Output Stabilization and Maneuver Regulation: A geometric approach

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Abstract

This paper investigates maneuver regulation for single input control affine systems from a geometric perspective. The maneuver regulation problem is converted to output stabilization and necessary and sufficient conditions are provided to solve the latter problem by feedback linearizing the dynamics transverse to a suitable embedded submanifold of the state space. When specialized to the linear time invariant setting, this work recovers well-known results on output stabilization.

Key words: maneuver regulation, path following, feedback linearization, zero dynamics, non-square systems, output stabilization

1 Introduction

The maneuver regulation (or path following) problem entails designing a smooth feedback making the trajectories of a system approach and traverse a pre-specified path, or *maneuver*. Unlike a tracking controller, a maneuver regulation controller drives the trajectories of a system to a maneuver *up to time reparameterization*. This difference is crucial in robotics and aerospace applications where the system dynamics impose constraints on the time parameterization of feasible maneuvers.

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This paper presents an approach to solving maneuver regulation problems inspired by the work of Banaszuk and Hauser [3]. There, the authors consider periodic maneuvers in the state space and present necessary and sufficient conditions for feedback linearization of the associated transverse dynamics. Feedback linearization is a natural framework for maneuver regulation, as evidenced by the body of work on path following which employs this approach (see for example [1,2,4,5,7,9]). In these papers, the maneuver regulation problem is converted to an input-output feedback linearization problem with respect to a *suitable* output. What are the underlying properties of the path and the system guaranteeing the existence of such an output function? In this paper we address this question for single-input control-affine systems. More specifically, the work presented here investigates systems with outputs and extends the results of [3] to the case of non-periodic maneuvers defined in the output space (rather than periodic maneuvers in the state space). The point of view we take in this paper is to pose maneuver regulation as an output stabilization problem. Under suitable assumptions (Assumption 2), solving the latter problem yields a solution to the former one. Hence our main focus is on output stabilization of nonlinear systems. To this end, we seek conditions for feedback linearization of dynamics transverse to a suitable controlled invariant submanifold of the state space. See the recent work in [16] for an alternative solution to the path following problem applicable to systems in strict feedback form. An interesting geometric approach for a class of non-holonomic systems is found in [6].

Our main results are necessary and sufficient conditions for global and local transverse feedback linearization (TFL) (Theorem 4.1 and Theorems 6.1-6.2). We also provide sufficient and easy-to-check conditions for global TFL (Theorem 4.4) as well as a sufficient condition for a system *not to be* transversely feedback linearizable (Corollary 6.3). In Section 5 we demonstrate that when specialized to the LTI setting, these results recover known conditions for output stabilization. We finally show (Lemmas 7.1 and 7.2) that, in the case of state maneuvers, our results recover the results of [3].

The following notation is used throughout the paper. We denote by $\Phi_t^v(x)$ the flow of a smooth vector field v. We let $\operatorname{col}(x_1, \ldots, x_k) := [x_1 \ \ldots \ x_n]^\top$ and, given two column vectors a and b, we let $\operatorname{col}(a, b) := [a^\top \ b^\top]^\top$. Given a smooth distribution D, we let $\operatorname{inv}(D)$ be its involutive closure (the smallest involutive distribution containing D) and $D^\perp$ be its annihilator. Given a map $f : \mathbb{R}^r \to \mathbb{R}^q$ and a point $p \in \mathbb{R}^r$, we denote $(df)_p := \frac{\partial f}{\partial x}(p)$. For brevity, the term *submanifold* is used in place of *embedded submanifold of* $\mathbb{R}^n$ throughout.

2 Introductory Example

Consider the kinematic unicycle with fixed translational speed $v \neq 0$

$$\begin{aligned} \dot{x} &= \begin{bmatrix} v\cos x_3 \\ v\sin x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \mathrm{col}(x_1, x_2). \end{aligned} \tag{1}$$

We are interested in making the output of (1) approach and follow a unit circle in output coordinates,

$$\{y \in \mathbb{R}^2 : y_1^2 + y_2^2 - 1 = 0\}.$$

The lift of the circle to the state space is the set $\Gamma = \{x \in \mathbb{R}^3 : x_1^2+x_2^2-1 = 0\}$. Making x approach Γ is equivalent to making y approach the circle. Thus our original problem can be reformulated as an output stabilization problem for (1) with output

$$y' = x_1^2 + x_2^2 - 1.$$

To stabilize y' we seek a smooth feedback stabilizing the largest controlled invariant submanifold of $\mathbb{R}^3$ contained in the zero level set of y', that is, the zero dynamics manifold of system (1) with output y'. This is the subset of the state space compatible with the motion of the unicycle on the unit circle. In this example, y' yields a uniform relative degree of 2 on the set $\{x \in \mathbb{R}^3 : \cos x_3 - x_1 \sin x_3 \neq 0\}$ and so the zero dynamics manifold is ($\dot{y}'$ denotes the time derivative of y along (1))

$$\Gamma^* = \{x \in \mathbb{R}^3 : y' = \dot{y}' = 0\}.$$

This is an embedded submanifold of dimension $n^* = 1$, it has two disconnected components, two helices, corresponding to clockwise and counter-clockwise motion along the circle. In coordinates, the associated zero dynamics vector fields are $\dot{x}_3 = -v$ and $\dot{x}_3 = v$, respectively. The coordinate transformation

$$\Xi : x \mapsto \begin{bmatrix} z \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \varphi(x) \\ x_1^2 + x_2^2 - 1 \\ 2v(x_1 \cos x_3 + x_2 \sin x_3) \end{bmatrix},$$

where $\varphi(x)$ is a suitable smooth function, brings the system into normal form

$$\begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} f_0(z,\xi) \\ \xi_2 \\ b(z,\xi) + a(z,\xi)u \end{bmatrix}.$$

Geometrically, the ξ-subsystem represents the dynamics *transverse* to Γ^*. In transformed coordinates, the output stabilization problem becomes the problem of stabilizing the equilibrium $\xi = 0$ of the ξ-subsystem, This is easily achieved by choosing, e.g.,

$$u(z,\xi) = -\frac{b(z,\xi) + \xi_1 + \xi_2}{a(z,\xi)}.$$

This smooth feedback locally stabilizes the set Γ^* and hence it makes the unicycle approach the unit circle. Depending on the initial condition, phase curves of the system may approach either one of the two connected components of Γ^*, that is, the unicycle may traverse the unit circle in the clockwise or counter-clockwise direction. Path traversal is guaranteed by the fact that the zero dynamics vector field has no equilibria.

This solution relies on the choice of an output function which is zero on Γ^* and yields a uniform relative degree of $n - n^* = 2$ on Γ^*, so that input-output linearization can be applied to stabilize Γ^* and hence solve the maneuver regulation problem. In this example the lift of the path constraint, y', happens to satisfy both properties above. In general, the lift of the path to the state space is characterized as the zero level set of several functions. If none of them yields a uniform relative degree of $n - n^*$, it is not clear whether there exists some other output function which is zero on Γ^* and yields a uniform relative degree of $n - n^*$ on Γ^*. For instance, consider the system

$$\begin{aligned} \dot{x}_1 &= x_3 + x_1 u \\ \dot{x}_2 &= 1 \qquad\qquad y = \mathrm{col}(x_1, x_2), \\ \dot{x}_3 &= u \end{aligned} \tag{2}$$

and the path $\{y \in \mathbb{R}^2 : y_1 = 0\}$. The lift of the path to the state space is $\Gamma = \{x \in \mathbb{R}^3 : x_1 = 0\}$. System (2) with output $y' = x_1$ does not have a well-defined relative degree anywhere on the set $\{x_1 = 0\}$ because $L_g y' = x_1$ changes sign in any neighborhood of $\{x_1 = 0\}$. Application of the zero dynamics algorithm gives that the largest controlled invariant submanifold contained in Γ is

$$\Gamma^* = \{x : x_1 = x_3 = 0\}.$$

Since y' does not yield a well-defined relative degree on Γ^*, it is not clear whether input-output linearization can be employed to stabilize Γ^* and hence solve the maneuver regulation problem. We will show in Section 4 that input-output linearization can in fact be used for this system. The output function

$$\alpha(x) = -x_1 e^{-x_3}$$

has the required properties.

