

Output Feedback Tracking: A Separation Principle Approach

Manfredi Maggiore and Kevin M. Passino

Abstract— We study the practical and asymptotic tracking problems for nonlinear systems when only the output of the plant and the reference signal are available for feedback. We provide sufficient conditions and a control topology yielding practical tracking. In the special case when the reference signal is generated by an exosystem and there exists an internal model satisfying suitable observability properties tracking becomes asymptotic.

I. INTRODUCTION

Consider the nonlinear system

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x)\end{aligned}\quad (1)$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^m$ is the measurable output. The vector field f and the function h are assumed to be sufficiently smooth. In this paper we address following problem.

Problem 1 (Output Feedback Practical Tracking): *Given the dynamical system (1), a sufficiently smooth reference trajectory¹ $r(t) = [r_1(t), \dots, r_m(t)]^\top$, and any real number $e_0 > 0$, find, if possible, an output feedback*

$$\begin{aligned}\dot{x}_c &= f_c(x_c, y, r) \\ u &= h_c(x_c, y)\end{aligned}\quad (2)$$

with the property that for the closed-loop system (1)-(2) there exists a positive real number T and a closed set \mathcal{A} such that any integral curve $(x(t), x_c(t))$ leaving from \mathcal{A} is defined for all $t \geq 0$, bounded, and $\|y(t) - r(t)\| \leq e_0$ for all $t \geq T$.

If Problem 1 can be solved with $e_0 = 0$ and $T = \infty$, we say that (2) solves the output feedback *asymptotic* tracking problem. Additionally, if the projection $\{x \in \mathbb{R}^n : (x, x_c) \in \mathcal{A}\}$ can be made arbitrarily large by a suitable choice of the controller, we say that the solution to Problem 1 is *semiglobal*.

Problem 1 has been solved globally and asymptotically for systems in *output feedback form* ([1], [2]). When the reference trajectory is generated by an exosystem, Problem 1 is included in the *more general* class of output regulation problems [3], where exosystem-generated disturbances and parametric uncertainties are allowed to affect the plant (our approach does not handle these). It has been shown, for special classes of nonlinear systems, how to solve the output regulation problem globally [4] or semi-globally [5], [6]. Other (non-output feedback) approaches to output tracking include differential flatness [7] and system inversion [8]. See also the more recent work in [9].

In this paper we cast Problem 1 as a nonautonomous stabilization problem and assume that there exists a smooth feedback stabilizing the system's state to the state of the stable inverse of the plant.

M. Maggiore is with the Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Rd., Toronto, ON, M5S 3G4, Canada (e-mail: maggiore@control.utoronto.ca). Part of this research was performed when the author was at the Ohio State University.

K. Passino is with the Department of Electrical Engineering, The Ohio State University, 2015 Neil Avenue, Columbus, OH 43210 USA (e-mail: k.passino@osu.edu).

This work was supported by NASA Glenn Research Center, Grant NAG3-2084 and the Natural Sciences and Engineering Research Council of Canada (NSERC).

¹We do not require to know the entire signal $r(t)$ in advance.

We use observers to estimate state and stable inverse of the plant and, in the spirit of the separation principle in [10], employ the resulting estimates to define an output feedback controller solving Problem 1. The estimation above can be carried out when the system is differentially flat or when the reference signal is generated by an exosystem and an internal model exists. In the latter case we show that asymptotic tracking can be achieved.

Interestingly, our approach may yield a *semiglobal* solution to Problem 1 even when the plant is *not globally flat*. More precisely, we show that a loss of relative degree (or a singularity in the coordinate transformation) yields a restriction on the reference signals to be tracked, but does not necessarily restrict the domain of operation of our controller. On the contrary, a loss of relative degree restricts the domain of operation of input-output linearizing controllers.

Throughout this paper we use $\text{col}(a, b)$ to indicate the vector $[a^\top, b^\top]^\top$. If v is a n -dimensional vector, v_i , $i = 1, \dots, n$, are its components. Given real numbers a, b, c , $\text{diag}[a, b, c]$ denotes the matrix with a, b, c on the diagonal and zeros elsewhere. Given matrices A, B, C , we denote by $\text{block-diag}[A, B, C]$ the matrix formed by placing A, B, C on the diagonal and zeros elsewhere. Given a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and a smooth vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we denote by $L_f g = \frac{\partial g}{\partial x} f(x)$ the Lie derivative of g along f . If $g = g(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$, then we denote $L_f g = \frac{\partial g(x, y)}{\partial x} f(x)$.

II. ASSUMPTIONS

In this section we state the assumptions we need throughout the paper. The assumptions are grouped into three categories associated with three different aspects of our control topology.

A. Stable Inverse Estimation

Assumption A1 (Stable Inverse): Given $r(t)$, there exists $x_0^r \in \mathbb{R}^n$ and a sufficiently smooth and bounded function $u^r(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ such that, when $u(t) = u^r(t)$, the integral curve of (1) leaving from x_0^r , $x^r(t)$, is bounded, defined for all $t \geq 0$, and such that $r(t) \equiv h(x^r(t))$. In other words, for all $t \geq 0$,

$$\begin{aligned}\dot{x}^r &= f(x^r(t), u^r(t)) \\ r(t) &= h(x^r(t)) \\ x^r(0) &= x_0^r.\end{aligned}\quad (3)$$

The following is the most restrictive assumption in this paper.

Assumption A2 (Compensator): One can find a compensator with input v

$$\begin{aligned}\dot{\zeta} &= a(\zeta, x, v) \\ u &= b(\zeta, x),\end{aligned}\quad (4)$$

where $\zeta \in \mathbb{R}^{q_1}$, $v \in \mathbb{R}^m$, and a, b are smooth, with the following properties.

