct extension of the equilibrium stabilization results by Byrnes, Isidori, and Willems in [7].

In this paper we develop a theory of set stabilization for passive systems which generalizes the equilibrium theory of [7], as well as the results in [8], [9], [10]. We investigate the stabilization of a closed set Γ, not necessarily compact, which is open-loop invariant and contained in the zero level set of the storage function. Our results answer this question: *when is it that a passivity-based controller makes Γ asymptotically stable for the closed-loop system?* Even in the special case when Γ is an equilibrium,

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our theory yields novel results, among them necessary and sufficient conditions for the passivity-based asymptotic stabilization of the equilibrium in question without imposing that the storage function be positive definite. The theory in [7], and [8], [9], [10] does not handle this situation.

The key insight behind the development of the results presented in this paper is the realization that at the heart of the stabilization problem by passivity-based feedback there lies a so-called reduction problem for a dynamical system $\Sigma : \dot{x} = f(x)$: *Consider two closed sets* Γ *and* $\mathcal{O}$*, with* $\Gamma \subset \mathcal{O}$*, which are invariant for* Σ*; suppose that* Γ *is stable, attractive, or asymptotically stable for the restriction of* Σ *to* $\mathcal{O}$*. When is it that* Γ *is stable, attractive, or asymptotically stable with respect to the whole state space?* We answer this question by presenting three novel reduction principles for attractivity, stability, and asymptotic stability that have independent interest and are applicable to other problems in control theory. The proofs of these and other results are omitted in this shortened paper. The interested reader is referred to the full version [11] and [12].

Outline: Section II presents stability definitions and reviews the notion of limit set and that of prolongational limit set. In Section III we state the passivity-based set stabilization problem and illustrate its relationship to the reduction problem. We then present the reduction principles. Section IV presents a novel notion of detectability using which, in Section V, we solve the passivity-based set stabilization problem. The main result, Theorem V.2, generalizes previous results on passivity-based stabilization. This fact is discussed in Section VI.

II. Preliminaries

In this paper we consider control-affine systems described by

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y &= h(x) \end{aligned} \tag{1}$$

with state space $\mathcal{X} \subset \mathbb{R}^n$, set of input values $\mathcal{U} \subset \mathbb{R}^m$ and set of output values $\mathcal{Y} \subset \mathbb{R}^m$. The set $\mathcal{X}$ is assumed to be either an open subset or a smooth submanifold of $\mathbb{R}^n$. We assume that f and g_i, $i = 1, \dots m$, are smooth vector fields on $\mathcal{X}$, and that $h : \mathcal{X} \to \mathcal{Y}$ is a smooth mapping.

Notation: Let $\mathbb{R}^+ = [0, +\infty)$. Given either a smooth feedback $u(x)$ or a piecewise-continuous open-loop control $u(t) : \mathbb{R}^+ \to \mathcal{U}$, we denote by $\phi_u(t, x_0)$ the unique solution of (1) with initial condition x_0. By $\phi(t, x_0)$ we denote the solution of the open-loop system $\dot{x} = f(x)$ with initial condition x_0. Given an interval I of the real line and a set $S \in \mathcal{X}$, we denote by $\phi_u(I, S)$ the set $\phi_u(I, S) := \{\phi_u(t, x_0) : t \in I, x_0 \in S\}$. The set $\phi(I, S)$ is defined analogously. Given a closed nonempty set

$S \subset \mathbb{R}^n$, a point $x \in \mathbb{R}^n$, and a vector norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, the point-to-set distance $\|x\|_S$ is defined as $\|x\|_S := \inf\{\|x-y\| : y \in S\}$. The state space $\mathcal{X}$, being a subset of $\mathbb{R}^n$, inherits a norm from $\mathbb{R}^n$, which we will denote $\|\cdot\| : \mathcal{X} \to \mathbb{R}$. For a constant $\alpha > 0$, a point $x \in \mathcal{X}$, and a set $S \subset \mathcal{X}$, define the open sets $B_\alpha(x) = \{y \in \mathcal{X} : \|y - x\| < \alpha\}$ and $B_\alpha(S) = \{y \in \mathcal{X} : \|y\|_S < \alpha\}$. We denote by $\mathrm{cl}(S)$ the closure of the set S, and by $\mathcal{N}(S)$ an open neighbourhood of S, that is, an open subset of $\mathcal{X}$ containing S. We use the standard notation $L_f V$ to denote the Lie derivative of a C^1 function V along a vector field f.