In this paper we seek checkable conditions for the existence of such an output function. Not surprisingly, our conditions involve the geometry of the path and structural properties of the system. For a collection of examples, the reader is referred to [13].

3 Problem Formulation

Consider the smooth dynamical system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3}$$

defined on $\mathbb{R}^n$, with f and g smooth vector fields, $h : \mathbb{R}^n \to \mathbb{R}^p$ $(p \geq 2)$ [3] smooth, and $u \in \mathbb{R}$.

Given a smooth parameterized curve $\sigma : \mathbb{D} \to \mathbb{R}^p$, where $\mathbb{D}$ is either $\mathbb{R}$ or S^1, the maneuver regulation problem entails finding a smooth control $u(x)$ making the output of the system approach the set $\sigma(\mathbb{D})$ and making sure that the curve is traversed in one direction. The feedback $u(x)$ should also be such that $\sigma(\mathbb{D})$ is invariant under the output dynamics, meaning that

$$\begin{pmatrix} h(x(0)) \in \sigma(\mathbb{D}) \\ L_f h(x(0)) + L_g h(x(0)) u(x(0)) \in T_{h(x(0))}\sigma(\mathbb{D}) \end{pmatrix} \Longrightarrow (\forall t \geq 0)\ h(x(t)) \in \sigma(\mathbb{D}).$$

When $\mathbb{D} = S^1$, $\sigma(\mathbb{D})$ is a periodic curve. Banaszuk and Hauser in [3] provide a solution to this problem in the special case when $\mathbb{D} = S^1$ and $h(x) = x$. Notice that one particular instance of maneuver regulation is the case when a controller is designed to make $y(t)$ asymptotically track a specific time *parameterization* of the curve $\sigma(t)$. Thus asymptotic tracking and maneuver regulation are closely related problems. In some cases, however, it may be undesirable to pose a maneuver regulation problem as one of tracking because tracking controllers don't make $\sigma(\mathbb{D})$ invariant under the output dynamics. Moreover, even if a maneuver regulation problem admits a solution, its time parameterized version may not (consider, for instance, the problem of maneuvering a wheeled vehicle with bounded translational speed by means of steering).

[3] We do not allow single output systems ($p = 1$) because in such case the problem investigated in this paper trivially requires y to follow the entire real line.

We impose geometric restrictions on the class of curves $\sigma(\cdot)$.

Assumption 1: There exists a C^1 map $\gamma : \mathbb{R}^p \to \mathbb{R}^{p-1}$ such that 0 is a regular value of γ and $\sigma(\mathbb{D}) = \gamma^{-1}(0)$, that is, $(d\gamma)_x$ is full rank for all $x \in \gamma^{-1}(0)$. Moreover, the *lift* of $\gamma^{-1}(0)$ to $\mathbb{R}^n$, $\Gamma := (\gamma \circ h)^{-1}(0)$, is a submanifold of $\mathbb{R}^n$.

The problem of maneuvering y to $\gamma^{-1}(0)$ is thus equivalent to maneuvering x to

$$\Gamma = \{x : \gamma_1(h(x)) = \cdots = \gamma_{p-1}(h(x)) = 0\}. \tag{4}$$

Under mild regularity conditions, this can be cast as an output stabilization problem for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y' &= (\gamma \circ h)(x). \end{aligned} \tag{5}$$

A natural question to ask is whether the path $\sigma(\mathbb{D})$ is *feasible* for (3). In other words, is there a subset of Γ which can be stabilized? In general one may only be able to stabilize the subset of Γ which can be made invariant by a suitable choice of the control input. Accordingly, let Γ^* be a connected component of the largest controlled invariant submanifold of Γ under (3) and let $n^* = \dim \Gamma^*$ ($n^* \leq \dim \Gamma \leq n-1$). Further, let u^* be a *friend of* Γ^*, i.e., a smooth feedback rendering Γ^* invariant, and define $f^* := (f + gu^*)|_{\Gamma^*}$. One of the hypotheses of our main result imply that u^* is unique, see Remark 4.1.

Assumption 2: Γ^* is a closed connected submanifold (with $n^* \geq 1$) and the following conditions hold

(i) $(\exists\, \epsilon > 0)(\forall\, x \in \Gamma^*) \quad \|L_{f^*} h(x)\| > \epsilon$.
(ii) $f^* : \Gamma^* \to T\Gamma^*$ is complete

In [3], $\Gamma^* = \Gamma = \sigma(S^1)$, and it is assumed that $f(x) \neq 0$ on Γ^*. Thus in that work Assumption 2 is automatically satisfied (the completeness of f^* follows from the periodicity of $\sigma(S^1)$).

We first focus on the well-definiteness part of the assumption. In order to derive conditions guaranteeing that Γ^* is a closed submanifold, associate with each constraint in (4) the single input, single output system $\{f, g, \gamma_i \circ h\}$ where $i \in \{1, \ldots, p-1\}$ and a corresponding zero dynamics manifold Γ_i^*.

Lemma 3.1. If $\bigcap_k \Gamma_k^*$ is a non-empty, closed, controlled invariant submanifold, then Γ^* exists and it is given by $\Gamma^* = \bigcap\limits_k \Gamma_k^*$.

Proof. ($\subset$) Choose any point $x \in \Gamma^*$. Since $\Gamma^* \subset \Gamma$,

$$(\forall k \in \{1, \ldots, p-1\}) \quad \gamma_k(h(x)) = 0.$$

This, together with the fact that, by definition, Γ^* is locally invariant around x, implies that

$$(\forall k \in \{1, \dots, p-1\}) \;\; x \in \Gamma_k^*$$

or $x \in \bigcap_k \Gamma_k^*$.

($\supset$) Since $\bigcap_k \Gamma_k^*$ is controlled invariant and output zeroing, and since $\Gamma^* \subset \bigcap_k \Gamma_k^*$, one has that, by the maximality of Γ^*, $\Gamma^* = \bigcap_k \Gamma_k^*$. □

Let r_i be the relative degree of system $\{f, g, \gamma_i \circ h\}$ and define $\mathcal{H}_i : x \mapsto \mathrm{col}(\gamma_i \circ h(x), L_f(\gamma_i \circ h(x)), \dots, L_f^{r_i-1}(\gamma_i \circ h)(x))$. If each r_i is well-defined and uniform over Γ, one has that each Γ_i^* is a closed submanifold and $\Gamma_i^* = \mathcal{H}_i^{-1}(0)$. This is not enough to guarantee that $\bigcap_k \Gamma_k^*$ is a submanifold, as the intersection of two submanifolds need not be a submanifold. A sufficient condition for the intersection $\Gamma_i^* \cap \Gamma_j^*$, $i \neq j$, to be a submanifold is that [8]

$$(\forall x \in \Gamma_i^* \cap \Gamma_j^*) \;\; T_x\Gamma_i^* + T_x\Gamma_j^* \simeq \mathbb{R}^n$$

or, equivalently, $\ker{(d\mathcal{H}_i)}_x + \ker{(d\mathcal{H}_j)}_x \simeq \mathbb{R}^n$. Using the fact that $T_x(\Gamma_i^* \cap \Gamma_j^*) = T_x\Gamma_i^* \cap T_x\Gamma_j^*$ one easily arrives at the following result.

