Reduction Theorems For Stability of Compact Sets in Time-Varying Systems

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Abstract

Reduction theorems provide a framework for stability analysis that consists in breaking down a complex problem into a hierarchical list of subproblems that are simpler to address. This paper investigates the following reduction problem for time-varying ordinary differential equations on $\mathbb{R}^n$. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Suppose that Γ_1 is uniformly asymptotically stable relative to Γ_2. Find conditions under which Γ_1 is uniformly asymptotically stable. We present a reduction theorem for uniform asymptotic stability that completely addresses the local and global version of this problem, as well as two reduction theorems for uniform stability and either local or global uniform attractivity. These theorems generalize well-known equilibrium stability results for cascade-connected systems as well as previous reduction theorems for time-invariant systems. We also present Lyapunov characterizations of the stability properties required in the reduction theorems that to date have not been investigated in the stability theory literature.

Key words: Time-varying systems, Cascades, Stability of sets

1 Introduction

The *reduction problem* was originally posed by P. Seibert in 1969 in the context of semidynamical systems [29,30]. In its most elementary formulation it concerns a differential equation with locally-Lipschitz right-hand side,

$$\dot{x} = f(x), \ x \in \mathbb{R}^n, \tag{1}$$

with no particular structure, and two nested subsets of the state space, $\Gamma_1 \subset \Gamma_2$, that are both positively invariant and have the property that Γ_1 is asymptotically stable *relative to* Γ_2. Loosely speaking, this means that solutions generated by (1) starting from initial states that are restricted to lie in Γ_2 converge, and remain close, to the set Γ_1. Then, the problem consists in finding conditions under which Γ_1 is asymptotically stable, so in particular attractive for solutions starting away from the set Γ_2. In addition, several refinements may be of interest; for instance, to admit arbitrarily large initial conditions, as well as versions addressing the properties of stability and attractivity, in place of asymptotic stability.

Such problems are far from being of pure academic interest. Their solution leads to the *reduction theorems* on stability, which are technical statements that form a framework of analysis and design of dynamical systems, based on breaking down a complex problem into a prioritized sequence of simpler sub-problems —one step at a time. Instances of following such a natural methodology in popular control methods such as *backstepping* [15] and sliding-modes [37], as well as in stability theory for cascaded systems,

$$\dot{x}_1 = f_1(x_1, x_2) \tag{2a}$$
$$\dot{x}_2 = f_2(x_2). \tag{2b}$$

This class of systems well illustrates the essence of the reduction problem. The basic (stability analysis) problem for the cascade (2) is to find conditions under which asymptotic stability of $\{x_1 = 0\}$ for $f_1(x_1, 0)$ and of $\{x_2 = 0\}$ for $f_2(x_2)$ leads to conclude that $\{x = 0\}$ is asymptotically stable for (1) with $f := [f_1^\top \ f_2^\top]^\top$. The extensive literature on cascaded systems originates, for time-invariant systems, with work by Vidyasagar in [38] focusing on local asymptotic stability of the zero equilibrium, followed by research aimed at establishing global results, e.g., [32,28,23,18,3,11,21,35]. Now, the stability questions investigated in the literature on time-invariant cascaded systems are, as a matter of fact, reduction problems such as asking under which conditions

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$\Gamma_1 = \{(x_1, x_2) = (0,0)\}$ is asymptotically stable provided that so is $\Gamma_2 = \{(x_1, x_2) : x_2 = 0\}$.

Stability analysis of cascaded systems is also important for control design; for instance, when one considers not only the control of a plant itself, but also of the actuators [26]. The central idea consists in constructing a controller that ensures that the systems trajectories converge *asymptotically* to an invariant manifold having the property that trajectories contained in it converge to the origin (or a set for that matter).

This rationale however, is not bound to cascaded systems. For instance, it is also reminiscent of the well-known result in [2] that a passive system is stabilizable via static output feedback if it is zero-state detectable (namely, if the state trajectories converge to the origin provided that so does the output). This connection was explored in [6]. Another clear example where the same rationale holds is the Slotine & Li Controller [33], one of the first tracking controllers for robot manipulators ensuring global asymptotic stability. The operation and stability properties of this controller can be naturally understood using the reduction viewpoint, and this is illustrated in Section 5.1.

The motivations to study the reduction problem go well beyond reinterpreting otherwise well established arguments for cascaded or passive systems, where only two sets, $\Gamma_1 \subset \Gamma_2$, are involved. Indeed, some control problems may be conveniently broken down into a prioritized sequence of more than two elementary sub-problems, which are then solved *separately*. That is, in general, the control specification of asymptotically stabilize a subset Γ of the state space may be solved by breaking it down in sub-tasks and defining a suitable collection of nested subsets $\Gamma_1 \subset \cdots \subset \Gamma_l \subset \Gamma_{l+1} := \mathbb{R}^n$ (a hierarchy of control specifications), with $\Gamma_1 = \Gamma$. Then, by asymptotically stabilizing Γ_i relative to Γ_{i+1}, for $i = 1, \ldots, l$. The reduction theorems allow to recursively deduce the asymptotic stability of Γ. Following such premise, in [7] was introduced the hierarchical control framework, which has direct implications on backstepping control.