Passivity: Throughout this paper it is assumed that (1) is passive with smooth nonnegative storage function $V : \mathcal{X} \to \mathbb{R}$, i.e., V is a C^r ($r \geq 1$) nonnegative function such that, for all piecewise-continuous functions $u : [0, \infty) \to \mathcal{U}$, for all $x_0 \in \mathcal{X}$, and for all t in the maximal interval of existence of $\phi_u(\cdot, x_0)$,

$$V(\phi_u(t, x_0)) - V(x_0) \leq \int_0^t u(\tau)^\top y(\tau) d\tau,$$

where $y(t) = h(\phi_u(t, x_0))$. It is well-known (see [3]) that the passivity property above is equivalent to the two conditions

$$(\forall x \in \mathcal{X}) \ L_f V(x) \leq 0 \text{ and } \ L_g V(x) = h(x)^\top, \tag{2}$$

where $L_g V$ denotes the row vector $[L_{g_1} V \ \cdots \ L_{g_m} V]$.

Set stability and attractivity: All definitions below are standard and can be found in [13]. Let $\Gamma \subset \mathcal{X}$ be a closed positively invariant for a dynamical system

$$\Sigma : \dot{x} = f(x), \quad x \in \mathcal{X}. \tag{3}$$

Definition II.1 (Set stability and attractivity)**.** (i) Γ is *stable* for Σ if for all $\varepsilon > 0$ there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.

(ii) Γ is an *attractor* for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\lim_{t\to\infty} \|\phi(t, x_0)\|_\Gamma = 0$.

(iii) Γ is a *global attractor* for Σ if it is an attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$.

(iv) Γ is *[globally] asymptotically stable* for Σ if it is stable and attractive [globally attractive] for Σ.

If Γ is a compact positively invariant set, then the concepts of stability, attractivity, and asymptotic stability, as defined above, are equivalent to the familiar ϵ-δ notions of uniform stability, attractivity, and asymptotic stability found, e.g., in [14, Definition 8.1]. In the unbounded case, however, our definitions of attractivity and asymptotic stability, referred to as semi-attractivity and semi-asymptotic stability in [13], are weaker than the corresponding ϵ-δ notions. For instance, the $\epsilon - \delta$ notion of attractivity requires that the domain of attraction of Γ contains a tube of radius δ, whereas the notion of attractivity in the definition above does not, and in fact if Γ is unbounded the width of its domain of attraction may shrink to zero at infinity.

Definition II.2 (Relative set stability and attractivity)**.** Let $\mathcal{O} \subset \mathcal{X}$ be such that $\mathcal{O} \cap \Gamma \neq \emptyset$. We say that Γ is *stable relative to* $\mathcal{O}$ for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma) \cap \mathcal{O}) \subset B_\varepsilon(\Gamma)$. Similarly, one modifies all other notions in Definition II.1 by restricting initial conditions to lie in $\mathcal{O}$.

Definition II.3 (Local stability and attractivity near a set)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be positively invariant sets. The set $\mathcal{O}$ is *locally stable near* Γ if for all $x \in \Gamma$, for all $c > 0$, and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma)$ and all $t > 0$, whenever $\phi([0,t], x_0) \subset B_c(x)$ one has that $\phi([0,t], x_0) \subset B_\varepsilon(\mathcal{O})$. The set $\mathcal{O}$ is *locally attractive near* Γ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\phi(t, x_0) \to \mathcal{O}$ at $t \to +\infty$.

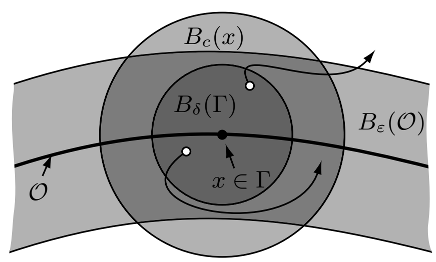

Fig. 1. An illustration of the notion of local stability near Γ

The property of local stability can be rephrased as follows. Given an arbitrary ball $B_c(x)$ centred at a point x in Γ, trajectories originating in $B_c(x)$ sufficiently close to Γ cannot travel far away from $\mathcal{O}$ before first exiting $B_c(x)$; see Figure 1. It is immediate to see that if Γ is stable, then $\mathcal{O}$ is locally stable near Γ.