- (i) There exist $\zeta_0^r \in \mathbb{R}^{q_1}$ and $v^r(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ such that, when $v(t) = v^r(t)$ and $x(t) = x^r(t)$, the integral curve of (4) leaving from ζ_0^r , $\zeta^r(t)$, is bounded, defined for all $t \geq 0$, and such that $u^r(t) \equiv b(\zeta^r(t), x^r(t))$.
- (ii) There exists a set of indices $\{k_1, \dots, k_m\}$, with $\sum k_i = n + q_1$, such that, for $i = 1, \dots, m$, the time derivatives $y_i^{(j)}$, $j = 0, \dots, k_i - 1$, calculated along the vector field of (1), (4), are independent of v . Moreover, the map

$$\begin{aligned}\mathcal{H}_X : \mathcal{X} &\rightarrow \mathcal{H}_X(\mathcal{X}), \quad \mathcal{X} \subset \mathbb{R}^n \times \mathbb{R}^{q_1} \\ (x, \zeta) &\mapsto \text{col}(y_1, \dots, y_1^{(k_1-1)}, \dots, y_m, \dots, y_m^{(k_m-1)}),\end{aligned}$$

has a smooth inverse $\mathcal{H}_X^{-1} : \mathcal{H}_X(\mathcal{X}) \rightarrow \mathcal{X}$.

Letting $X^r = \text{col}(x^r, \zeta^r)$ and

$$Y^r = \text{col}(r_1, \dots, r_1^{(k_1-1)}, \dots, r_m, \dots, r_m^{(k_m-1)}),$$

part (ii) in A2 implies that, if $Y^r \in \mathcal{H}_X(\mathcal{X})$, $X^r = \mathcal{H}_X^{-1}(Y^r)$.

The condition $Y^r \in \mathcal{H}_X(\mathcal{X})$ restricts the class of reference signals $r(t)$. The following assumption further requires that $Y^r(t)$ be contained in a convex compact set contained in $\mathcal{H}_X(\mathcal{X})$. This allows one to use the dynamic projection-based observer in [10] to estimate the stable inverse of the plant.

Assumption A3 (Reference Trajectory): The reference trajectory $r(t)$ is such that, for all $t \geq 0$,

$$Y^r(t) \in \mathcal{C}_1 \subset \mathcal{H}_X(\mathcal{X}),$$

for some convex compact set \mathcal{C}_1 whose boundary $\partial\mathcal{C}_1$ is an $n+q_1-1$ dimensional C^1 submanifold, i.e., $\partial\mathcal{C}_1 = \{Y^r \in \mathbb{R}^{n+q_1} : g_1(Y^r) = 0\}$, where $g_1 : \mathbb{R}^{n+q_1} \rightarrow \mathbb{R}$ is a C^1 function for which 0 is a regular value, i.e., $\forall Y^r \in \partial\mathcal{C}_1$, $\partial g_1 / \partial Y^r \neq 0$.

We remark that A3 can be relaxed by requiring that $Y^r(t) \in \mathcal{C}_1$ for all $t \geq T$, for some positive real T , without affecting the results of this paper.

B. Nonlinear Stabilization

Consider the change of coordinates $\tilde{x} = x - x^r$, rewrite (1) in new coordinates as

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, u), \quad (5)$$

and notice that asymptotic stability of the origin of (5) implies asymptotic tracking for (1).

Assumption A4 (Stabilizer): There exist a smooth function $\bar{u}(x, x^r, u^r)$, a C^1 function $V'(\tilde{x})$, $V' : D' \rightarrow \mathbb{R}^+$, and a real number $c' \geq 1$ such that $\bar{u}(x^r, x^r, u^r) = u^r$, $\{\tilde{x} \in \mathbb{R}^n : V'(\tilde{x}) \leq c'\}$ is a compact subset of D' , and the time derivative of V' along the trajectories of

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, \bar{u}(x, x^r, u^r))$$

satisfies

$$\dot{V}' \leq -\Phi'(\tilde{x}),$$

where $\Phi'(\tilde{x})$ is continuous on D' and positive definite on the set $\{\tilde{x} \in \mathbb{R}^n : V'(\tilde{x}) \leq c'\}$.

This assumption, derived from Assumption ULP in [11], implies that the smooth feedback $\bar{u}(x, x^r, u^r)$ uniformly asymptotically stabilizes the origin of (5) and any integral curve leaving from the set $\{\tilde{x} \in \mathbb{R}^n : V'(\tilde{x}) \leq c'\}$ approaches the origin. A4 can be relaxed by allowing V' to depend on time provided it satisfies suitable bounding properties. Also, the results in Section III-B remain unchanged if the origin $\tilde{x} = 0$ is assumed to be practically stable by requiring Φ' to be positive definite over the set $\{\tilde{x} : \underline{c}' \leq V'(\tilde{x}) \leq c'\}$, for some $\underline{c}' \in (0, c')$. Next, in preparation for the application of the separation principle in [10], following the idea of Tornambé in [12] we augment (1) with m chains of integrators² - one chain for every input channel u_i - of order n_1, \dots, n_m , respectively (the indices n_i are defined in A6),

$$\begin{aligned} \dot{\xi} &= A_c \xi + B_c w, \quad \xi \in \mathbb{R}^{q_2}, q_2 = \sum_i n_i, w \in \mathbb{R}^m \\ u &= C_c \xi \end{aligned} \quad (6)$$

where the triple (A_c, B_c, C_c) is in Brunovsky normal form. Next, using the stabilizer \bar{u} in A4, we seek to design a stabilizer \bar{w} for the augmented system (5), (6). To this end, we need the following.

²One may replace the chain of integrators (6) by any *stable* linear system with vector relative degree $\{n_1, \dots, n_m\}$. With appropriate notational changes, the results of this paper still hold.