Literature on the reduction problem. In [31], Seibert and Florio proved reduction theorems for stability and asymptotic stability of compact sets for time-invariant semidynamical systems. See also work by B.S. Kalitin and co-workers [10,12]. In [7], the results of [31] were generalized to closed, non-compact sets under a uniform boundedness assumption on solutions. That paper also presents a new reduction theorem for attractivity. The recent work [20] presents reduction theorems for stability, attractivity, and asymptotic stability of compact sets in hybrid dynamical systems. All the results just mentioned concern time-invariant systems. For time-varying systems, in [13] Kalitin and Chabour proved some reduction theorems for uniform stability and uniform asymptotic stability *of the origin*, and used them to establish such properties using positive-semidefinite Lyapunov functions. Yet, aside from [13], which is limited to stability of the origin, the general reduction problem on stability of closed sets for time-varying systems remains substantially open. This is what we address here.

Contributions of this paper. This paper presents a complete solution to the reduction problem for the time-varying differential equation

$$\dot{x} = f(t, x), \tag{3}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies a uniform Lipschitz continuity assumption (the Basic Assumption presented in Section 2). The set $\Gamma_1 \subset \mathbb{R}^n$ in the reduction is assumed to be compact, while $\Gamma_2 \subset \mathbb{R}^n$ is assumed be closed. We present three reduction theorems for uniform stability (Theorem 14), uniform attractivity (Theorem 16), and uniform asymptotic stability (Theorem 18). Both local and global versions of these properties are characterized.

Crucial for the development of this paper is the elucidation of the relationship between different notions of uniform stability and attractivity of compact sets for time-varying systems under the assumption of uniform Lipschitz continuity. In particular, we establish that uniform attractivity is equivalent to uniform asymptotic stability, see Proposition 9.

All the reduction theorems in the literature reviewed above rely on notions of stability and attractivity near a set that are absent in the literature on Lyapunov stability and, with the exception of Theorem 1 in [13], have not been given Lyapunov characterizations. This paper presents three Lyapunov characterizations of these properties. The first is a generalization of Theorem 1 in [13] (see Proposition 25), the second gives a Lyapunov characterization of a notion of uniform attractivity (see Proposition 27), and the third gives a characterization of a notion of uniform attractivity near a set (see Corollary 29).

Thus, together, the reduction theorems and the Lyapunov characterizations presented in this paper constitute a set of tools allowing one to assess the uniform asymptotic stability of compact sets using a modular approach that simplifies the analysis. This fact is illustrated in Section 5 through a number of examples. In particular, we provide a formal stability analysis of the Slotine & Li controller for fully-actuated robots [33] that follows faithfully the original intuitive arguments behind the controller design.

Comparison with existing literature. The papers most relevant to our work are [31,7,13]. The reduction theorems for uniform stability and uniform (global) asymptotic stability presented in this paper recover the results of [31], which are restricted to time-invariant systems. The reduction theorem for attractivity, on the other hand, has no counterpart in [31].

The reduction theorems in [7] involve closed and non-compact sets and rely on an hypothesis that solutions enjoy a uniform boundedness property. Nonetheless, extending the applicability of these theorems to time-varying systems is by no means straightforward. For instance, one might consider augmenting the state

x with t and correspondingly, system (3) with the equation $\dot{t} = 1$. Then, a reduction problem for the non-compact sets $\{(x,t) : x \in \Gamma_i\} \subset \mathbb{R}^n \times \mathbb{R}$, $i = 1,2$, in the augmented state space would follow naturally. This approach, however, fails because the solutions of the system with state (x,t) are unbounded (because t is) and this violates the hypotheses in [7]. Furthermore, even if the results of [7] were applicable to the system with extended state, they would not guarantee uniformity of various stability properties with respect to the initial time, and an unnecessarily conservative Lipschitz assumption with respect to t would be automatically imposed.

Finally, the paper [13] investigates uniform stability and uniform attractivity of equilibria, rather than compact sets as we do in this paper, and does not present reduction theorems for attractivity. The proofs of three results in this paper (Theorem 14, Proposition 25, and Lemma 31) follow the lines of analogous proofs found in [13] and are therefore omitted. The interested reader may find these proofs in the extended version of this paper [19]. Detailed comparisons with the work in [13] are found in Remarks 15, 19, and 26.

Organization. In Section 2 we present definitions of relative stability properties and other stability notions. Section 3 provides a precise formulation of the reduction problem. In Section 4 we present our reduction theorems for uniform stability, uniform attractivity, and uniform asymptotic stability; then, some useful implications of these theorems; and finally, Lyapunov characterizations of the key stability properties used in the reduction theorems. In Section 5 we provide examples illustrating the use and rationale of reduction theory, and in Section 6 we prove the three reduction theorems. The paper is wrapped up with concluding remarks in Section 7, and completed with a technical appendix containing the proof of Proposition 9.