Definition II.4 (Local uniform boundedness)**.** The system Σ is *locally uniformly bounded near* Γ if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$.

Limit Sets: In order to characterize the asymptotic properties of bounded solutions, we will use the well-known notion of limit set, due to G. D. Birkhoff (see [15]), and that of prolongational limit set, due to T. Ura (see [16]). Given a smooth feedback $u(x)$ and a point $x_0 \in \mathcal{X}$, the *positive limit set* (or ω-limit set) of the closed-loop solution $\phi_u(t, x_0)$ is defined as

$$L_u^+(x_0) := \{p \in \mathcal{X} : (\exists \{t_n\} \subset \mathbb{R}^+)\, t_n \to +\infty, \\ \phi_u(x_0, t_n) \to p\}.$$

The positive limit set of the open-loop solution $\phi(t, x_0)$, defined in an analogous way, is denoted $L^+(x_0)$. We let $L_u^+(S) := \bigcup_{x_0 \in S} L_u^+(x_0)$ and $L^+(S) := \bigcup_{x_0 \in S} L^+(x_0)$.

If $U \subset \mathcal{X}$ and $x_0 \in \mathrm{cl}(U)$, the *prolongational limit set relative to* U of an open-loop solution $\phi(t, x_0)$ is defined as

$$J^+(x_0, U) := \{p \in \mathcal{X} : (\exists \{(x_n, t_n)\} \subset U \times \mathbb{R}^+),\, x_n \to x_0, \\ t_n \to +\infty,\, \phi(x_n, t_n) \to p\}.$$

We denote $J^+(S, U) := \bigcup_{x_0 \in S} J_u^+(x_0, U)$. One can show that if $x_0 \in \mathrm{cl}(U)$, then $L^+(x_0) \subset J^+(x_0, U)$.

III. Stabilization problem and reduction principles

The main objective of this paper is the stabilization of a closed set Γ using *passivity-based feedbacks* of the form

$$u = -\varphi(x), \text{ with } \varphi(\cdot)\Big|_{h(x)=0} = 0,\ h(x)^\top \varphi(x)\Big|_{h(x)\neq 0} > 0, \tag{4}$$

where $\varphi : \mathcal{X} \to \mathcal{U}$ is a smooth function. The class of passivity-based feedbacks in (4) includes that of output feedback controllers $u = -\varphi(h(x))$ commonly used in the literature on passive systems.

Set Stabilization Problem. Given a closed set $\Gamma \subset V^{-1}(0) = \{x \in \mathcal{X} : V(x) = 0\}$ which is positively invariant for the open-loop system in (1), and given a passivity-based feedback of the form (4), find conditions guaranteeing that Γ is [globally] asymptotically stable for the closed-loop system.

The rationale behind passivity-based feedback is the following. Using (2) and the properties of the passivity-based feedback (4), the time derivative of the storage function V along trajectories of the closed-loop system formed by (1) with feedback (4) is given by

$$\begin{aligned} \frac{dV(\phi_u(t, x_0))}{dt} =& L_f V(\phi_u(t, x_0)) \\ & - L_g V(\phi_u(t, x_0)) \varphi(\phi_u(t, x_0)) \\ \leq & - h(\phi_u(t, x_0))^\top \varphi(\phi_u(t, x_0)) \leq 0. \end{aligned} \tag{5}$$

Thus, a passivity-based feedback renders the storage function V nonincreasing along solutions of the closed-loop system. One expects that if the system enjoys suitable properties, then the storage function should decrease asymptotically to zero and the solutions should approach a subset of $V^{-1}(0)$, hopefully the set Γ.

Our point of departure in understanding what system properties yield the required result is the well-known property, found in the proof of Theorem 3.2 in [7], that, for all $x_0 \in \mathcal{X}$, the positive limit set $L_u^+(x_0)$ of the closed-loop system is invariant for the open-loop system and such that $L_u^+(x_0) \subset h^{-1}(0)$. Let $\mathcal{O}$ denote the *maximal* set contained in $h^{-1}(0)$ which is invariant for the open-loop system. In light of the property above, if $L_u^+(x_0)$ is non-empty, then it must be contained in $\mathcal{O}$. Therefore, all bounded trajectories of the closed-loop system asymptotically approach $\mathcal{O}$. Since $L_f V \leq 0$, V is nonincreasing along solutions of the open-loop system, and so $V^{-1}(0)$ is an invariant set for the open-loop system. Moreover, since V is nonnegative, any point $x \in V^{-1}(0)$ is a local minimum of V and hence $dV(x) = 0$. Therefore, $L_g V(x) = h(x)^\top = 0$ on $V^{-1}(0)$, and so $\Gamma \subset V^{-1}(0) \subset h^{-1}(0)$. Since $V^{-1}(0)$ is invariant and contained in $h^{-1}(0)$, it is necessarily a subset of $\mathcal{O}$ (this implies that $\mathcal{O}$ is not empty). Putting everything together, we conclude that