Assumption A5 (Dynamic Extension): For the system with output $\alpha \in \mathbb{R}^m$,

$$\begin{aligned} \dot{x} &= f(x, C_c \xi) & \dot{x}^r &= f(x^r, b(\zeta^r, x^r)) \\ \dot{\xi} &= A_c \xi + B_c w & \dot{\zeta}^r &= a(\zeta^r, x^r, v^r) \\ \alpha(x, x^r, \zeta^r) &= \bar{u}(x, x^r, b(\zeta^r, x^r)) \end{aligned} \quad (7)$$

the time derivatives $\dot{\alpha}_1, \dots, \alpha_1^{(n_1)}, \dots, \dot{\alpha}_m, \dots, \alpha_m^{(n_m)}$ calculated along the vector field of (7), do not depend on w and v^r .

Recalling that $X^r = \text{col}(x^r, \zeta^r)$, we rewrite the (x^r, ζ^r) dynamics in (7) as

$$\dot{X}^r = F(X^r, v^r), \quad r = H(X^r),$$

with obvious definition of F and H . Since $\bar{u}(x^r, x^r, u^r) = u^r$, A5 implies that the time derivatives of u^r and

$$\xi^r \triangleq \text{col}(u_1^r, \dots, (u_1^r)^{(n_1-1)}, \dots, u_m^r, \dots, (u_m^r)^{(n_m-1)}),$$

calculated along the vector field F (i.e., for $i = 1, \dots, m$, $(u_i^r)^{(j)} = L_F^j c_i$, $j = 0, \dots, n_i - 1$), are independent of v^r . The following lemma shows that, under assumptions A4 and A5, there exists a stabilizer for the augmented system (5), (6).

Lemma 1 *Assume that A4 and A5 hold. Then there exist a smooth function $\bar{w}(x, \xi, X^r)$, a C^1 function $V(\tilde{x}, \tilde{\xi}) : D \rightarrow \mathbb{R}^+$, with $\tilde{\xi} = \xi - \xi^r$, and a real number $c^* \geq 1$ such that $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q_2} : V(\tilde{x}, \tilde{\xi}) \leq c^*\}$ is a compact subset of D , and the time derivative of V along the trajectories of*

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, u^r + C_c \tilde{\xi}) \\ \dot{\tilde{\xi}} &= A_c \tilde{\xi} + B_c \bar{w}(x, \xi, X^r) - \text{col}((u_1^r)^{(n_1)}, \dots, (u_m^r)^{(n_m)}) \end{aligned}$$

satisfies $\dot{V} \leq -\Phi(\tilde{x}, \tilde{\xi})$, where $\Phi(\tilde{x}, \tilde{\xi})$ is continuous on D and positive definite on the set $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q_2} : V(\tilde{x}, \tilde{\xi}) \leq c^\}$. Moreover, if c' in A4 can be chosen arbitrarily large and V' is radially unbounded, then c^* and V have the same properties.*

Proof: Omitted. ■

C. State Estimation

The next few definitions and assumptions are taken from [10]. Consider system (1) and, given a set of indices $\{l_1, \dots, l_m\}$, with $\sum l_i = n$, let $y_x := \text{col}(y_1, \dots, y_1^{(l_1-1)}, \dots, y_m, \dots, y_m^{(l_m-1)})$ (all derivatives are calculated along f) and define

$$\mathcal{H}_x : (x, u_1, \dots, u_1^{(n_1-1)}, \dots, u_m, \dots, u_m^{(n_m-1)}) \mapsto y_x$$

where the indices n_j , $j = 1, \dots, m$ indicate the number of time derivatives of u_j that end up appearing in \mathcal{H}_x (when \mathcal{H}_x does not depend on u_i , we set $n_i = 0$). By using the dynamic extension (6) we have $y_x = \mathcal{H}_x(x, \xi)$. For any positive real number $c \leq c^*$, let

$$\Omega_c = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^{q_2} : V(\tilde{x}, \tilde{\xi}) \leq c\}.$$

Note that the properties of V in Lemma 1, the boundedness of $(x^r(t), u^r(t))$, and the smoothness of $u^r(t)$ imply that Ω_c is a bounded set.

Assumption A6 (Observability): System (1) is observable over an open set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ containing the origin, i.e., there exists a set of indices $\{l_1, \dots, l_m\}$ such that the mapping $\mathcal{F} : \mathcal{O} \rightarrow \mathcal{Y}$ (where $\mathcal{Y} = \mathcal{F}(\mathcal{O})$) defined by

$$(x, \xi) \mapsto Y := \begin{bmatrix} y_x \\ \xi \end{bmatrix} = \begin{bmatrix} \mathcal{H}_x(x, \xi) \\ \xi \end{bmatrix} \quad (8)$$

has a smooth inverse $\mathcal{F}^{-1} : \mathcal{Y} \rightarrow \mathcal{O}$,

$$\mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}(y_x, \xi) = \begin{bmatrix} \mathcal{H}_x^{-1}(y_x, \xi) \\ \xi \end{bmatrix}. \quad (9)$$