Notation. We denote by 0_k, $k \in \mathbb{N}$, the vector of zeros in $\mathbb{R}^k$, and for $x \in \mathbb{R}^k$, we denote by $\|x\|:=(x^\top x)^{1/2}$, its Euclidean norm. We denote by $\mathbb{S}^1$ the set of real numbers modulo 2π. If $\Gamma \subset \mathbb{R}^n$ is a closed set, we denote by $\|x\|_\Gamma:=\inf_{y\in\Gamma}\|x-y\|$ the point-to-set distance of $x \in \mathbb{R}^n$ to Γ. If $A, B \subset \mathbb{R}^n$, we define $d(A,B):=\sup_{x\in A}\{\|x\|_B\}$. If $\delta > 0$, we let $B_\delta(\Gamma):=\{x \in \mathbb{R}^n : \|x\|_\Gamma < \delta\}$. For a set K, ∂K denotes the boundary of K, $\mathrm{int}(K)$ its interior, and $\overline{K}$ its closure. For $t_0 \in \mathbb{R}$, we denote $\mathbb{R}_{\geq t_0}:=\{t \in \mathbb{R} : t \geq t_0\}$. A function $\alpha : [0,r) \to \mathbb{R}$, with $r > 0$, belongs to class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$.

2 Preliminaries

We investigate the time-varying differential equation with state space[3] $\mathbb{R}^n$. With an abuse of notation, we denote by $x(t,t_0,x_0)$ the solution of (3) satisfying $x(t_0,t_0,x_0) = x_0$, where t_0 is the *initial time* and x_0 is the *initial state*. The pair (t_0,x_0) is called the *initial data* of the solution. We denote by $T^+_{t_0,x_0}$ the *right maximal interval of existence* of the solution with initial data (t_0,x_0), i.e., the maximal interval contained in $\mathbb{R}_{\geq t_0}$ on which the solution $x(t,t_0,x_0)$ is defined. If $I \subset \mathbb{R}$ and $U \subset \mathbb{R}^n$, we define $x(I,t_0,U):=\{x(t,t_0,x_0) \in \mathbb{R}^n : t \in I, x_0 \in U\}$. This set is well-defined as long as $I \subset T^+_{t_0,x_0}$ for all $(t_0,x_0) \in \mathbb{R} \times U$.

[3] The main results of this paper continue to hold if the state space is a smooth complete Riemannian manifold, see Remark 22.

We require the time-varying vector field f in (3) to possess a basic continuity property, stated in the next assumption.

Basic Assumption. The function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous with respect to its first argument and satisfies the following Lipschitz continuity property with respect to its second argument. For any compact set $K \subset \mathbb{R}^n$, there exists a constant $L > 0$ such that for each $x_1, x_2 \in K$ and for each $t \in \mathbb{R}$, $\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|$. $\triangle$

Remark 1. The Lipschitz continuity requirement in the Basic Assumption cannot be relaxed as it is a fundamental ingredient in the proofs of Proposition 9 and the main results in Theorems 14, 16, and 18. As a matter of fact, Lipschitz continuity is often imposed in the literature on stability of nonlinear time-varying systems. To illustrate, the Basic Assumption appears in the paper [13] reviewed in the introduction. Further, the paper [17] presents a converse Lyapunov theorem for uniform global asymptotic stability of compact sets relying on the assumption that the function $f(t,x)$ has the form $f(t,x) = f_1(x, f_2(t))$, where $f_1 : \mathbb{R}^n \times D \to \mathbb{R}^n$ is C^1, $D \subset \mathbb{R}^k$ is a bounded open set, and $f_2 : \mathbb{R} \to D$ is a piecewise continuous function whose image is contained in a compact subset of D. This is a special case of the Basic Assumption. $\triangle$

Definition 2 (positive invariance)**.** A set $\Gamma \subset \mathbb{R}^n$ is *positively invariant* for (3) if $x(T^+_{t_0,x_0}, t_0, x_0) \subset \Gamma$ for all $t_0 \in \mathbb{R}$ and all $x_0 \in \Gamma$. In other words, for any initial data $(t_0,x_0) \in \mathbb{R} \times \Gamma$, the solution remains in Γ for all $t \geq t_0$ for which the solution is defined. $\triangle$

Next, we present some notions of uniform stability and uniform attractivity of compact sets. Table 1 summarizes all stability-related acronyms used in this paper.

Definition 3 (uniform stability and attractivity of compact sets)**.** Consider system (3) and let $\Gamma \subset \mathbb{R}^n$ be a compact set.

- Γ is *uniformly stable* (US) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is *uniformly globally stable* (UGS) if Γ is US and for each $\delta > 0$ there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is *uniformly attractive* (UA) if there exists $r > 0$

Table 1
List of stability-related acronyms used in the paper.

Acronym	**Meaning**	**Where**
US	uniformly stable	Defn. 3
UGS	uniformly globally stable	Defn. 3
UA	uniformly attractive	Defn. 3
UGA	uniformly globally attractive	Defn. 3
UAS	uniformly asymptotically stable	Defn. 3
UGAS	uniformly globally asymptotically stable	Defn. 3
t_0-US	t_0-uniformly stable	Defn. 5
t_0-UA	t_0-uniformly attractive	Defn. 5
t_0-UGA	t_0-uniformly globally attractive	Defn. 5
t_0-UAS	t_0-uniformly asymptotically stable	Defn. 5
t_0-UGAS	t_0-uniformly globally asymptotically stable	Defn. 5
LUS-Γ	locally uniformly stable near Γ	Defn. 10

such that for each $\varepsilon > 0$ there exists $T > 0$ such that $x(\mathbb{R}_{\geq t_0+T}, t_0, B_r(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$.
- Γ is *uniformly globally attractive* (UGA) if the UA property holds for all $r > 0$.
- Γ is *uniformly asymptotically stable* (UAS) if it is US and UA.
- Γ is *uniformly globally asymptotically stable* (UGAS) if it is UGS and UGA.