Corollary 3.2. Γ^* is a closed submanifold if each system $\{f, g, \gamma_i \circ h\}$, $i \in \{1 \dots p-1\}$ has a uniform relative degree r_i over Γ and, if $p > 2$, the following conditions are satisfied.

(i) For $k = 1, \dots, p-2$,

$$\left(\forall x \in \bigcap_{j=1}^{k+1} \Gamma_j^*\right) \;\; H_x^k + \ker(d\mathcal{H}_{k+1})_x \simeq \mathbb{R}^n,$$

where H_x^k is defined inductively as

$$\begin{aligned} H_x^1 &:= \ker(d\mathcal{H}_1)_x, \qquad\qquad k = 1 \\ H_x^k &:= H_x^{k-1} \cap \ker(d\mathcal{H}_k)_x,\; k > 1. \end{aligned}$$

(ii) Letting $u_k^* := -\frac{L_f^{r_k}(\gamma_k \circ h)}{L_g L_f^{r_k-1}(\gamma_k \circ h)}$, $1 \leq k \leq p-1$,

$$(u_1^*)\Big|_{\bigcap_i \Gamma_i^*} = \cdots = (u_{p-1}^*)\Big|_{\bigcap_i \Gamma_i^*}.$$

In this case, $n^* = n - \sum_{i=1}^{p-1} r_i$.

Remark 3.1. Rather than using transversality to derive the sufficient conditions of Corollary 3.2, one can employ a slight modification of the zero dynamics algorithm of [11] (see also [10]) or the constrained dynamics algorithm presented in [15]. In both cases a feasible initial condition for the algorithm should be defined to be any point $x_0 \in \Gamma^*$ such that $f(x_0)+g(x_0)u_0 \in T_{x_0}\Gamma^*$ for some real number u_0. If the sufficient conditions of Corollary 3.2 are not satisfied, the zero dynamics algorithm may still find a *locally* maximal controlled invariant submanifold of Γ.

Remark 3.2. Condition (i) in the Corollary above can be weakened by assuming, instead that, for $k = 1, \dots, p-2$,

$$\left(\text{on a neighborhood of } \bigcap_{j=1}^{k+1} \Gamma_j^*\right) \ \dim(H_x^k + \ker(d\mathcal{H}_{k+1})_x) = \text{constant}.$$

The condition, in Assumption 2, that $\|L_{f^*}h(x)\| > \epsilon$ on Γ^* implies that there are no equilibria on Γ^* and that, whenever $x \in \Gamma^*$, $\|\dot{y}\| = \|L_{f^*}h(x)\| > \epsilon$. This condition ensures that the output of (3) traverses the curve $\sigma(\mathbb{D})$. The next example illustrates that this condition is not strictly necessary for the feasibility of the maneuver regulation problem.

Example 3.1. Consider the dynamical system and path

$$\begin{aligned}
\dot{x} &= \operatorname{col}(x_2, u, x_3) \\
y &= \operatorname{col}(x_1, x_2), \quad \sigma : \lambda \in \mathbb{R} \mapsto \operatorname{col}(\lambda, \lambda)
\end{aligned}$$

Here $\mathbb{D} = \mathbb{R}$ and $\sigma(\mathbb{D}) = \{y : y_1 - y_2 = 0\}$. The lift Γ is given by $\Gamma = \{x : x_1 - x_2 = 0\}$ and it is readily seen that $\Gamma^* = \Gamma$ and a friend of Γ^* is $u^* = x_1$. Assumption 2 is not satisfied since there exists a single point on Γ^* where $L_{f^*}h(x) = \operatorname{col}(x_2, x_1) = 0$. Yet, almost all initial conditions on Γ^* result in path traversal. Specifically, the only case where the path is *not* traversed is when $x_1(0) = x_2(0) = 0$. △

Example 3.1 shows that even if Assumption 2 is violated, it may still be possible to traverse the path. However, if $\|L_{f^*}h(x)\|$ fails to be bounded away from zero, then the situation becomes problematic.

Example 3.2. Consider the dynamical system and path

$$\begin{aligned}
\dot{x} &= \operatorname{col}\left(x_1x_3, \frac{-2x_1^2x_3}{(x_1^2+1)^2}, x_3\right) + \operatorname{col}(0, u, 0) \\
y &= \operatorname{col}(x_1, x_2), \quad \sigma : \lambda \in \mathbb{R} \mapsto \operatorname{col}\left(\lambda, \frac{1}{\lambda^2+1}\right)
\end{aligned}$$

Here $\mathbb{D} = \mathbb{R}$, $\Gamma = \Gamma^* = \{x : x_2 - \frac{1}{x_1^2+1} = 0\}$, and $u^* = 0$. Assumption 2 is not satisfied since

$$L_f h(x) = \operatorname{col}\left(x_1x_3, \frac{-2x_1^2x_3}{(x_1^2+1)^2}\right)$$

is zero on the set $\{x : x_1 x_3 = 0\}$. Let $u = -x_2 + \frac{1}{x_1^2+1}$. The result is a closed-loop system where any initial condition

$$x^0 = \mathrm{col}\,(\delta, \star, \epsilon)$$

where $\epsilon\delta = 0$ will not result in path traversal. However, initial conditions with $\epsilon\delta \neq 0$ *will* result in path traversal. This example illustrates the fact that Assumption 2(i) avoids pathological situations whereby some phase curves originating outside of Γ^* may approach points of Γ^* where $L_{f^*}h = 0$, thus not traversing the path $\sigma(\mathbb{D})$. $\triangle$

We are now ready to formulate the main problems investigated in this paper. The following are a direct generalization of analogous statements found in [3].

Problem 1: *Find, if possible, a diffeomorphism*

$$\begin{aligned} \Xi : \mathcal{N} &\to \Xi(\mathcal{N}) \subset \Gamma^* \times \mathbb{R}^{n-n^*} \\ x &\mapsto (z, \xi)\,, \end{aligned}$$

where $\mathcal{N}$ is a neighborhood of Γ^*, such that

(i) The restriction of Ξ to Γ^* is

$$\Xi|_{\Gamma^*} : z \mapsto (z, 0)$$

(ii) The dynamics of system (3) *take the form*

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{n-n^*-1} &= \xi_{n-n^*} \\ \dot{\xi}_{n-n^*} &= b(z, \xi) + a(z, \xi)u \end{aligned} \tag{6}$$

where $a(z, \xi) \neq 0$ in $\Xi(\mathcal{N})$.

The following is the local version of Problem 1.

Problem 2: *Given $x^0 \in \Gamma^*$, find, if possible, a diffeomorphism $\Xi^0 : U^0 \to \Xi(U^0) \subset (\Gamma^* \cap U^0) \times \mathbb{R}^{n-n^*}$, $x \mapsto (z^0, \xi^0)$, where U^0 is a neighborhood of x^0, such that properties (i) and (ii) of Problem 1 are satisfied.*

It is clear that if one can solve Problem 1 or 2, and the map $x \mapsto \xi$ is such that $\xi \to 0$ implies $x \to \Gamma^*$ then the smooth feedback

$$u = -\frac{1}{a(z, \xi)}(b(z, \xi) + K\xi). \tag{7}$$

achieves local output stabilization of (5) and local stabilization of (3) to Γ^* (resp., $\Gamma^* \cap U^0$). In turn, stabilization of (3) to Γ^* implies, by Assumption 2(i), traversal of $\sigma(\mathbb{D})$ in output coordinates. It is also not difficult to see that (7) makes $\sigma(\mathbb{D})$ invariant under the output dynamics (see Section 3). In other words, if one can solve Problem 1 or 2 then one can solve the maneuver regulation problem defined above.