Assumption A7 (Topology of \mathcal{O}): There exists a positive scalar $\bar{c} \leq c^*$ and a set \mathcal{C}_2 such that

$$\mathcal{F}(\Omega_{\bar{c}}) \subset \mathcal{C}_2 \subset \mathcal{Y} (= \mathcal{F}(\mathcal{O})), \quad (10)$$

where \mathcal{C}_2 has the following properties

- (i) The boundary of \mathcal{C}_2 , $\partial\mathcal{C}_2$, is an $n-1$ dimensional, C^1 submanifold of \mathbb{R}^n , i.e., there exists a C^1 function $g_2 : \mathcal{C}_2 \rightarrow \mathbb{R}$ such that $\partial\mathcal{C}_2 = \{Y \in \mathcal{C}_2 : g_2(Y) = 0\}$, and $(\partial g_2 / \partial Y)^\top \neq 0$ on $\partial\mathcal{C}_2$.
- (ii) $\mathcal{C}_2^\xi = \{y_x \in \mathbb{R}^n : (y_x, \bar{\xi}) \in \mathcal{C}_2\}$ is convex for all $\bar{\xi} \in \mathbb{R}^{n_u}$.
- (iii) 0 is a regular value of $g_2(\cdot, \bar{\xi})$ for each fixed $\bar{\xi} \in \mathbb{R}^{n_u}$, i.e., for all $y_x \in \mathcal{C}_2^\xi$, $\frac{\partial g_2}{\partial y_x}(y_x, \bar{\xi}) \neq 0$.
- (iv) $\bigcup_{\bar{\xi} \in \mathbb{R}^{n_u}} \mathcal{C}_2^\xi$ is compact.

See [10] for a detailed explanation of A6 and A7.

Condition (10) yields the following implications

$$V(\tilde{x}, \tilde{\xi}) = 0 \Leftrightarrow (\tilde{x}, \tilde{\xi}) = (0, 0) \Rightarrow (x^r, \xi^r) \in \Omega_{\bar{c}} \stackrel{\Delta}{=} (x^r, \xi^r) \in \mathcal{C}_2.$$

III. PROPOSED SOLUTION

We now provide a solution to Problem 1 using the separation principle presented in [10]. In Section III-A we present the control topology. In Section III-B we show that systems which are differentially flat (dynamic feedback linearizable) automatically satisfy A1, A2 and thus naturally lend themselves to the estimation of the stable inverse of the system. Finally, we focus our attention to the case when the reference signal $r(t)$ is generated by an exosystem. We show that if an internal model exists, it can be used as the compensator in A2.

A. Control Topology

Consider the dynamic output feedback controller

$$\begin{aligned} \dot{\xi} &= A_c \xi + B_c \bar{w}(\hat{x}, \xi, \hat{X}^r) \\ u &= C_c \xi, \end{aligned} \quad (11)$$

where $\hat{X}^r = \text{col}(\hat{x}^r, \hat{\xi}^r)$ and \hat{x} are the states of two estimators

$$\dot{\hat{X}}^r = \begin{cases} \left[\frac{\partial \mathcal{H}_X}{\partial \hat{X}^r} \right]^{-1} \left\{ L_{F_1} \mathcal{H}_X - \frac{\Gamma^1 N_{Y^r}(\hat{Y}^r) L_{G_1} g_1}{N_{Y^r}(\hat{Y}^r)^\top \Gamma^1 N_{Y^r}(\hat{Y}^r)} \right\} \\ \quad \text{if } L_{G_1} g_1 \geq 0 \text{ and } \hat{Y}^r \in \partial\mathcal{C}_1 \\ F_1(\hat{X}^r, r) \\ \triangleq F(\hat{X}^r, 0) + \left[\frac{\partial \mathcal{H}_X}{\partial \hat{X}^r} \right]^{-1} (\mathcal{E}^1)^{-1} L^1 (r - H(\hat{X}^r)) \\ \text{otherwise} \end{cases} \quad (12)$$

$$\dot{\hat{x}} = \begin{cases} \left[\frac{\partial \mathcal{H}_x}{\partial \hat{x}} \right]^{-1} \left\{ L_{F_2} \mathcal{H}_x - \frac{\Gamma^2 N_{y_x}(\hat{Y}) L_{G_2} g_2}{N_{y_x}(\hat{Y})^\top \Gamma^2 N_{y_x}(\hat{Y})} \right\} \\ \quad \text{if } L_{G_2} g_2 \geq 0 \text{ and } \hat{Y} \in \partial\mathcal{C}_2 \\ F_2(\hat{x}, \xi, y) \\ \triangleq f(\hat{x}, C_c \xi) + \left[\frac{\partial \mathcal{H}_x}{\partial \hat{x}} \right]^{-1} (\mathcal{E}^2)^{-1} L^2 (y - h(\hat{x})) \\ \text{otherwise} \end{cases} \quad (13)$$

and the various parameters are defined in the next table (where $i = 1, 2$). The estimators (12) and (13) incorporate high-gain parameters ρ_1, ρ_2 to guarantee convergence and a dynamic projection to avoid peaking and confine \hat{X}^r and (\hat{x}, ξ) to within the observable regions \mathcal{X}

$\hat{Y}^r = \mathcal{H}_X(\hat{X}^r), \hat{Y} = \mathcal{F}(\hat{x}, \xi)$	$G_1 = L_{F_1} \mathcal{H}_X,$ $G_2 = \text{col}(L_{F_2} \mathcal{H}_x, \xi)$
$N_{Y^r}(\hat{Y}^r) = \left(\frac{\partial g_1}{\partial \hat{Y}^r} \right)^\top$	$\mathcal{E}^i = \text{block-diag}[\mathcal{E}_1^i, \dots, \mathcal{E}_m^i]$
$N_{y_x}(\hat{Y}) = \left(\frac{\partial g_2}{\partial \hat{y}_x} \right)^\top$	$\mathcal{E}_j^1 = \text{diag}[\rho_1, \dots, \rho_1^{k_j}]$ $\mathcal{E}_j^2 = \text{diag}[\rho_2, \dots, \rho_2^{l_j}]$
$L^i = \text{block-diag}[L_1^i, \dots, L_m^i]$	$\Gamma^i = (S^i \bar{\mathcal{E}}^i)^{-1} (S^i \bar{\mathcal{E}}^i)^{-\top}$
L_j^1 Hurwitz ($k_j \times 1$)	
L_j^2 Hurwitz ($l_j \times 1$)	$S^i = (P^i)^{1/2}$ P^i satisfies:
$\bar{\mathcal{E}}^i = \text{block-diag}[\bar{\mathcal{E}}_1^i, \dots, \bar{\mathcal{E}}_m^i]$	
$\bar{\mathcal{E}}_j^1 = 1/\rho_1^{k_j} \mathcal{E}_j^1, \bar{\mathcal{E}}_j^2 = 1/\rho_2^{l_j} \mathcal{E}_j^2$	
$A^i{}^\top P^i + P^i A^i = -I_{(n+q_i) \times (n+q_i)}$	
$A^i = \begin{bmatrix} 0_{(n+q_i-1) \times 1} & I_{(n+q_i-1) \times (n+q_i-1)} \\ 0_{1 \times (n+q_i)} & \end{bmatrix} - L^i [1, 0_{1 \times n+q_i-1}]$	