$\triangle$

Remark 4. All properties in Definition 3 are analogous to familiar definitions concerning equilibria found, e.g., in [14, Section 4.5], and the definition that a compact set Γ is UGAS is equivalent to the one found, e.g., in [17]. We remark that the definition implies that $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$ for each $x_0 \in B_\delta(\Gamma)$ and each $t_0 \in \mathbb{R}$. This is justified because if a solution remains in the *bounded* set $B_\varepsilon(\Gamma)$, then its right-maximal interval of existence is $\mathbb{R}_{\geq t_0}$. $\triangle$

Next, we present some notions of stability and attractivity of closed, but not necessarily compact sets. The notion of t_0-UA used in this paper is taken from [27].

Definition 5 (t_0-uniform stability and t_0-uniform attractivity of closed sets)**.** Consider system (3), and let $\Gamma \subset \mathbb{R}^n$ be a closed set.
- Γ is *t_0-uniformly stable* (t_0-US) if for each $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $\Gamma \subset U$, and for each $x_0 \in U$, for each $t_0 \in \mathbb{R}$, and each $t \in T^+_{t_0,x_0}$, it holds that $x(t, t_0, x_0) \in B_\varepsilon(\Gamma)$.
- The *basin of t_0-uniform attraction of* Γ is the set $\mathbb{B}(\Gamma)$ of initial states for which solutions converge to Γ uniformly with respect to t_0:
$$\begin{aligned}\mathbb{B}(\Gamma) := \{x_0 \in \mathbb{R}^n : (\forall\varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\\ t_0 + T \in T^+_{t_0,x_0} \text{ and}\\ x(\mathbb{R}_{\geq t_0+T} \cap T^+_{t_0,x_0}, t_0, x_0) \subset B_\varepsilon(\Gamma)\}\,.\end{aligned}$$
- Γ is *t_0-uniformly attractive* (t_0-UA) if $\Gamma \subset \operatorname{int}(\mathbb{B}(\Gamma))$.
- Γ is *t_0-uniformly globally attractive* (t_0-UGA) if $\mathbb{B}(\Gamma) = \mathbb{R}^n$.
- Γ is *t_0-uniformly asymptotically stable* (t_0-UAS) if Γ is t_0-US and t_0-UA.
- Γ is *t_0-uniformly globally asymptotically stable* (t_0-UGAS) if Γ is t_0-US and t_0-UGA.

$\triangle$

Remark 6. (On US) US is defined for compact sets only (See Def. 3), but an identical definition may be formulated for closed and unbounded sets. In such case US implies t_0-US, but not vice versa; only for compact sets these properties are equivalent —see item (i) of Proposition 9 below. More precisely, for the US property, given $\varepsilon > 0$ one requires the existence of a neighborhood of initial states of the form $B_\delta(\Gamma)$ whose associated solutions remain in $B_\varepsilon(\Gamma)$ for arbitrary initial times. For the t_0-US property, the neighborhood of initial states is only required to be an open set U containing Γ. When Γ is compact, there is no loss of generality in assuming that U has the form $B_\delta(\Gamma)$, which is the reason why US and t_0-US are equivalent properties for compact sets. On the other hand, if Γ is unbounded then Γ may be t_0-US without being US. This is illustrated in Figure 1, in which it is showed that solutions starting close to U, or even to Γ but laying out of U, may leave the band $B_\varepsilon(\Gamma)$. Unlike Definition 3, Definition 5 allows for finite escape times in the property of t_0-US, and this is because the set Γ is no longer assumed to be compact.

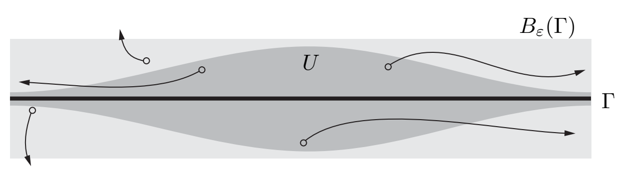

Fig. 1. The set Γ is t_0-US but not US.

$\triangle$

Remark 7 (On UA and t_0-UA)**.** The notions of UA and t_0-UA (and their global counterparts) are both uniform with respect to the initial time t_0, but differ in their requirements on initial states. For the UA property, all solutions with initial states in a neighborhood $B_r(\Gamma)$ get to an arbitrarily small neighborhood $B_\varepsilon(\Gamma)$ of Γ in some time $T > 0$ which depends on ε and is independent of t_0. For the t_0-UA property, the time T depends on x_0 and ε, and is independent of t_0. Even when Γ is compact, UA and t_0-UA are non-equivalent properties. In particular, UA implies t_0-UA, but not vice versa. If system (3) is time-invariant, i.e., f does not depend on t, the t_0-UA property in Definition 5 coincides with the notion of semi-attractivity in [1], and in this case $\mathbb{B}(\Gamma)$ defined above coincides with the basin of attraction of Γ in [1]. Our definition of basin of t_0-uniform attraction does not require the set Γ to be attractive. For instance, in our setting the origin of a saddle point given by the ODE $\dot{x}_1 = -x_1$, $\dot{x}_2 = x_2$, has a well-defined basin of attraction given by the x_1-axis. While our definition agrees with ones commonly found in the literature (e.g., [9,14,27]), some references require Γ to be attractive, or even asymptotically stable (e.g, [8,39]). In the latter cases, the basin of attraction is necessarily open, while $\mathbb{B}(\Gamma)$ is not.