$$\Gamma \subset V^{-1}(0) \subset \mathcal{O} \subset h^{-1}(0). \tag{6}$$

It is then clear that if the trajectories of the closed-loop system in a neighbourhood of Γ are bounded, the least a passivity-based feedback can guarantee is the attractivity of $\mathcal{O}$; but this is not sufficient for our purposes. Notice that, on $\mathcal{O}$, $\varphi(\cdot) = 0$ and so the closed-loop dynamics on $\mathcal{O}$ coincide with the open-loop dynamics. In particular, then, $\mathcal{O}$ is an invariant set for the closed-loop system. In order to ensure the property of asymptotic stability of Γ, the open-loop system *must* enjoy the same property *relative to* $\mathcal{O}$. Therefore, a necessary condition for Γ to be asymptotically stable for the closed-loop system is that Γ be asymptotically stable relative to $\mathcal{O}$ for the open-loop system. Is this condition also sufficient or are extra-properties needed? This question leads to the reduction problem stated in the introduction: *If* $\Gamma \subset \mathcal{O}$ *is stable, attractive, or asymptotically stable relative to* $\mathcal{O}$*, what extra conditions guarantee that* Γ *is stable, attractive, or asymptotically stable with respect to the whole state space?* This problem was originally formulated by P. Seibert and J.S. Florio in 1969-1970. Seibert and Florio developed reduction principles for stability (see Theorem 3.4 in [17]) and asymptotic stability (see Theorem 4.13 and Corollary 4.11 in [17]) for dynamical systems on metric spaces assuming that Γ is compact. Their conditions first appeared in [18] and [19], while the proofs are found in [17] (see also the work in [20] for related results).

The reduction problem arises in many areas of nonlinear control theory, including the stability of cascade-connected systems, the separation principle of output feedback control, and the adaptive control problem. It also plays a role in singular perturbations and center manifold theory. The theorems below, which extend Seibert and Florio's results in the finite dimensional setting, are relevant to all these problems. We omit all proofs in this shortened paper, but refer the interested reader to the full version [11] and [12]. Consider the dynamical system

$$\Sigma: \ \dot{x} = f(x), \ x \in \mathcal{X}, \tag{7}$$

with f locally Lipschitz on $\mathcal{X}$, and let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be closed sets which are positively invariant for system Σ. We have the following

Theorem III.1 (Reduction principle for attractivity)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ is attractive if the following conditions hold:

(i) Γ is asymptotically stable relative to $\mathcal{O}$

(ii) $\mathcal{O}$ is locally attractive near Γ,

(iii) there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all initial conditions in $\mathcal{N}(\Gamma)$, the associated solutions are bounded and such that the set $\mathrm{cl}(\phi(\mathbb{R}^+, \mathcal{N}(\Gamma))) \cap \mathcal{O}$ is contained in the domain of attraction of Γ relative to $\mathcal{O}$.

The set Γ is globally attractive if:

(i)' Γ is globally asymptotically stable relative to $\mathcal{O}$,

(ii)' $\mathcal{O}$ is a global attractor,

(iii)' all trajectories in $\mathcal{X}$ are bounded.

Conditions (ii) and (ii') above are also necessary. Theorem III.1 is novel in that Seibert and Florio did not investigate a reduction principle for attractivity.

Theorem III.2 (Reduction principle for asymptotic stability)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ is [globally] asymptotically stable if the following conditions hold:

(i) Γ is [globally] asymptotically stable relative to $\mathcal{O}$,

(ii) $\mathcal{O}$ is locally stable near Γ,

(iii) $\mathcal{O}$ is locally attractive near Γ [$\mathcal{O}$ is globally attractive],

(iv) if Γ is unbounded, then Σ is locally uniformly bounded near Γ,

(v) [all trajectories of Σ are bounded.]