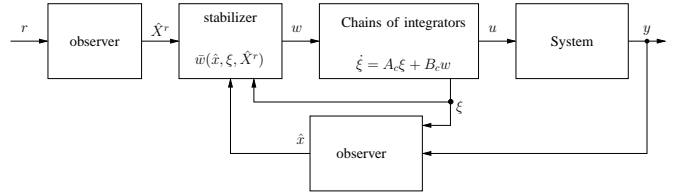


Fig. 1. Block diagram of the controller solving Problem 1.

and \mathcal{O} , respectively (see [10] for more details). Note that the unknown input v^r is replaced by 0 in (12). These estimates are used in \bar{w} (see Lemma 1) to replace X^r and x . The resulting control topology is illustrated in Figure 1. The properties of the two estimators are summarized in the following lemma.

Lemma 2 Consider (12) and (13) and suppose that $\forall t \geq 0$, $(x(t), \xi(t)) \in \Omega_{\bar{c}}$. Assume that A1-A3, and A6-A7 hold. Then, the estimates \hat{X}^r and \hat{x} enjoy the following properties

- (i) The sets $\mathcal{H}_X^{-1}(\mathcal{C}_1)$ and $\mathcal{F}^{-1}(\mathcal{C}_2)$ are positively invariant for (12) and (13), respectively.
- (ii) For all $\delta > 0$, there exist $\bar{\rho}_i$ and $T_i(\rho_i)$, $i = 1, 2$, such that $\|\hat{X}^r(t) - X^r(t)\| \leq \delta$ for all $t \geq T_1(\rho_1)$ and $\|\hat{x}(t) - x(t)\| \leq \delta$ for all $t \geq T_2(\rho_2)$, with $T_i(\rho_i) \rightarrow 0$ as $\rho_i \rightarrow 0$, whenever $\rho_i \in (0, \bar{\rho}_i)$, $i = 1, 2$. Moreover, for sufficiently small ρ_2 , $\|\hat{x}(t) - x(t)\| \rightarrow 0$. If the vector field F does not depend on v^r , i.e., $F(X^r, v^r) = F(X^r)$, then $\|\hat{X}^r(t) - X^r(t)\| \rightarrow 0$.

Proof: The properties of \hat{x} are proven in [10], Theorem 1 and Lemma 1. A variation of the same proofs can be used to prove the properties of \hat{X}^r . ■

The following result is a direct consequence of the separation principle in [10].

Theorem 1 Suppose A1-A7 hold. Then, for any $\underline{c} \in (0, \bar{c})$, there exist positive real numbers ρ_1^*, ρ_2^* such that, for all $\rho_1 \in (0, \rho_1^*)$, $\rho_2 \in (0, \rho_2^*)$, the dynamic output feedback controller

$$\begin{aligned} \dot{\xi} &= A_c \xi + B_c \bar{w}(\hat{x}, \xi, \hat{X}^r) \\ u &= C_c \xi \end{aligned} \quad (14)$$

solves Problem 1 over the set

$$\mathcal{A} = \left\{ (x, \xi, \hat{X}^r, \hat{x}) \in \mathbb{R}^{3n+q_1+q_2} : (x, \xi) \in \Omega_{\underline{c}}, \hat{X}^r \in \mathcal{H}_X^{-1}(\mathcal{C}_1), (\hat{x}, \xi) \in \mathcal{F}^{-1}(\mathcal{C}_2) \right\}$$

Sketch of the proof: Since $r(t) = h(x^r(t))$ and h is continuous, for all $\varepsilon_0 > 0$ there exists $\varepsilon^* > 0$ such that for all $x \in \mathbb{R}^n$ such that $\|x - x^r\| \leq \varepsilon^*$, $\|h(x) - h(x^r)\| \leq \varepsilon_0$. Using an argument similar to that of the proof of Lemma 2 in [10] one finds that for any $\vartheta \in (0, \underline{c})$, there exist positive real numbers ρ_1^*, ρ_2^* such that if $\rho_i \in (0, \rho_i^*)$, $i = 1, 2$, every integral curve $(x(t), \xi(t))$ leaving from $\Omega_{\underline{c}}$ cannot exit the set Ω_{ε} and converges in finite time to the residual set Ω_{ϑ} . Clearly ϑ can be chosen so that

$$(x, \xi) \in \Omega_{\vartheta} \Rightarrow \|x - x^r\| \leq \varepsilon^*,$$

thus proving the practical tracking property. From this discussion and the positive invariance of $\mathcal{H}_X^{-1}(\mathcal{C}_1)$ and $\mathcal{F}^{-1}(\mathcal{C}_2)$ we get that Problem 1 is solved over the set \mathcal{A} . ■

Corollary 1 *Under the assumptions of Theorem 1, if A4 holds for arbitrarily large c' and a radially unbounded V' , and A6 holds globally (i.e., $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^{q_2}$) with $\mathcal{F}(\mathcal{O})$ a convex set, then the solution to Problem 1 is semiglobal.*

Proof: From Lemma 1, if c' can be chosen arbitrarily large and V' is radially unbounded, c^* and V have the same properties. Since $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^{q_2}$ and $\mathcal{F}(\mathcal{O})$ is a convex set, we have that A7 is satisfied for an arbitrarily large \bar{c} and a sufficiently large set \mathcal{C}_2 (see Remark 5 in [10]). Thus, the set $\Omega_{\bar{c}}$ can be made arbitrarily large. ■

Notice that, in order to solve Problem 1 semiglobally, we do not require \mathcal{X} in A2 to be all of \mathbb{R}^{n+q_1} . The advantages of this feature are illustrated in the next section.