4 Solution to Problem 1

Theorem 4.1. Problem 1 is solvable if and only if there exists a smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $\Gamma^* \subset \{x \in \mathbb{R}^n : \alpha(x) = 0\}$
(2) α yields a uniform relative degree $n - n^*$ over Γ^*.

Proof. ($\Rightarrow$) Consider system (6) and let $\alpha = \xi_1$. Conditions (i) and (ii) follow immediately.

($\Leftarrow$) From a slight modification[4] of the proof of [10, Proposition 9.1.1] one obtains a coordinate transformation $\Xi : \mathbb{R}^n \to \mathcal{Z}^* \times \mathbb{R}^{n-n^*}$, valid in a neighborhood of $\mathcal{Z}^*$, yielding the normal form (6) and such that $\Xi|_{\mathcal{Z}^*} : z \mapsto (z, 0)$. $\mathcal{Z}^*$ is the zero dynamics manifold associated with the output function α. We are left to show that $\mathcal{Z}^* = \Gamma^*$. First notice that $\Gamma^* \subset \mathcal{Z}^*$ for if $x \in \Gamma^*$ then $\alpha(x) = 0$. Since through x there passes a controlled invariant submanifold, Γ^*, and x is output zeroing, it follows that $x \in \mathcal{Z}^*$ as well. Finally, since Γ^* and $\mathcal{Z}^*$ are two connected, closed submanifolds of the same dimension and $\Gamma^* \subset \mathcal{Z}^*$, one has that $\Gamma^* = \mathcal{Z}^*$. □

The function α is used to generate the feedback (7) by setting

$$\begin{aligned} a(\Xi(x)) &= L_g L_f^{n-n^*-1} \alpha(x) \\ b(\Xi(x)) &= L_f^{n-n^*} \alpha(x). \end{aligned}$$

The conditions in Theorem 4.1, although rather intuitive, are difficult to check in practice. However, they are instrumental in deriving *checkable* sufficient conditions for the existence of a solution to Problem 1.

[4] Here the main difference is that we do not require that the vector fields $\{\tau_i\}_{i \in \{1 \ldots n-n^*\}}$ in [10, Proposition 9.1.1] be complete. This implies that the normal form (6) is valid over a neighborhood $\mathcal{N}$ of Γ^*, rather than $\mathbb{R}^n$. If the vector fields τ_i $i \in \{1 \ldots n - n^*\}$ are complete, then the transformation is globally valid on $\mathbb{R}^n$.

Corollary 4.2. If one of the *path constraints* in (4), $\gamma_{\bar{k}} \circ h$, yields a uniform relative degree $n - n^*$ over Γ^*, then Problem 1 is solved by setting $\alpha = \gamma_{\bar{k}} \circ h$.

Thus, it may be possible to solve Problem 1 by performing input-output linearization choosing as output one of the path constraints.

Lemma 4.3. A necessary condition for the solvability of Problem 1 (resp. Problem 2) is that, for all $x \in \Gamma^*$ (resp., for $x = x^0$),

$$T_x\Gamma^* + \mathrm{span}\{g, \ldots, ad_f^{n-n^*-1} g\}(x) \simeq \mathbb{R}^n. \tag{8}$$

Proof. Condition (8) is coordinate and feedback invariant. So it is enough to show that it holds in (z, ξ)-coordinates with the feedback transformation $u = (-b(z,\xi) + v)/a(z,\xi)$. For any $x \in \Gamma^*$ (resp., for $x = x^0$), let $(z, 0) = \Xi(x)$. Choose local coordinates (W, ψ) for Γ^* around z and let $\tilde{g}$ and $\tilde{f}$ denote the system vector fields after coordinate and feedback transformation. We have

$$\begin{aligned} &T_{\psi(z)}\psi(W) + \mathrm{span}\{\tilde{g}, \ldots, ad_{\tilde{f}}^{n-n^*-1}\tilde{g}\}(\psi(z), 0) \\ &= \mathrm{Im}\left(\begin{bmatrix} I_{n^*} & 0 & \star & \cdots & \star \\ 0_{n-n^*\times n^*} & b & Ab & \cdots & A^{n-n^*-1}b \end{bmatrix}\right) \end{aligned}$$

where the pair (A, b) is in Brunovky normal form. The claim immediately follows. ☐

Remark 4.1. Condition (8) is a generalization of the notion of transverse linear controllability to the case of controlled invariant submanifolds of any dimension. It is useful in deriving checkable sufficient conditions for the existence of a solution to Problem 1. It also implies that

$$(\forall x \in \Gamma^*)\ T_x\Gamma^* \cap \mathrm{span}\{g\} = 0$$

and hence that the friend u^* of Γ^* is unique (see [10]).

The notion of transverse linear controllability was introduced in [3]. See also [12] for a more general notion. In both papers, $n^* = 1$, $\mathbb{D} = S^1$, and $T_x\Gamma^* = \mathrm{span}\,\{f^*(x)\}$.

Theorem 4.4. Problem 1 is solvable if

(1) Γ^* is diffeomorphic to a generalized cylinder ($\Gamma^* \cong T^k \times \mathbb{R}^{n^*-k}$, $k \in \{0, \ldots, n^*\}$, T^k is the k-torus)
(2) $T_x\Gamma^* + \mathrm{span}\,\{g \ldots ad_f^{n-n^*-1} g\}(x) = \mathbb{R}^n$ on Γ^*

(3) $(n-n^* \geq 2)$ [5] $\implies$ (span $\{g \ldots ad_f^{n-n^*-2}g\}$ is involutive).

Proof. We will show that if the above conditions hold, then a function α can be constructed satisfying the conditions of Theorem 4.1. Since $\Gamma^* \cong T^k \times \mathbb{R}^{n^*-k}$ there exists a diffeomorphism $\Theta : T^k \times \mathbb{R}^{n^*-k} \to \Gamma^*$. Let $w_1, \ldots, w_k$ be vector fields on T^k whose integral curves form curvilinear coordinates on T^k. Let $w_{k+1}, \ldots, w_{n^*}$ be the natural basis of $\mathbb{R}^{n^*-k}$. Push $w_1, \ldots, w_{n^*}$ forward by Θ to obtain $v_1, \ldots, v_{n^*} : \Gamma^* \to T\Gamma^*$. Note that for $p \in \Gamma^*$ the domain of $t \mapsto \phi_t^{v_i}(p)$ is S^1 if $i \in \{1, \ldots, k\}$ and $\mathbb{R}$ if $i \in \{k+1, \ldots, n^*\}$.