Conditions (i), (ii), and (iii) above are necessary.

Theorem III.3 (Reduction principle for stability)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be two closed positively invariant sets. If assumptions (i), (ii), and (iv) of Theorem III.2 hold, then Γ is stable.

If Γ is a compact set, then Theorems III.2 and III.3 are equivalent to the results presented in Theorems 3.4, 4.13, and Corollary 4.11 in [17].

IV. Detectability

For convenience, we repeat the definition of the set $\mathcal{O}$ given in Section III

Definition IV.1 (Set $\mathcal{O}$)**.** Given the control system (1), we denote by $\mathcal{O}$ the maximal set contained in $h^{-1}(0)$ which is invariant for the open-loop system $\dot{x} = f(x)$.

When system (1) is linear time-invariant (LTI), the set $\mathcal{O}$ is the unobservable subspace. As discussed in Section III, as long as the trajectories of the closed-loop system in a neighbourhood of Γ are bounded, a passivity-based feedback renders the set $\mathcal{O}$ attractive. In order to guarantee asymptotic stability of $\Gamma \subset \mathcal{O}$, the reduction principle in Theorem III.2 suggests that Γ should be asymptotically stable relative to $\mathcal{O}$ for the open-loop system. We call this property Γ-detectability.

Definition IV.2 (Γ-detectability)**.** System (1) is *locally* Γ*-detectable* if Γ is asymptotically stable relative to $\mathcal{O}$ for the open-loop system. The system is Γ*-detectable* if Γ is globally asymptotically stable relative to $\mathcal{O}$ for the open-loop system.

Our notion of detectability is parameterized by Γ, and not by $\mathcal{O}$, although the set $\mathcal{O}$ figures in its definition. This is due to the fact that $\mathcal{O}$ is entirely determined by the open-loop vector field f and the output function h. In the case of LTI systems, when $\Gamma = \{0\}$, the above definition requires that all trajectories on the unobservable subspace $\mathcal{O}$ converge to 0. Therefore, in the LTI setting, Γ-detectability coincides with the classical notion of detectability. Further, the notion of Γ-detectability generalizes that of zero-state detectability in [7]. As a matter of fact, when V is positive definite, and thus $\Gamma = \{0\}$, the two detectability notions coincide.

Lemma IV.3. If V is positive definite and $\Gamma = V^{-1}(0) = \{0\}$, then the following three conditions are equivalent:

(a) System (1) is locally zero-state detectable [zero-state detectable],

(b) the equilibrium $x = 0$ is [globally] attractive relative to $\mathcal{O}$ for the open-loop system,

(c) system (1) is locally Γ-detectable [Γ-detectable].

Proof: The set of points $x_0 \in \mathcal{X}$ such that the open-loop solution satisfies $h(\phi(t, x_0)) \equiv 0$ is precisely the maximal open-loop invariant subset of $h^{-1}(0)$, i.e., the set $\mathcal{O}$. Thus, conditions (a) and (b) are equivalent. Since (1) is passive, by (2) we have $L_f V \leq 0$. By the assumption that V is positive definite, it follows that $x = 0$ is a stable equilibrium of the open-loop system. Thus, $x = 0$ is [globally] asymptotically stable relative to $\mathcal{O}$ for the open-loop system if and only if $x = 0$ is [globally] attractive relative to $\mathcal{O}$ for the open-loop system, proving that conditions (b) and (c) are equivalent. ■

The next lemma shows that Γ-detectability also encompasses the notion of V-detectability in [10].

Lemma IV.4. If $\Gamma = V^{-1}(0)$ is a compact set, then the following three conditions are equivalent:

(a) System (1) is locally V-detectable,

(b) the set Γ is attractive relative to $\mathcal{O}$ for the open-loop system,

(c) system (1) is locally Γ-detectable.

Moreover, if V is proper, then the global versions of conditions (a)-(c) are equivalent.