Summarizing the results presented in this section, we have found that if there exists a compensator (4) satisfying A2 and if suitable observability/stabilizability properties are satisfied, there exists a dynamic output feedback controller solving Problem 1. *When can the compensator (4) be found?* A partial answer to this question is provided in the next two sections.

B. Differentially Flat Systems

Assume now that (1) is differentially flat (dynamic feedback linearizable) with respect to the flat output y (see [7]), i.e., there exists a regular compensator³ with input w (referred to as the *linearizing compensator*)

$$\dot{\eta} = \varphi(\eta, x, w), \quad u = \gamma(\eta, x, w), \quad \eta \in \mathbb{R}^p, \quad w \in \mathbb{R}^m \quad (15)$$

and a set $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^p$ such that the plant augmented with such compensator yields (up to a feedback transformation) the trivial system in output coordinates, $y_i^{(m_i)} = w_i$, $\sum m_i = n + q$, and the mapping $T(x, \eta) : \mathcal{D} \rightarrow T(\mathcal{D})$ defined as

$$T(x, \eta) = \text{col}(\mathbf{h}_1, \dots, L_{\mathbf{f}}^{m_1-1} \mathbf{h}_1, \dots, \mathbf{h}_m, \dots, L_{\mathbf{f}}^{m_m-1} \mathbf{h}_m),$$

where

$$\mathbf{f}(x, \eta, w) = \text{col}(f(x, \gamma(\eta, x, w)), \varphi(\eta, x, w)), \quad \mathbf{h}(x, \eta) = h(x),$$

are the vector field and output function of the augmented system (1), (15), is a diffeomorphism on \mathcal{D} . Assume for the moment that the output function of (15) is independent of w , i.e., $\gamma(\eta, x, w) = \gamma(\eta, x)$. It is then clear that A1 holds in that letting $Y^r = \text{col}(r_1, \dots, r_1^{(m_1-1)}, \dots, r_m, \dots, r_m^{(m_m-1)})$ we have $(x^r(t), \eta^r(t)) = T^{-1}(Y^r(t))$ and $u^r(t) = \gamma(\eta^r(t), x^r(t))$. It is also clear that (15) satisfies A2. As a matter of fact, part (i) in

³For each fixed x the map $\gamma(\eta, x, w) : (\eta, w) \mapsto u$ is a submersion, in other words (η, w) can be uniquely determined from (x, u) , see [13].

A2 is implied by the regularity of (15), and part (ii) is satisfied with $k_i = m_i$, $\mathcal{H}_X(\cdot, \cdot) = T(\cdot, \cdot)$, and $\mathcal{X} = \mathcal{D}$. In the general case when $\gamma(\cdot, \cdot, \cdot)$ depends on w we just add integrators: $\dot{z}_j = v_j$, $j = 1, \dots, m$, $w = z$. Letting $\zeta = \text{col}(\eta, z)$, $a(\zeta, x, v) = \text{col}(\varphi(\eta, x, z), v)$, $b(\zeta, x) = \gamma(\eta, x, z)$, and $k_i = m_i + 1$, we have that $a(\cdot, \cdot, \cdot)$ and $b(\cdot, \cdot)$ satisfy A2 on $\mathcal{X} = \mathcal{D} \times \mathbb{R}^m$. The previous considerations are summarized in the following.

Fact 1 *A sufficient condition for A2 to hold is that (1) is differentially flat (dynamic feedback linearizable) with respect to y .*

However, differential flatness is not a necessary condition for A2 to hold, as it is shown in the next section.

When (1) is differentially flat, so that in output coordinates the system is in Brunovsky normal form, one can design an input-output linearizing controller which employs the derivatives of the output and the reference signal to yield tracking. Such derivatives can be estimated by means of high-gain observers and thus Problem 1 can be solved, in the spirit of Teel and Praly [14] or Khalil and co-workers [15], [16], by replacing the derivatives by their estimates and saturating the control input. On the other hand, even when (1) is differentially flat, the control topology presented in Section III-A *does not rely on input-output linearization* and the linearizing compensator is used only for the estimation of the stable inverse of the plant. *Wouldn't it be better to use input-output linearization rather than the technique presented in this paper?* We use an example to answer this question.