Definition 8 (uniform boundedness of solutions)**.** Let $x_0 \in \mathbb{R}^n$. The solutions with initial state x_0 are t_0-*uniformly bounded* if there exists a constant $c > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)$ for all $t_0 \in \mathbb{R}$. The set of initial states giving rise to t_0-uniformly bounded solutions is defined as

$$\mathbb{BS} := \{x_0 \in \mathbb{R}^n : (\exists c > 0)(\forall t_0 \in \mathbb{R}) \; x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)\}. \tag{4}$$

$\triangle$

The next result clarifies the relationships between the concepts of stability and attractivity in Definitions 3 and 5.

Proposition 9. *Consider the differential equation* (3)*, in which the function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Basic Assumption. Let $\Gamma \subset \mathbb{R}^n$ be a compact, positively-invariant set. Then:*

(i) Γ *is* US *if and only if* Γ *is* t_0-US*;*
(ii) Γ *is* UAS *if and only if* Γ *is* UA*;*
(iii) Γ *is* UGAS *if and only if* Γ *is* UGA *and all solutions are t_0-uniformly bounded, i.e., $\mathbb{BS} = \mathbb{R}^n$;*
(iv) Γ *is* UAS *if and only if* Γ *is* t_0-UAS*;*
(v) Γ *is* UGAS *if and only if* Γ *is* t_0-UGAS *and all solutions are t_0-uniformly bounded, i.e., $\mathbb{BS} = \mathbb{R}^n$.*

The proof is provided in Appendix A.

We conclude this section with definitions of local uniform stability, local t_0-uniform attractivity, and relative stability and attractivity. These are adaptations of notions found in [31,7,20] to the time-varying setting. Unlike the stability notions reviewed earlier, the notions in the next definitions are not widespread in the stability theory literature, but they turn out to be important for the formulation and solution of the reduction problem investigated in this paper.

Definition 10 (local uniform stability of Γ_2 near Γ_1)**.** Let $\Gamma_1 \subset \Gamma_2$ be two closed subsets of $\mathbb{R}^n$, with Γ_1 compact. The set Γ_2 is *locally uniformly stable near* Γ_1 (LUS-Γ_1) for (3) if there exists $r > 0$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_0 \in \mathbb{R}$ and any $x_0 \in B_\delta(\Gamma_1)$ the following implication holds:

$$(\forall t \in T^+_{t_0,x_0}) \, (x([t_0,t], t_0, x_0) \subset B_r(\Gamma_1) \implies x([t_0,t], t_0, x_0) \subset B_\varepsilon(\Gamma_2)) \,. \tag{5}$$

$\triangle$

In other words, the set Γ_2 is LUS-Γ_1 if solutions starting sufficiently close to Γ_1 remain arbitrarily close to Γ_2 so long as they are contained $B_r(\Gamma_1)$. Roughly speaking, the definition allows solutions starting close to Γ_1 to move away from Γ_2 only after they have exited the neighborhood $B_r(\Gamma_1)$. We refer the reader to Figure 1 in [20] and the discussion therein for a depiction of this property.

Definition 11 (t_0-uniform attractivity near a set)**.** Consider system (3). The closed set $\Gamma_2 \subset \mathbb{R}^n$ is t_0-*uniformly attractive near* Γ_1 (t_0-UA near Γ_1) if there exists $r > 0$ such that $B_r(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$. $\triangle$

Definition 12 (relative properties)**.** Consider system (3), and let $\Gamma_1 \subset \Gamma_2$ be two closed subsets of $\mathbb{R}^n$, with Γ_1 compact.

- Γ_1 is US *relative to* Γ_2 for (3) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$.
- Γ_1 is UGS *relative to* Γ_2 if Γ_1 is US relative to Γ_2 and for each $\delta > 0$ there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$.
- Γ_1 is UA *relative to* Γ_2 if there exists $r > 0$ such that for each $\varepsilon > 0$ there exists $T > 0$ such that $x(\mathbb{R}_{\geq t_0+T}, t_0, B_r(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$.
- Γ_1 is UGA [4] *relative to* Γ_2 if $r > 0$ can be chosen arbitrarily large in the definition of UA relative to Γ_2.
- Γ_1 is, respectively, UAS *relative to* Γ_2 or UGAS *relative to* Γ_2, if Γ_1 is US (resp., UGS) and UA (resp., UGA) relative to Γ_2.