Proof: Suppose that (1) is locally V-detectable. Then, for all $x_0 \in V^{-1}([0, c]) \cap \mathcal{O}$, we have $V(x(t)) \to 0$. Since $V^{-1}(0)$ is compact, in a sufficiently small neighbourhood of Γ, $V^{-1}(\phi(t, x_0)) \to 0$ implies $\phi(t, x_0) \to V^{-1}(0)$, and thus $\Gamma = V^{-1}(0)$ is attractive relative to $\mathcal{O}$ for the open-loop system, showing that condition (a) implies (b). Since $L_f V \leq 0$, Γ is also stable for the open-loop system. Thus, condition (b) implies (c). Now suppose that (1) is locally Γ-detectable. Then, there exists a neighbourhood S of Γ such that, for all $x_0 \in S \cap \mathcal{O}$, $\phi(t, x_0) \to \Gamma$. Since $\Gamma = V^{-1}(0)$ is compact and V is continuous, there exists $c > 0$ such that $V^{-1}([0, c]) \subset S$. Hence, for all $x_0 \in V^{-1}([0, c]) \cap \mathcal{O}$ or, equivalently for all $x_0 \in V^{-1}([0, c])$ such that $h(\phi(t, x_0)) \equiv 0$, we have $\phi(t, x_0) \to V^{-1}(0)$. By the continuity of V and the compactness of $V^{-1}(0)$ the latter fact implies that $V(\phi(t, x_0)) \to 0$. This proves that condition (c) implies (a). The proof of equivalence of the global notions of detectability follows directly from the fact that if V is proper, then $V(\phi(t, x_0)) \to 0$ if and only if $\phi(t, x_0) \to V^{-1}(0)$. ■

We now give sufficient conditions for (1) to be Γ-detectable. The proof is in [11]. Let

$$\mathcal{S} = \{x \in \mathcal{X} : L_f^m h(x) = 0, 0 \leq m \leq r + n - 2\}.$$

Notice that the definition of $\mathcal{S}$ does not directly involve the storage function (but recall that $h^\top = L_g V$, so it does indirectly depend on V).

Proposition IV.5. Suppose that all open-loop trajectories that originate and remain on $\mathcal{S}$ are bounded and that the open-loop system in (1) is locally uniformly bounded near Γ. If

$$\mathcal{S} \cap J^+(\mathcal{S}, \mathcal{S}) \subset \Gamma, \tag{8}$$

then system (1) is Γ-detectable. Moreover, if $\Gamma = V^{-1}(0)$, then condition (8) may be replaced by the following one:

$$\mathcal{S} \cap L^+(\mathcal{S}) \subset V^{-1}(0). \tag{9}$$

Remark. Proposition IV.5 relaxes the sufficient conditions for detectability found in [7, Proposition 3.4] and [9, Theorem 10]. We refer the reader to [11] for a discussion. The natural way to check Γ-detectability is to compute the set $\mathcal{O}$ in Definition IV.1, and then assess the asymptotic stability of Γ relative to $\mathcal{O}$. Should the computation of the set $\mathcal{O}$ be too difficult, Proposition IV.5 above provides an alternative, but conservative, criterion for Γ-detectability that may prove useful in some cases. It is important to notice that condition (8) may be hard to check in practice because it involves the computation of the prolongational limit set $J^+(\mathcal{S}, \mathcal{S})$. The conditions found in [7, Proposition 3.4] and [9, Theorem 10] suffer from the same limitation because they too involve the computation of limit sets.

V. Solution of the set stabilization problem

We are now ready to solve the stabilization problem, by presenting conditions that guarantee that a passivity-based controller of the form (4) makes Γ stable, attractive, or asymptotically stable for the closed-loop system. All results are straightforward consequences of the reduction principles presented in Section III, and they rely on the next fundamental observation, whose proof is found in [11].

Proposition V.1. Consider the passive system (1) with a passivity-based feedback of the form (4), and the set $\mathcal{O}$ in Definition IV.1. Then, the set $\mathcal{O}$ is locally stable near Γ for the closed-loop system.

Theorem V.2 (Asymptotic stability of Γ)**.** Consider system (1) with a passivity-based feedback of the form (4). If Γ is compact, then

- Γ is asymptotically stable for the closed-loop system if, and only if, system (1) is locally Γ-detectable,
- if all trajectories of the closed-loop system are bounded, then Γ is globally asymptotically stable for the closed-loop system if, and only if, system (1) is Γ-detectable.

If Γ is unbounded and the closed-loop system is locally uniformly bounded near Γ, then

- Γ is asymptotically stable for the closed-loop system if, and only if, system (1) is locally Γ-detectable.
- if all trajectories of the closed-loop system are bounded, Γ is globally asymptotically stable for the closed-loop system if, and only if, system (1) is Γ-detectable.