Condition (2) can be rewritten as

$$\text{span}\{v_1, \ldots, v_{n^*}, g, \ldots, ad_f^{n-n^*-1}g\}(x) \simeq \mathbb{R}^n.$$

We use the flows of these vector fields to generate s-coordinates. Choose any point $x^0 \in \Gamma^*$ and consider the mapping F defined as

$$s \mapsto \Phi^g_{s_n} \circ \cdots \circ \Phi^{ad_f^{n-n^*-1}g}_{s_{n^*+1}} \circ \Phi^{v_{n^*}}_{s_{n^*}} \circ \cdots \circ \Phi^{v_1}_{s_1}(x^0).$$

The map $F : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$ is a neighborhood of $T^k \times \mathbb{R}^{n^*-k}$ in $\mathbb{R}^{n^*}$ and $\mathcal{N}$ is a neighborhood of Γ^*, is a diffeomorphism. Let $T_1 = \text{col}(s_1, \ldots, s_{n^*+1})$, $T_2 = \text{col}(s_{n^*+2}, \ldots, s_n)$, and define

$$\begin{aligned}
H_1^{T_1}(x^0) &:= \Phi^{ad_f^{n-n^*-1}g}_{s_{n^*+1}} \circ \Phi^{v_{n^*}}_{s_{n^*}} \circ \cdots \circ \Phi^{v_1}_{s_1}(x^0) \\
H_2^{T_2}(x^1) &:= \begin{cases} \Phi^g_{s_n} \circ \cdots \Phi^{ad_f^{n-n^*-2}g}_{s^{n^*+2}}(x^1) & \text{if } n-n^* \geq 2 \\ x^1 & \text{if } n-n^* = 1. \end{cases}
\end{aligned}$$

With these definitions, rewrite $F(s)$ as

$$F(s) = H_2^{T_2} \circ H_1^{T_1}(x^0).$$

Choose $\alpha(x) = s_{n^*+1}(x)$. By construction, any point $x \in \Gamma^*$ can be reached by flowing along $v_1, \ldots v_{n^*}$. Therefore, in s-coordinates, any point $x \in \Gamma^*$ is represented as

$$F^{-1}(x) = \text{col}\Big(\underbrace{\star \ \cdots \ \star}_{n^* \text{ elements}} \ 0 \ \cdots \ 0 \Big).$$

Thus, on Γ^*, $\alpha(x) = 0$, which proves that condition (1) of Theorem 4.1 is satisfied. Further, notice that, by construction

$$L_{ad_f^{n-n^*-1}g}\alpha = 1$$

on $\mathcal{N}$. If $n - n^* = 1$, this shows that α yields a uniform relative degree 1 over Γ^*, as required. If $n - n^* \geq 2$, let $D = \text{span}\,\{ad_f^i g\}_{i \in \{0,\ldots,n-n^*-2\}}$. By

[5] Notice that since $p \geq 2$, one has that $n - n^* \geq 1$.

assumption, D is a non-singular and involutive distribution. Let S denote the integral manifold of D passing through the point $H_1^{T_1}(x^0)$. In s-coordinates

$$S = \{s \in \mathcal{M} \cap V \subset \mathbb{R}^n : s_1 = c_1, \dots, s_{n^*+1} = c_{n^*+1}\}$$

where c_i, $i = 1, \dots, n^* + 1$ are constants and V is an open set containing the point $H_1^{T_1}(x^0)$. Thus, for any $s \in S$,

$$T_s S = \operatorname{Im} \begin{bmatrix} 0 \\ \\ I_{n-n^*-1} \end{bmatrix}.$$

Since S is an integral manifold of D it follows that, in s-coordinates, $D(s) = T_s S$, implying that in s-coordinates the vector fields $ad_f^i g$, $i \in \{0, \dots, n-n^*-2\}$, have the form:

$$ad_f^i g = \operatorname{col}\Big(0 \quad \cdots \quad 0 \quad \underbrace{\star \quad \cdots \quad \star}_{n-n^*-1 \text{ elements}} \Big).$$

It readily follows that, on $\mathcal{N}$, $L_{ad_f^i g}\alpha(x) = 0$, $i = 0, \dots, n - n^* - 2$. Thus, α yields a uniform relative degree $n - n^*$ over Γ^*. □

Corollary 4.5. If $n - n^* \in \{1, 2\}$ and $\Gamma^* \cong T^k \times \mathbb{R}^{n^*-k}$, then Problem 1 is solvable if and only if (3) is transversely linearly controllable.

Proof. This follows directly from Lemma 4.3 and Theorem 4.4. □

Example 4.1. We return to the unicycle system (1) of Section 2 and show that Problem 1 is always solvable for (1). Let $\sigma : \mathbb{D} \to \mathbb{R}^2$ be any curve satisfying assumption 1. It is easy to show, for example using the zero dynamics algorithm [11], that Γ^* is closed connected and $n^* = 1$. These facts together imply that Γ^* is parallelizable satisfying condition (1) of Theorem 4.4.

Furthermore, since $n-n^*-2 = 0$, condition (3) of Theorem 4.4 is automatically satisfied. Simple geometric considerations reveal that condition (2) also holds. We conclude that the conditions of Theorem 4.4 are all satisfied and thus Problem 1 is always solvable for the kinematic unicycle (1). △

Example 4.2. Go back to system (2) in Section 2. Recall that

$$\Gamma^* = \{x : x_1 = x_3 = 0\}.$$

Thus $n^* = 1$ and the friend of Γ^* is $u^* = 0$, yielding $f^* = \frac{\partial}{\partial x_2}$. Check the sufficient conditions of Theorem 4.4. Parallelizability of Γ^* is obvious. We have

$$g = x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad ad_f g = (x_3 - 1)\frac{\partial}{\partial x_1}.$$

Thus, for all $x \in \Gamma^*$,

$$T_x\Gamma^* + \mathrm{span}\{g, ad_f g\}(x) = \mathrm{span}\left\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, -\frac{\partial}{\partial x_1}\right\} \simeq \mathbb{R}^3$$

showing that the system is transversely linearly controllable and condition (1) in Theorem 4.4 is satisfied. Condition (2) is automatically satisfied by Corollary 4.5. We conclude that there exists a solution to Problem 1. One obtains the function α given in Section 2 by following the semi-constructive procedure outlined in the proof of Theorem 4.4. $\triangle$

5 Linear Time Invariant Systems

In this section we specialize our results to the case of LTI systems with paths given by straight lines passing through the origin (i.e., one dimensional subspaces as opposed to one dimensional submanifolds). The value in this analysis is that it better illustrates some of the ideas of this work, and shows how our solution recovers a well-known necessary and sufficient condition for output stabilization of LTI systems.

Consider the following n-dimensional single input linear system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx, \end{aligned} \tag{9}$$

(with $y \in \mathbb{R}^p$) and, given a full rank $D \in \mathbb{R}^{(p-1)\times p}$, define the path as $\sigma(\mathbb{R}) := \ker(D)$. It is readily seen that $\sigma(\mathbb{R})$ satisfies Assumption 1. The lift of the path to the state space is $\Gamma = \ker(DC)$, and hence Assumption 1 is satisfied. In the linear setting, Γ^* becomes the largest (A, b)-invariant subspace contained in $\ker(DC)$. Let F be a *friend* of Γ^*, i.e., a feedback matrix that makes Γ^* an invariant subspace for $(A + bF)$. Then, we have $f^*(x) = (A + bF)x$, which is a complete vector field, and so Assumption 2(ii) is satisfied. Notice, however, that Assumption 2(i) is not satisfied in this simplified setup because

$$L_{f^*}h(x) = C(A + bF)x$$

is always zero at the origin. Hence, we focus our attention solely on the output stabilization problem. We thus seek to stabilize the output of the LTI system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y' &= C'x := DCx. \end{aligned} \tag{10}$$

by means of state feedback. Following [17], we refer to this as the *output stabilization problem (OSP)*. In Problem 1, we require that the ξ dynamics be in Brunovky normal form. Thus in the spirit of Problem 1, we further require that the rate of decay of the output to zero can be arbitrarily assigned (another way to say this is that the observable modes of (A, b, C') can be pole-shifted). We refer to this as the *output stabilization with controllability problem (OSCP)*. In this section we show that OSCP is *equivalent* to Problem 1.