Example 1 The nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1^2 + u_1, \\ \dot{x}_3 &= x_4 - u_1 - x_1^2, \\ \dot{x}_4 &= -x_3 - x_4 + x_4 u_2 \\ y &= \text{col}(x_1, x_3), \end{aligned} \quad (16)$$

is differentially flat (dynamic feedback linearizable). A linearizing compensator is

$$\dot{\zeta}_1 = \zeta_2 + \zeta_3, \quad \dot{\zeta}_2 = v_1, \quad \dot{\zeta}_3 = v_2, \quad u = \text{col}(\zeta_1, \zeta_2).$$

The decoupling matrix of the augmented system is

$$\begin{bmatrix} 1 & 1 \\ x_4 - 1 & -1 \end{bmatrix},$$

and hence the vector relative degree of the augmented system, $\{4, 3\}$, is well-defined on the set $\{(x, \zeta) : x_4 \neq 0\}$. Given a smooth reference signal $r(t)$ such that $\ddot{r}_1 + \dot{r}_2 > 0$ (so that $x_4^r > 0$), an input-output linearizing controller for the augmented system is given by

$$v = \begin{bmatrix} r_1^{(4)} \\ r_2^{(3)} \end{bmatrix} - \frac{1}{x_4} \begin{bmatrix} -1 & 1 \\ 1 - x_4 & 1 \end{bmatrix} K e,$$

where $e = \text{col}(e_1, \dots, e_1^{(3)}, e_2, \dots, e_2^{(2)})$, with $e = y - r$, and K a suitable 2×7 matrix. This controller solves Problem 1 over a set \mathcal{A} which does not contain any point (x^0, ζ^0) such that $x_4^0 < 0$, and hence does not yield semiglobal output feedback tracking.

On the other hand, we now show that semiglobal output feedback tracking can be achieved using the control topology in Figure 1. It is quite clear that A1 holds for any smooth reference signal $r(t)$ satisfying $\ddot{r}_1 + \dot{r}_2 \neq 0$ and that A2 holds with $k_1 = 4$, $k_2 = 3$, and $\mathcal{X} = \{(x, \zeta) : x_4 \neq 0\}$. Consequently, since the set $\{(x, \zeta) : x_4 > 0\}$ is already convex, the set \mathcal{C}_1 satisfying A3 can be taken to

be any convex inner approximation with smooth boundary. Letting $\tilde{x} = x - x^r$, the stabilizer

$$\bar{u}(x, x^r, u^r) = \begin{bmatrix} u_1^r + (x_1^r)^2 - x_1^2 + M\tilde{x} \\ u_2^r \end{bmatrix}, \quad M = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

satisfies A4 globally (i.e., $c' = \infty$) with

$$V'(\tilde{x}) = \frac{1}{4}\tilde{x}^\top \begin{bmatrix} 29 & 9 & 7 & 7 \\ 9 & 27 & -1 & 2 \\ 7 & -1 & 6 & 2 \\ 7 & 2 & 2 & 4 \end{bmatrix} \tilde{x}.$$

Letting $y_x = \text{col}(y_1, \dot{y}_1, y_2, \dot{y}_2)$ we have $y_x = \mathcal{H}_x(x, u_1) = \text{col}(x_1, x_2, x_3, x_4 - x_1^2 - u_1)$. Since the mapping $\mathcal{F} : (x, u_1) \mapsto (y_x, u_1)$ is a global diffeomorphism, A6 is satisfied with $l_1 = 2, l_2 = 2, n_1 = 1, n_2 = 0$, and $\mathcal{O} = \mathbb{R}^4 \times \mathbb{R}$. It follows that A7 is satisfied by an arbitrarily large c^* and a sufficiently large set \mathcal{C}_2 . Since $n_1 = 1, n_2 = 0$, we need the following dynamic extension

$$u_1 = \xi, \quad \dot{\xi} = w_1, \quad u_2 = w_2.$$

It is easy to see that \dot{u}_1 is independent of w and v^r , and hence A5 holds. From Corollary 1 we conclude that the controller (11), (12), (13) yields semiglobal output feedback tracking.

△

Returning to the question posed earlier, this example shows that semiglobal output tracking may be achieved even when the plant is not globally differentially flat. In other words, in our framework *global differential flatness is not a necessary condition for semiglobal output feedback tracking*. When the system is not globally flat because either the relative degree of the augmented system (1), (15) is not everywhere well-defined or the change of coordinates $T(\cdot, \cdot)$ is not a global diffeomorphism, we restrict the class of reference signals to be tracked (in the example we imposed $\ddot{r}_1 + \dot{r}_2 > 0$). However, since the linearizing compensator is only employed for estimation, one may well find a global (or semiglobal) stabilizer yielding a semiglobal solution to Problem 1. On the other hand, the input-output linearization approach employs the linearizing compensator as a dynamic controller. In this framework the domain of operation of the closed-loop system is unnecessarily restricted and hence Problem 1 cannot be solved semiglobally.

A further advantage of the technique proposed here is that it naturally lends itself to the estimation of input disturbances. This topic has been investigated in [17].

C. Tracking With an Exosystem

Assume that the reference signal is generated by a *neutrally stable* exosystem (see [3]) $\dot{w} = s(w)$, $r = q(w)$, where $w \in \mathbb{R}^r$, and $s(\cdot), q(\cdot)$ are smooth. We now show that if there exists an internal model with suitable observability properties, then A1 and A2 are satisfied and the controller (11), (12), (13) yields asymptotic tracking. The notion of internal model we use in the next assumption is due to Isidori (see Section 8.4 in [3]).

Assumption A8 (Internal Model): (i) There exist mappings $x = \pi(w)$, $u = c(w)$, with $\pi(0) = 0$, $c(0) = 0$, satisfying the *regulator equations*

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w)) \quad 0 = h(\pi(w)) - q(w)$$

and such that the autonomous system $\dot{w} = s(w)$, $u = c(w)$ is immersed into a system (the *internal model*) $\dot{\zeta} = a(\zeta)$, $u = b(\zeta)$, with $\zeta \in \mathbb{R}^{q_1}$, i.e., there exists a smooth mapping $\tau : \mathbb{R}^r \rightarrow \mathbb{R}^{q_1}$ such that

$$\frac{\partial \tau}{\partial w} s(w) = a(\tau(w)), \quad c(w) = b(\tau(w)) \\ c(w_1) \neq c(w_2) \Rightarrow b(\tau(w_1)) \neq b(\tau(w_2))$$

for all $w \in \mathbb{R}^r$.