Proof: The sufficiency part of the theorem follows from the following considerations. By Proposition V.1, $\mathcal{O}$ is locally stable near Γ. If Γ is compact, by Theorem III.3 local Γ-detectability implies stability of Γ. The stability of Γ and its compactness in turn imply that all closed-loop trajectories in some neighbourhood of Γ are bounded. Since all bounded trajectories asymptotically approach $\mathcal{O}$, $\mathcal{O}$ is locally attractive near Γ. If all trajectories of the closed-loop system are bounded, then $\mathcal{O}$ is globally attractive. Theorem III.2 yields the required result.

Now suppose that Γ is unbounded. By local uniform boundedness near Γ we have that all closed-loop solutions in some neighbourhood of Γ are bounded and hence $\mathcal{O}$ is locally attractive near Γ. Once again, if all closed-loop trajectories are bounded, then $\mathcal{O}$ is globally attractive. The required result now follows from Theorem III.2.

The various necessity statements follow from the following basic observation. Any passivity-based feedback of the form (4) makes $\mathcal{O}$ an invariant set for the closed-loop system (see Section III). Therefore, if Γ is [globally] asymptotically stable for the closed-loop system, necessarily Γ is [globally] asymptotically stable relative to $\mathcal{O}$ for the closed-loop system. In other words, (1) is necessarily locally Γ-detectable [Γ-detectable]. ■

We conclude this section with the following result, which gives conditions that are alternatives to the Γ-detectability assumption.

Proposition V.3. Theorem V.2 still holds if the local Γ-detectability [Γ-detectability] assumption is replaced by the following condition:

(i') Γ is stable relative to $V^{-1}(0)$ and Γ is [globally] attractive relative to $\mathcal{O}$.

We omit the proof of this proposition. If the sufficient conditions for Γ-detectability in Proposition IV.5 fail, rather than checking for Γ-detectability one may find it easier to check condition (i') in Proposition V.3, because verifying whether Γ is stable relative to $V^{-1}(0)$ does not require finding the maximal open-loop invariant subset $\mathcal{O}$ of $h^{-1}(0)$; moreover, checking that Γ is attractive relative to $\mathcal{O}$ amounts to checking the familiar condition in [7]

$$h(\phi(t, x_0)) \equiv 0 \implies \phi(t, x_0) \to \Gamma \text{ as } t \to +\infty.$$

Note that, in the framework of [7] and [10], the requirement that Γ be stable relative to $V^{-1}(0)$ is trivially satisfied because in these references it is assumed that $\Gamma = V^{-1}(0)$.

VI. DISCUSSION

Theorem 3.2 in [7] and Theorem 2.3 in [10], dealing with the special case when $\Gamma = V^{-1}(0)$ $(= \{0\})$ and Γ is compact, become corollaries of our main result, Theorem V.2. We have already shown (see Lemmas IV.3 and IV.4) that in this special case the properties of zero-state detectability (when $\Gamma = \{0\}$), and V-detectability coincide with our notion of Γ-detectability. In this context, then, the results in [7] and [10] state that local Γ-detectability is a sufficient condition for the asymptotic stabilization of the origin using a passivity-based feedback. We have shown that actually this condition is also *necessary*. An analogous remark can be made for the global solution of the set stabilization problem.

The theory in [7] and [10] does not handle the special case when Γ is compact and $\Gamma \subsetneq V^{-1}(0)$, while our theory does. This case includes the important situation when one wants to stabilize an equilibrium ($\Gamma = \{0\}$) but the storage is only positive semi-definite. Based on the results in [7] and [10], it may be tempting to conjecture that Theorem 3.2 in [7] and Theorem 2.3 in [10] still hold if one employs the following notion of detectability:

$$\begin{aligned}(\forall x_0 \in \mathcal{N}(\Gamma))\ h(\phi(t, x_0)) = 0 \text{ for all } t \in \mathbb{R}& \\ \implies \phi(t, x_0) \to \Gamma,&\end{aligned} \tag{10}$$