$\triangle$

3 Problem formulation and motivation

Consider the system (3) under the Basic Assumption and let $\Gamma_1 \subset \Gamma_2$ be two closed, positively-invariant sets, with Γ_1 compact. Suppose that Γ_1 is P *relative to* Γ_2, where P corresponds to any of the following properties: US, t_0-UA, t_0-UGA, UAS, or UGAS. In its general form, the reduction problem consists in finding conditions under which the property P holds in $\mathbb{R}^n$. As it turns out, however, this problem is meaningful only if it is assumed that Γ_1 is UAS or UGAS relative to Γ_2. The reason is that the properties of uniform stability of Γ_1 relative to Γ_2 and t_0-uniform attractivity of Γ_1 relative to Γ_2 are fragile, in the sense that, in general, they may fail to hold in the whole $\mathbb{R}^n$, even if Γ_2 possesses strong stability properties. This was first pointed out in [31,4] in the time-invariant setting and, for the purpose of motivation, it is illustrated below with two examples.

[4] Similarly, one may define the notion that Γ_1 is t_0-UA or t_0-UGA relative to Γ_2, but it is not used in this paper.

Example 1. (Uniform stability of Γ_1 relative to Γ_2 is a fragile property). Consider the cascade-connected system with state $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}\dot{x}_1 &= x_2 f(t)\\ \dot{x}_2 &= -x_2^3,\end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function such that $f(t) \geq 1$ for all $t \in \mathbb{R}$. Let $\Gamma_1 = \{0_2\}$ and $\Gamma_2 = \{(x_1, x_2) : x_2 = 0\}$. The set Γ_2 is positively invariant because $x_2 = 0$ is an equilibrium of the subsystem with state x_2. On Γ_2, the subsystem with state x_1 reduces to $\dot{x}_1 = 0$, and therefore Γ_1 is US relative to Γ_2.

Since the equilibrium $x_2 = 0$ is globally asymptotically stable for the differential equation $\dot{x}_2 = -x_2^3$, and since the system has no finite escape times, the set Γ_2 is t_0-UGAS (in fact, UGAS). Yet, Γ_1 is unstable. To see why this is the case, pick $\epsilon > 0$ and $t_0 \in \mathbb{R}$, and let $(x_1(t), x_2(t))$ be the solution with initial state $x(t_0) = [0 \;\; \epsilon]^\top$. Then $x_2(t) \to 0$ at a rate of $t^{-1/2}$, and using the fact that $x_2(t) > 0$, we deduce that

$$x_1(t) = \int_{t_0}^{t} x_2(\tau) f(\tau) d\tau \geq \int_{t_0}^{t} x_2(\tau) d\tau \to \infty \text{ as } t \to \infty.$$

Since $\epsilon > 0$ is arbitrary, the origin is unstable.

In conclusion, Γ_1 is US relative to Γ_2 and Γ_2 is t_0-UGAS, but Γ_1 is not US in $\mathbb{R}^2$ because $t \mapsto x_2(t)$ is not integrable. In [24, Theorem 1, condition (10)] it is shown that the integrability of $t \mapsto x_2(t)$ plays a crucial role in the UGS property. $\triangle$

Example 2. (t_0-Uniform attractivity of Γ_1 relative to Γ_2 is a fragile property). This example is adapted from [6]. Consider the time-varying system

$$\dot{x}_1 = x_2(x_1 - 1) - x_1(x_1^2 + x_2^2 - 1) - x_2 x_3 \sin(t)^2 \quad (6a)$$

$$\dot{x}_2 = -x_1(x_1 - 1) - x_2(x_1^2 + x_2^2 - 1) + x_1 x_3 \sin(t)^2 \quad (6b)$$

$$\dot{x}_3 = -x_3^3, \quad (6c)$$

and let $\Gamma_1 = \{(x_1, x_2, x_3) = (1, 0, 0)\}$ and $\Gamma_2 = \{(x_1, x_2, x_3) : x_3 = 0\}$. As in the previous example, Γ_2 is positively invariant and t_0-UGAS. We claim that Γ_1 is t_0-UA relative to Γ_2. To see why this is the case, let $(r, \theta) \in \mathbb{R}_{>0} \times \mathbb{S}^1$ be polar coordinates for the (x_1, x_2) plane, excluding the origin, so that $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. In (r, θ, x_3) coordinates, the above time-varying system reads as

$$\dot{r} = -r(r^2 - 1) \quad (7a)$$

$$\dot{\theta} = 1 - r\cos(\theta) + x_3 \sin(t)^2 \quad (7b)$$

$$\dot{x}_3 = -x_3^3. \quad (7c)$$

In (r, θ, x_3) coordinates, the sets Γ_1, Γ_2 are given by, respectively, $\tilde{\Gamma}_1 = \{(r, \theta, x_3) = (1, 0, 0)\}$ and $\tilde{\Gamma}_2 = \{(r, \theta, x_3) : x_3 = 0\}$. The dynamics on $\tilde{\Gamma}_2$ are described by the time-invariant system

$$\dot{r} = -r(r^2 - 1) \quad (8a)$$

$$\dot{\theta} = 1 - r\cos(\theta). \quad (8b)$$

For each $t_0 \in \mathbb{R}$, if $r(t_0) \neq 0$ then the solution $r(t) \to 1$ uniformly with respect to t_0, and if $\theta(t_0) \neq \pi$, then $\theta(t) \to (0 \bmod 2\pi)$. This proves that Γ_1 is t_0-UA relative to Γ_2. On the other hand, Γ_1 is *not* US relative to Γ_2 because the unit circle is a homoclinic orbit of system (8) (see the left-hand side of Figure 2) which implies that there are initial states in Γ_2 arbitrarily close to Γ_1 leading to solutions following the whole circle before converging to Γ_1.