Following [17], partition $\mathbb{C}$ as the disjoint union $\mathbb{C}^+ \cup \mathbb{C}^-$ where $\mathbb{C}^+$ denotes the closed right half complex plane. Let $m(\lambda)$ denote the minimal polynomial of A and factor $m(\lambda)$ as $m(\lambda) = m^+(\lambda)m^-(\lambda)$, where the zeros of m^+ and m^- are in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. Let $V^+(A) = \ker(m^+(A))$ and $V^-(A) = \ker(m^-(A))$ be the associated unstable and stable modal subspaces of A. Then, $\mathbb{R}^n = V^+(A) \oplus V^-(A)$. Theorem 4.4 in [17] gives a necessary and sufficient condition for output stabilizability of (10):

Theorem 5.1. OSP is solvable if and only if

$$V^+(A) \subset \Gamma^* + \mathrm{Im}([b \; Ab \; \cdots A^{n-1}b]).$$

The theorem can be rephrased as follows ([17]): OSP is solvable if and only if the unstable modes of A can be made unobservable at the output or they can pole-shifted. The following is an obvious consequence of Theorem 5.1.

Corollary 5.2. OSCP is solvable if and only if

$$\mathbb{R}^n = \Gamma^* + \mathrm{Im}([b \; Ab \; \cdots \; A^{n-1}b]). \tag{11}$$

Intuitively (11) states that all modes of A can be made unobservable at the output or they can be pole-shifted.

It turns out that condition (11) is precisely transverse linear controllability specialized to the LTI setting.

Lemma 5.3. Condition (11) is equivalent to

$$\mathbb{R}^n = \Gamma^* \oplus \mathrm{Im}([b \; Ab \; \cdots \; A^{n-n^*-1}b]). \tag{12}$$

We omit the easy proof of this lemma. Since $f(x) = Ax$ and $g(x) = B$,

$$\mathrm{span}\{g, \ldots, ad_f^{n-n^*-1}g\} = \mathrm{Im}([b \; Ab \; \cdots \; A^{n-n^*-1}b]),$$

so (12) coincides with the transverse linear controllability condition (8). In conclusion, problem OSCP is solvable if and only if (9) is transversely linearly controllable. On the other hand, applying Theorem 4.1 and Lemma 4.3 to the LTI system (9), it follows that transverse linear controllability is also a necessary and sufficient condition for the solvability of Problem 1. We conclude that, in the LTI case, Problem 1 is equivalent to OSCP.

6 Solution to Problem 2

The following is an obvious result in the light of Theorem 4.1.

Theorem 6.1. Problem 2 is solvable if and only if there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ defined in a neighborhood U^0 of some $x^0 \in \Gamma^*$ such that

(1) $\Gamma^* \cap U^0 \subset \{x \in U^0 : \alpha(x) = 0\}$
(2) α yields a relative degree $n - n^*$ at x^0.

Proof. ($\Rightarrow$) Let $\alpha = \xi_1^0$, conditions (1) and (2) follow.

($\Leftarrow$) Let $\xi_1^0 = \alpha(x)$. A partial coordinate transformation on U^0 is given by

$$\xi_k^0 = L_f^{k-1}\alpha,\ k \in \{1 \dots n - n^*\}.$$

We seek n^* more independent functions to complete the transformation and yield the correct form. This can always be done [10, Proposition 4.1.3]. From the proof of Theorem 4.1 we have that the zero dynamics of the resulting normal form coincide, on U^0, with Γ^*. □

When $n - n^* \geq 2$, let

$$D = \text{span}\,\{g \dots ad_f^{n-n^*-2}g\}. \tag{13}$$

Theorem 4.4 proves that if $\Gamma^* \cong T^k \times \mathbb{R}^{n^*-k}$, then the involutivity of D, together with transverse linear controllability, are sufficient conditions for the existence of a function α satisfying conditions (1) and (2) in Theorem 4.1 and hence solving Problem 1. When the involutive closure of D, $\text{inv}(D)$, is regular at $x^0 \in \Gamma^*$, the next result provides necessary and sufficient conditions to solve Problem 2. These conditions are easier to check than those in Theorem 6.1.

Theorem 6.2. Assume that $\text{inv}(D)$ is regular at $x^0 \in \Gamma^*$. Then Problem 2 is solvable if and only if

(1) $T_{x^0}\Gamma^* + \text{span}\{g, \dots, ad_f^{n-n^*-1}g\}(x^0) \simeq \mathbb{R}^n$
(2) $(n - n^* \geq 2) \Longrightarrow ad_f^{n-n^*-1}g(x^0) \notin T_{x^0}\Gamma^* + \text{inv}(D)(x^0)$.

Proof. ($\Rightarrow$) Assume that the conditions of Theorem 6.1 hold. By Lemma 4.3, (1) holds. By definition of relative degree, in a neighborhood of $\Gamma^* \cap U^0$,

$d\alpha \in D^\perp$ and $L_{ad_f^{n-n^*-1}g}\alpha \neq 0$. Recall that $d\alpha \in D^\perp$ implies $d\alpha \in (\mathrm{inv}\, D)^\perp$. Since $L_{ad_f^{n-n^*-1}g}\alpha \neq 0$, one has that $ad_f^{n-n^*-1}g \notin \mathrm{span}\{d\alpha\}^\perp$ and thus also $ad_f^{n-n^*-1}g \notin \mathrm{inv}(D)$. This fact and condition (1) easily imply condition (2).

($\Leftarrow$) Assume conditions (1) and (2) hold. Condition (2) implies $\dim(\mathrm{inv}(D)) \leq n-1$. Clearly $\dim(\mathrm{inv}\, D) \geq n - n^* - 1$. If $\dim(\mathrm{inv}\, D) = n - n^* - 1$ then essentially the same proof of Theorem 4.4 applies and we are done. Hence, we focus on the case $n - n^* \leq \dim(\mathrm{inv}\, D) \leq n-1$. Let $\{v_1, \ldots, v_{n^*}\}$ be a set of vector fields defined locally on $\Gamma^* \cap V$ (where V is some open neighborhood of $\mathbb{R}^n$ containing x^0) such that

$$(\forall\, x \in \Gamma^* \cap V)\ T_x\Gamma^* = \mathrm{span}\{v_1, \ldots, v_{n^*}\}(x).$$

As in the proof of Theorem 4.4, generate s-coordinates by flowing along the vector fields $v_1, \ldots, v_{n^*}, ad_f^{n-n^*-1}g, \ldots, g$ with times $s_1, \ldots, s_{n^*}, s_{n^*+1}, \ldots, s_n$, respectively. By condition (1), there exists a neighborhood $U \subset V$ of x^0 such that the map F defined as

$$s \mapsto \Phi^g_{s_n} \circ \cdots \circ \Phi^{ad_f^{n-n^*-1}g}_{s_{n^*+1}} \circ \Phi^{v_{n^*}}_{s_{n^*}} \circ \cdots \circ \Phi^{v_1}_{s_1}(x^0),$$

is a diffeomorphism of $F^{-1}(U)$ onto U. Define the set

$$M := \{x \in U : s_{n^*+2}(x) = \cdots = s_n(x) = 0\}$$

which is a submanifold of U containing $\Gamma^* \cap U$ of dimension $n^* + 1$. The submanifold M is the set of points reachable from Γ^* by flowing along $ad_f^{n-n^*-1}g$. Therefore, for $x \in \Gamma^* \cap U$, $T_xM = T_x\Gamma^* \oplus \mathrm{span}\{ad_f^{n-n^*-1}g(x)\}$. Without loss of generality, we assume that $\mathrm{inv}(D)$ has constant dimension on U. Given $x \in \Gamma^* \cap U$ we have

$$\dim(T_x\Gamma^* \cap \mathrm{inv}(D))(x) = \dim(T_x\Gamma^*) + \dim(\mathrm{inv}(D)(x)) - \dim(T_x\Gamma^* + \mathrm{inv}(D)(x)).$$