(ii) There exists a set of indices $\{k_1, \dots, k_m\}$, with $\sum k_i = n + q_1$, such that the map

$$(x, \zeta) \mapsto \text{col}(y_1, \dots, y_1^{(k_1-1)}, \dots, y_m, \dots, y_m^{(k_m-1)})$$

is a diffeomorphism on $\mathcal{X} \subset \mathbb{R}^n \times \mathbb{R}^{q_1}$.

The solvability of the regulator equations in A8 implies the existence of a stable inverse, which is given by $(x^r(t), u^r(t)) = (\pi(w(t)), c(w(t)))$. Moreover, the internal model can be used as the compensator (4) in A2 with $v = 0$. To see why this is true, notice that (i) in A2 is satisfied with $\zeta_0^r = \tau(w(0))$, while (ii) in A2 directly follows from property (ii) in A8 and the fact that the internal model is an autonomous system. The previous considerations are summarized in the following.

Fact 2 *A sufficient condition for A2 to hold is that there exists an internal model satisfying A8.*

As remarked earlier, this fact shows that differential flatness is not a necessary condition for A2 to hold. We now turn our attention to the *asymptotic tracking* problem.

Corollary 2 *Suppose A3-A8 hold. Then, for any $\underline{c} \in (0, \bar{c})$, there exist positive real numbers ρ_1^*, ρ_2^* such that, for all $\rho_1 \in (0, \rho_1^*)$, $\rho_2 \in (0, \rho_2^*)$, the dynamic output feedback controller (11)-(13) solves the output feedback asymptotic tracking problem over the set \mathcal{A} defined in Theorem 1.*

Proof: From A8 and Fact 2 we have that A1, A2 are satisfied and the vector field F does not depend on v^r , i.e., $F(X^r, v^r) = F(x^r)$. Thus, from Lemma 2, $\dot{X}^r(t) - X^r(t) \rightarrow 0$ as $t \rightarrow \infty$, so that, in the proof of Theorem 1, $\vartheta = 0$ and the origin $(\tilde{x}, \tilde{\xi})$ is attractive. Thus, in particular, by the continuity of $h(\cdot)$ we have $h(x(t)) - h(x^r(t)) \rightarrow 0$ as $t \rightarrow \infty$. ■

CONCLUDING REMARKS

We presented an approach, based on a separation principle, to solve the output feedback practical tracking problem for systems which are not affected by uncertainties or disturbances. When the reference signal is generated by an exosystem and an internal model satisfying suitable assumptions exists, this approach yields a solution to the output feedback *asymptotic* tracking problem. Since this approach relies on the on-line estimation of the stable inverse of the plant, it is susceptible to degradation in performance when uncertainties or disturbances are present.

REFERENCES

- [1] R. Marino and P. Tomei, *Nonlinear Control Design: Geometric, Adaptive & Robust*. NJ: Prentice Hall, 1995.
- [2] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design*. NY: John Wiley & Sons, Inc., 1995.
- [3] A. Isidori, *Nonlinear Control Systems*, 3rd ed. London: Springer-Verlag, 1995.
- [4] A. Serrani and A. Isidori, "Global robust output regulation for a class of nonlinear systems," *Systems and Control Letters*, vol. 39, pp. 133–139, 2000.
- [5] A. Isidori, "A remark on the problem of semiglobal nonlinear output regulation," *IEEE Transactions on Automatic Control*, vol. 42, no. 12, pp. 1734–1738, 1997.
- [6] A. Serrani, A. Isidori, and L. Marconi, "Semiglobal robust regulation of minimum-phase nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol. 10, pp. 379–396, 2000.
- [7] M. Fliess, M. Lévin, P. Martin, and P. Rouchon, "Flatness and defect of non-linear systems: Introductory theory and examples," *International Journal of Control*, vol. 61, no. 6, pp. 1327–1361, 1995.

- [8] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Transactions on Automatic Control*, vol. 41, no. 7, pp. 930–942, July 1996.
- [9] C. Qian and W. Lin, "Practical output tracking of nonlinear systems with uncontrollable unstable linearization," *IEEE Transactions on Automatic Control*, vol. 47, no. 1, pp. 21–36, 2002.
- [10] M. Maggiore and K. M. Passino, "A separation principle for a class of non-UCO systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1122–1133, July 2003.
- [11] A. Teel and L. Praly, "Tools for the semiglobal stabilization by partial state and output feedback," *SIAM J. Control Optim.*, vol. 33, no. 5, pp. 1443–1488, 1995.
- [12] A. Tornambè, "Output feedback stabilization of a class of non-minimum phase nonlinear systems," *Systems & Control Letters*, vol. 19, pp. 193–204, 1992.
- [13] M. Van Nieuwstadt, M. Rathinam, and R. M. Murray, "Differential flatness and absolute equivalence of nonlinear control systems," *SIAM Journal on Control and Optimization*, vol. 36, no. 4, pp. 1225–1239, 1998.
- [14] A. Teel and L. Praly, "Global stabilizability and observability imply semi-global stabilizability by output feedback," *Systems & Control Letters*, vol. 22, pp. 313–325, 1994.
- [15] F. Esfandiari and H. Khalil, "Output feedback stabilization of fully linearizable systems," *International Journal of Control*, vol. 56, no. 5, pp. 1007–1037, 1992.
- [16] A. Atassi and H. Khalil, "A separation principle for the stabilization of a class of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 9, pp. 1672–1687, September 1999.
- [17] L. Consolini and M. Maggiore, "Robust output feedback tracking with a matching condition," in *Proc. of 42nd Conf. Decision Contr.*, Maui, Hawaii, December 2003.
- [18] M. Vidyasagar, "On the stabilization of nonlinear systems using state detection," *IEEE Transactions on Automatic Control*, vol. 25, no. 3, pp. 504–509, 1980.