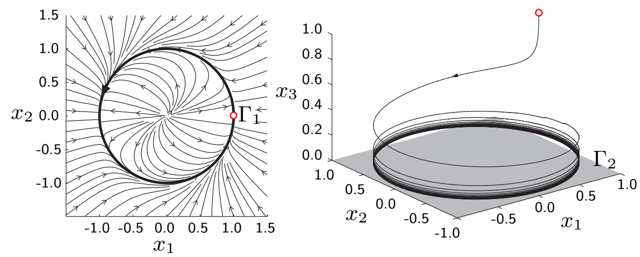

Fig. 2. On the left-hand side, the phase portrait of system (8) in $(x_1, x_2) = (r\cos(\theta), r\sin(\theta))$ coordinates, representing the dynamics on Γ_2. The set Γ_1, an equilibrium, is t_0-UA relative to Γ_2, but unstable. On the right-hand side, an orbit of the time-varying system (6) converging to Γ_2, but not to Γ_1.

Consider initial data (t_0, x_0) where $t_0 \in \mathbb{R}$ is arbitrary and $x_0 \in \mathbb{R}^3$ is a vector whose third component is positive and whose first two components lie on the unit circle, i.e., in (r, θ, x_3) coordinates, $r(t_0) = 1$, $x_3(t_0) > 0$. The corresponding solution $(r(t), \theta(t), x_3(t))$ has the property that $r(t) \equiv 1$, and $x_3(t)$ tends to zero with rate $t^{-1/2}$. Thus, Equation (7b) may be rewritten as

$$\dot{\theta} = 1 - \cos(\theta) + \mu(t),$$

where $\mu(t) \geq 0$ converges to 0 with rate $t^{-1/2}$. The solution $\theta(t)$ satisfies

$$\begin{aligned}\theta(t) &= \theta(t_0) + \int_{t_0}^{t} 1 - \cos(\theta(\tau)) d\tau + \int_{t_0}^{t} \mu(\tau) d\tau\\ &\geq \theta(t_0) + \int_{t_0}^{t} \mu(\tau) d\tau \to \infty \text{ as } t \to \infty.\end{aligned}$$

Thus, in (x_1, x_2, x_3) coordinates, the solution does not converge to Γ_1, and in fact it converges to the unit circle on Γ_2, see the right-hand side of Figure 2. This proves that Γ_1 is not t_0-UA.

In conclusion, Γ_1 is t_0-UA relative to Γ_2 and Γ_2 is t_0-UGAS, but Γ_1 is not t_0-UA in $\mathbb{R}^3$. $\triangle$

We are now ready to precisely state the reduction problem.

Reduction Problem. Suppose that Γ_1 is UAS or UGAS relative to Γ_2. Find conditions under which a property P$\in$ {US, t_0-UA, t_0-UGA, UAS, UGAS} holds in $\mathbb{R}^n$. $\triangle$

Remark 13. Note that in the list of properties P of interest, we did not include uniform attractivity (UA). The reason is that, by Proposition 9, uniform attractivity of compact sets is equivalent to uniform asymptotic stability, therefore there is no need to state a separate reduction problem for uniform attractivity. The t_0-uniform attractivity property (t_0-UA), on the other hand, is complementary to uniform stability (US) in that, together, these two properties are equivalent to uniform asymptotic stability (see Proposition 9, parts (i) and (iv)). An analogous remark holds for the global version of these properties. △

4 The reduction theorems

First, we present three reduction theorems for the properties of uniform stability, t_0-uniform attractivity, uniform attractivity, and their global counterparts for time-varying systems. Then, we present useful consequences of these theorems. For clarity of exposition, the proofs of the main statements are provided in Section 6.

4.1 Main statements for time-varying systems

Our first theorem generalizes [13, Lemmas 1 and 2] on stability of the origin to the setting in this paper.

Theorem 14 (Reduction theorem for uniform stability)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$*. Then* Γ_1 *is* US *if*

(i) Γ_1 *is* UAS *relative to* Γ_2*, and*
(ii) Γ_2 *is* LUS-Γ_1*.*

Remark 15. The proof of Theorem 14 follows similar lines as the proofs of [13, Lemmas 1 and 2]. In turn, the proofs in [13] extend to the time-varying setting an argument established by Seibert and Florio in [31]. Thus, even though Theorem 14 is original and has interest of its own (below, in the proof of necessity of Theorem 18, it is shown that assumption (ii) is a necessary condition for Γ_1 to be US), the proof is omitted. △

Theorem 16 (Reduction theorem for t_0-uniform (global) attractivity)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$*. Assume that*

(i) Γ_1 *is* UAS *relative to* Γ_2*,*
(ii) Γ_2 *is* t_0-UA *near* Γ_1*, and*
(iii) there exists $\delta > 0$ *such that the set*

$$K_\delta := \overline{\bigcup_{t_0 \in \mathbb{R}} x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1))}$$

is compact and such that $K_\delta \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$*.*

Then, Γ_1 *is* t_0-UA *and* $B_\delta(\Gamma_1) \subset \mathbb{B}(\Gamma_1)$*.*

Moreover, if

(i)' Γ_1 *is* UGAS *relative to* Γ_2*, and*
(ii)' Γ_2 *is* t_0-UGA*,*

then all initial states giving rise to t_0*-uniformly bounded solutions are contained in the basin of* t_0*-uniform attraction of* Γ*, i.e.,* $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$*. In particular, if all solutions of* (3) *are* t_0*-uniformly bounded, i.e.,* $\mathbb{BS} = \mathbb{R}^n$*, then* Γ_1 *is* t_0-UGA*.*

Theorem 16 is proved in Section 6.1.