Conditions (1) and (2) imply that $\dim(T_x\Gamma^* + \mathrm{inv}(D)(x)) = n-1$. Hence $\dim(T_x\Gamma^* \cap \mathrm{inv}(D))$ is constant dimensional on U. Let $\hat{n} = \dim(T_x\Gamma^* \cap \mathrm{inv}(D))$. By conditions (1) and (2), for $x \in \Gamma^* \cap U$,

$$\begin{aligned}&\mathrm{span}\{ad_f^{n-n^*-1}g(x)\} \cap (T_x\Gamma^* + \mathrm{inv}(D)(x)) \\ &\quad = (\mathrm{span}\{ad_f^{n-n^*-1}g(x)\} \cap T_x\Gamma^*) + (\mathrm{span}\{ad_f^{n-n^*-1}g(x)\} \cap \mathrm{inv}(D)(x)) = 0.\end{aligned}$$

Since the operation of set intersection above distributes, we also have that, for $x \in \Gamma^* \cap U$,

$$\begin{aligned}T_xM \cap \mathrm{inv}(D)(x) &= (T_x\Gamma^* + \mathrm{span}\{ad_f^{n-n^*-1}g(x)\}) \cap \mathrm{inv}(D)(x) \\ &= (T_x\Gamma^* \cap \mathrm{inv}(D)(x)) + (\mathrm{span}\{ad_f^{n-n^*-1}g(x)\} \cap \mathrm{inv}(D)(x)) \\ &= T_x\Gamma^* \cap \mathrm{inv}(D)(x).\end{aligned}$$

Thus, for $x \in \Gamma^* \cap U$, $\dim(T_xM \cap \mathrm{inv}(D)(x)) = \hat{n}$. Next we argue that $T_xM \cap \mathrm{inv}(D)(x)$ has constant dimension $\hat{n}$ for all $x \in M$. For,

$$\dim(T_xM \cap \mathrm{inv}(D)(x)) = \dim(T_xM) + \dim(\mathrm{inv}(D)(x)) - \dim(T_xM + \mathrm{inv}(D)(x))$$

and $\dim(T_xM + \mathrm{inv}(D)(x)) = n$ on $\Gamma^* \cap U$ and hence also, without loss of generality, U. Since we are considering the case $n - n^* \leq \dim(\mathrm{inv}\, D) \leq n-1$, we have that $1 \leq \hat{n} \leq n^*$. Making, if needed, U smaller, let $\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}\}$ be a set of vector fields defined on M spanning $T_xM \cap \mathrm{inv}(D)$. Choose additional $n^* - \hat{n}$ vector fields $\{\hat{v}_{\hat{n}+1}, \ldots, \hat{v}_{n^*}\}$ defined on $\Gamma^* \cap U$ such that $T_x\Gamma^* = \mathrm{span}\{\hat{v}_1, \ldots, \hat{v}_{n^*}\}(x) \ \ \forall x \in \Gamma^* \cap U$. Then, on $\Gamma^* \cap U$,

$$\begin{gathered}\mathrm{span}\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}, \hat{v}_{\hat{n}+1}, \ldots, \hat{v}_{n^*}, g, \ldots, ad_f^{n-n^*-1} g\} \simeq \mathbb{R}^n \\ \mathrm{inv}(D) = D \oplus \mathrm{span}\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}\}.\end{gathered}$$

By making, if necessary, U smaller we can assume that the decomposition of $\mathrm{inv}(D)$ above holds on M. The domain of definition of the vector fields involved in our construction is summarized as:

$$\begin{aligned}&\{\hat{v}_{\hat{n}+1}, \ldots, \hat{v}_{n^*}\} \text{ on } \Gamma^* \cap U \\ &\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}\} \text{ on } M \\ &\{g, \ldots, ad_f^{n-n^*-1} g\} \text{ on } U.\end{aligned}$$

We use these vector fields to define the map $G : G^{-1}(U^0) \to U^0$ ($U^0 \subset U$ is a neighborhood of x^0),

$$\begin{aligned}p \mapsto & \Phi^g_{p_n} \circ \cdots \circ \Phi^{ad_f^{n-n^*-2}}_{p_{n^*+2}} \circ \Phi^{\hat{v}_{\hat{n}}}_{p_{n^*+1}} \circ \cdots \circ \Phi^{\hat{v}_1}_{p_{n^*-\hat{n}+2}} \\ & \circ \Phi^{ad_f^{n-n^*-1} g}_{p_{n^*-\hat{n}+1}} \circ \Phi^{\hat{v}_{n^*}}_{p_{n^*-\hat{n}}} \circ \cdots \circ \Phi^{\hat{v}_{\hat{n}+1}}_{p_1}(x^0).\end{aligned}$$

Let $P_1 = (p_1, \ldots, p_{n^*-\hat{n}})$, $P_2 = (p_{n^*-\hat{n}+1}, \ldots, p_{n^*+1})$, $P_3 = (p_{n^*+2}, \ldots, p_n)$, and define

$$\begin{aligned}G_1^{P_1}(x^0) &:= \Phi^{\hat{v}_{n^*}}_{p_{n^*-\hat{n}}} \circ \cdots \circ \Phi^{\hat{v}_{\hat{n}+1}}_{p_1}(x^0) \\ G_2^{P_2}(x^1) &:= \Phi^{\hat{v}_{\hat{n}}}_{p_{n^*+1}} \circ \cdots \circ \Phi^{\hat{v}_1}_{p_{n^*-\hat{n}+2}} \circ \Phi^{ad_f^{n-n^*-1} g}_{p_{n^*-\hat{n}+1}}(x^1) \\ G_3^{P_3}(x^2) &:= \begin{cases}\Phi^g_{p_n} \circ \cdots \circ \Phi^{ad_f^{n-n^*-2}}_{p_{n^*+2}}(x^2) & \text{if } n - n^* \geq 2 \\ x^2 & \text{if } n - n^* = 1.\end{cases}\end{aligned}$$

so that $G(p) = G_3^{P_3} \circ G_2^{P_2} \circ G_1^{P_1}(x^0)$. For a fixed x^0, $x^1 \in \Gamma^* \cap U^0$, and $x^2 \in M \cap U^0$, each $G_i^{P_i}$ is a diffeomorphism onto its image, thus G is a diffeomorphism onto U^0. This can be most easily seen by examining the order in which the various flows are composed. In particular, since $\hat{v}_{\hat{n}+1}, \ldots, \hat{v}_{n^*}$ are independent on Γ^*, the set of points reached by flowing along these vector fields is a submanifold, $\bar{S}$, of dimension $n^* - \hat{n}$, contained in Γ^*. Next, since

$ad_f^{n-n^*-1}g, \hat{v}_1, \ldots, \hat{v}_{\hat{n}}$ are independent on $M \cap U^0$, the set of points reachable from $\bar{S}$ by flowing along these vector fields is precisely $M \cap U^0$. Thus $M \cap U^0 = \{x \in U^0 : P_3(x) = 0\}$ and $\Gamma^* \cap U^0 = \{x \in U^0 : p_{n^*-\hat{n}+1}(x) = 0, P_3(x) = 0\}$. Finally, the set of points reachable from $M \cap U^0$ by flowing along $g, \ldots, ad_f^{n-n^*-2}g$ is the entire U^0.