which corresponds to requiring that $\mathcal{O}$ in Definition IV.1 is an attractor for the open-loop system. This conjecture is false: we have shown that (local) Γ-detectability (i.e., the asymptotic stability of Γ relative to $\mathcal{O}$ for the open-loop system) is a necessary condition for the stabilization of Γ. Even if one relaxes the asymptotic stability requirement and just asks for attractivity of Γ relative to $\mathcal{O}$, the above conjecture is still false. As a matter of fact, Theorem III.1 suggests that even in this case (local) Γ-detectability is a key property. A counter-example illustrating this loss of attractivity is the pendulum. The upright equilibrium is globally attractive, but unstable, relative to the homoclinic orbit of the pendulum. Despite the fact that a passivity-based feedback can be used to asymptotically stabilize the homoclinic orbit (see, e.g., [21], [22]), the upright equilibrium is unstable for the closed-loop system. This well-known phenomenon finds explanation in the theory developed in this paper: the cause of the problem is the instability of the upright equilibrium relative to the homoclinic orbit. We next present another explicit counter-example illustrating our point.

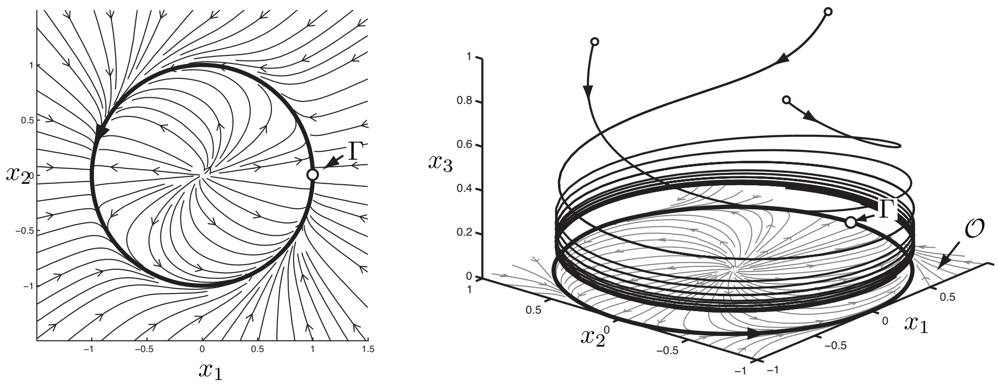

Fig. 2. On the left-hand side, phase portrait on $\mathcal{O}$ for the open-loop system (12). Γ is globally attractive relative to $\mathcal{O}$. On the right-hand side, closed-loop system (11) with feedback $u = -y$. Γ is *not* attractive relative to the whole state space.

Example. Consider the control system with state (x_1, x_2, x_3),

$$\begin{aligned} \dot{r} &= -r(r-1) \\ \dot{\theta} &= \sin^2(\theta/2) + x_3 \\ \dot{x}_3 &= u \\ y &= x_3^3, \end{aligned} \tag{11}$$

where $(r, \theta) \in (0, +\infty) \times S^1$ represent polar coordinates for (x_1, x_2). The control system is passive with storage $V(x) = x_3^4/4$. Let Γ be the equilibrium point $\{(x_1, x_2, x_3) : x_1 = 1, x_2 = x_3 = 0\}$ and note that $\mathcal{O} = \{(x_1, x_2, x_3) : x_3 = 0\}$. On $\mathcal{O}$, the open-loop dynamics read as

$$\begin{aligned} \dot{r} &= -r(r-1) \\ \dot{\theta} &= \sin^2(\theta/2), \end{aligned} \tag{12}$$

and it is easily seen that the equilibrium Γ attracts every point in $\mathcal{O}$ except the origin. Hence, Γ is attractive relative to $\mathcal{O}$, but unstable (indeed, the unit circle is a homoclinic orbit of the equilibrium); see Figure 2. Therefore, condition (10) holds but the system is not locally Γ-detectable. Consider the passivity-based feedback $u = -y$, which renders $\mathcal{O}$ globally asymptotically stable. Now for any initial condition off of $\mathcal{O}$ such that $(x_1(0), x_2(0)) \neq (0, 0)$, $x_3(0) > 0$, the corresponding trajectory is bounded, but its positive limit set is the unit circle on $\mathcal{O}$, and therefore it is not a subset of Γ; see Figure 2. In conclusion, Γ is not attractive for the closed-loop system (and neither is it stable). This example illustrates the fact that, when $\Gamma \subsetneq V^{-1}(0)$ is compact, simply requiring condition (10) in place of Γ-detectability may not be enough for attractivity of Γ.

In the light of Theorem V.2 and the example above, it is clear that the addition of the stability requirement on Γ, relative to $\mathcal{O}$, is a crucial enhancement to the notions of detectability in [7] and [10].

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