Choose $\alpha(x) = p_{n^*-\hat{n}+1}(x)$. Then $\Gamma^* \cap U^0 \subset \{x \in U^0 : \alpha(x) = 0\}$ and thus condition (1) in Theorem 6.1 is satisfied. By the same reasoning used the in the proof of Theorem 4.4, the involutivity of $\text{inv}(D) = \text{span}\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}\} + D$, implies that the vector fields $ad_f^i g$, $i = 0, \ldots, n - n^* - 2$, in s-coordinates have the form

$$ad_f^i g = \text{col}\Big(\underbrace{0 \ \cdots \ 0}_{n^*-\hat{n}+1 \text{ zeros}} \ \star \ \cdots \ \star \Big)$$

and thus, on U^0, $L_{ad_f^i g}\alpha = 0$, $i = 0, \ldots, n - n^* - 2$. It is also clear that $L_{ad_f^{n-n^*-1}g}\alpha = 1$ on U^0. Thus the assumptions of Theorem 6.1 are satisfied. $\square$

Note that if $\dim(\text{inv}\, D(x^0)) = n$, condition (2) in Theorem 6.1 is violated and Problem 2 is unsolvable at x^0. This easily implies the following

Corollary 6.3. If there exists $x^0 \in \Gamma^*$ such that $\dim(\text{inv}\, D(x^0)) = n$, then Problem 1 is unsolvable.

Corollary 6.4. Assume that $\text{inv}\, D$ is regular on Γ^* and that, for all $x \in \Gamma^*$,

(1) $T_x\Gamma^* + \text{span}\{g, \ldots, ad_f^{n-n^*-1}g\}(x) \simeq \mathbb{R}^n$
(2) $ad_f^{n-n^*-1}g(x) \notin T_x\Gamma^* + \text{inv}(D)(x)$.

Then there exists an open covering $\{U^{(i)}\}$ of Γ^* and a collection of transformations $\{\Xi^{(i)}\}$, with $\Xi^{(i)} : x \mapsto (z^{(i)}, \xi^{(i)}) \in \Gamma^* \cap U^{(i)} \times \mathbb{R}^{n-n^*}$ such that $\Gamma^* \cap U^{(i)} = \{\xi^{(i)} = 0\}$ and in $(z^{(i)}, \xi^{(i)})$ coordinates the systems has the form (6).

7 State Maneuvers

In this section, we show that when $y = x$ in (3) and $\mathbb{D} = S^1$ the results obtained thus far are equivalent to the results presented in [3]. See also [14]. The conditions presented in [3, Theorem 2.1] for a global solution are

(a) $\dim\Big(\text{span}\,\{f^*, g, \ldots, ad_{f^*}^{n-2}g\}\Big) = n$ on Γ^*
(b) There exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that
  (i) $d\alpha \neq 0$ on Γ^*.
  (ii) $\alpha = 0$ on Γ^*

(iii) $L_{ad^i_{f^*}g}\alpha = 0$ near Γ^* for $i = 0 \ldots n-3$.

Lemma 7.1. Conditions (a) and (b) above hold if and only if the conditions of Theorem 4.1 hold.

Proof. ($\Rightarrow$) Assume conditions (a) and (b) hold. Condition (b.ii) is the same as condition (1) in Theorem 4.1. Next, since f^* is by definition tangent to Γ^*, condition (b.ii) implies that $L_{f^*}\alpha = 0$ on Γ^*. This, together with condition (b.iii), implies that, on Γ^*, $\text{span}\{d\alpha\}^\perp = \text{span}\{f^*, g, \ldots, ad^{n-3}_{f^*}g\}$. By condition (a), necessarily $L_{ad^{n-2}_{f^*}g}\alpha \neq 0$ on Γ^*. This, together with condition (b.iii) shows that α yields a relative degree $n-1$, which is precisely condition (2) in Theorem 4.1.

($\Leftarrow$) Assume the conditions of Theorem 4.1 hold. Condition (a) holds by Lemma 4.3. Condition (b.ii) is identical to condition (1) in Theorem 4.1. Finally, since α yields a relative degree $n-1$ (recall that here $n^* = 1$), conditions (b.i) and (b.iii) are satisfied. $\square$

In the local case, consider the distribution D, in (13), with $n^* = 1$. The conditions presented in [3, Theorem 2.4] are

(a) $\dim\left(\text{span}\left\{f^*, g, \ldots, ad^{n-2}_{f^*}g\right\}\right) = n$ on Γ^*
(b) The distribution D is either
  (i) involutive or
  (ii) $\dim(\text{inv}\, D) = n-1$ in a neighborhood of Γ^* and $f^* \in \text{inv}\, D$ on Γ^*.

Lemma 7.2. Conditions (a) and (b) above hold if and only if the conditions of Corollary 6.4 hold.

Proof. ($\Rightarrow$) Assume (a) and (b) above hold. Then we just have to show that condition (2) of Corollary 6.4 holds. If D is involutive then (a) immediately gives (2). Otherwise, since $f^* \in \text{inv}\, D$ on Γ^*, condition (a) implies condition (2).

($\Leftarrow$) Obvious. $\square$

A key difference between the normal form presented in this paper (6) and the one presented in [3] lies in the structure given to the vector field f_0 in (6). In the case $n^* = 1$ the following procedure illustrates how to obtain the normal form presented in [3]. Fix a point $x_0 \in \Gamma^*$ and define the map $t \mapsto \Phi^{f^*}_t(x^0)$ and its inverse $\varphi' : \Gamma^* \to \varphi'(\Gamma^*)$. Note that, by Assumption 2(ii), φ' is globally defined and that, when $\mathbb{D} = S^1$, $\varphi'(\Gamma^*) = S^1$. By construction $L_{f^*}\varphi' = 1$ on Γ^*. Let $z = \varphi'(x)$ and let $\xi_i = L^{i-1}_f\alpha$, $i = 1 \ldots n-1$. With this transformation, together with the feedback

$$u = \frac{-L_f^{n-1}\alpha + v}{L_g L_f^{n-2}\alpha}$$

one obtains

$$\begin{aligned}
\dot{z} &= 1 + f_1(z,\xi) + g_0(z,\xi)v \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{n-2} &= \xi_{n-1} \\
\dot{\xi}_{n-1} &= v,
\end{aligned} \tag{14}$$

$(f_1(z,0) = 0)$ which is the normal form as presented in [3]. It is interesting to note that the normal form of [3] is also valid when $\mathbb{D} = \mathbb{R}$ (in such a case, the domain of z is $\varphi'(\Gamma^*) = \mathbb{R}$ rather than S^1).

When $n^* > 1$, the normal form (14), could perhaps be generalized by finding a partial coordinate transformation $z = \varphi(x)$ yielding $\dot{z} = \mathrm{col}(1, 0, \ldots, 0)$ on Γ^*. This is always possible locally. Doing so globally amounts to finding a *global* rectification for a vector field on a manifold.

8 Conclusions

Feedback linearizing transverse dynamics is one approach to designing maneuver regulation controllers. Clearly one may be able to solve maneuver regulation problems for systems that are not transversely feedback linearizable. However, when achievable, transverse feedback linearization is an attractive approach to simplify the control design. This paper presented conditions for transverse feedback linearization to be achievable. Extension of the present work to multi-input systems is under way.

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