Remark 17. Assumption (ii) is a necessary condition for Γ_1 to be t_0-UA. Indeed, since $\Gamma_1 \subset \Gamma_2$, $\mathbb{B}(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$. If Γ_1 is t_0-UA, then by definition $\Gamma_1 \subset \operatorname{int} \mathbb{B}(\Gamma_1) \subset \operatorname{int} \mathbb{B}(\Gamma_2)$. Since Γ_1 is compact, the latter inclusion implies that there exists $r > 0$ such that $B_r(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$, and thus Γ_2 is t_0-UA near Γ_1. Similarly, Assumption (ii)' is necessary for Γ_1 to be t_0-UGA. Assumption (iii) is hard to check in general, but the first part of Theorem 16, which asserts that Γ_1 is t_0-UA, is useful to establish other statements, such as the first one in Theorem 18 below.

Theorem 18 (Reduction theorem for uniform (global) asymptotic stability)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$*. Then* Γ_1 *is* UAS *if and only if*

(i) Γ_1 *is* UAS *relative to* Γ_2*,*
(ii) Γ_2 *is* LUS-Γ_1*, and*
(iii) Γ_2 *is* t_0-UA *near* Γ_1*.*

Moreover, Γ_1 *is* UGAS *if and only if*

(i)' Γ_1 *is* UGAS *relative to* Γ_2*,*
(ii) Γ_2 *is* LUS-Γ_1*,*
(iii)' Γ_2 *is* t_0-UGA*, and*
(iv) all solutions are t_0*-uniformly bounded, i.e.,* $\mathbb{BS} = \mathbb{R}^n$*.*

Finally, if assumptions (i)', (ii), and (iii)' hold, then Γ_1 *is* UAS *and all initial states giving rise to* t_0*-uniformly bounded solutions are contained in the basin of* t_0*-uniform attraction of* Γ*, i.e.,* $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$*.*

Theorem 18 is proved in Section 6.2.

Remark 19. Lemma 3 in [13] establishes sufficiency in the special case of Theorem 18 in which Γ_1 is the origin. The lemma in question, however, does not establish necessity of the various assumptions and it does not characterize the basin of attraction of the origin, as Theorem 18 does for the set Γ_1. Moreover, the proof of sufficiency of Theorem 18 presented in Section 6.2 is different than the proof of [13, Lemma 3] because we leverage the reduction theorem for t_0-uniform attractivity, Theorem 16, which is not present in [13]. △

Remark 20. Theorems 14, 16, and 18 may be used recursively to analyze the stability of chains of nested closed positively invariant sets $\Gamma_1 \subset \cdots \subset \Gamma_k \subset \mathbb{R}^n$ in which Γ_1 is compact. This was done in the context of the hierarchical control problem in [7, Proposition 14] and then applied to backstepping. See also [20, Theorem 4.9]. Furthermore, the results of this paper can be directly used to extend the method proposed in [7] to the context of time-varying systems with minimal modifications. △

Remark 21. Theorems 14, 16, and 18 establish uniform stability and attractivity properties in $\mathbb{R}^n$. If $\mathcal{X} \subset \mathbb{R}^n$ is a positively invariant set such that $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, and if the assumptions of these theorems hold for initial states restricted to lie in $\mathcal{X}$, then the results of the theorems hold relative to $\mathcal{X}$. In Section 5.2, we illustrate this fact with an example. △

Remark 22. The observation made in Remark 21 is important as it implies that Theorems 14, 16, and 18 apply to time-varying systems whose state spaces are smooth complete Riemannian manifolds $\mathcal{X}$, not necessarily diffeomorphic to $\mathbb{R}^n$ (see for instance the kinematic unicycle example in Section 5.3). The reason that our results can be applied to such state spaces is this. The Nash isometric embedding theorem [22, Theorem 3] guarantees the existence of a C^1 embedding of $\mathcal{X}$ in a Euclidean space $\mathbb{R}^n$ of suitable dimension, in such a way that the Riemannian metric of $\mathcal{X}$ is the restriction to $\mathcal{X}$ of the Euclidean metric of $\mathbb{R}^n$. By smoothly extending the vector field $f(t,x)$ from $\mathcal{X}$ to $\mathbb{R}^n$ (which can be done globally in virtue of [16, Lemma 8.6]), one obtains a time-varying differential equation on $\mathbb{R}^n$ for which $\mathcal{X} \subset \mathbb{R}^n$ is a positively invariant set. Thus, Theorems 14, 16, and 18 apply and all the statements in this section continue to hold if the state space of the differential equation is a smooth complete Riemannian manifold. △

4.2 Consequences of the reduction theorems

Next, we present some useful consequences of the reduction theorems. The first statement, which is a straightforward consequence of the reduction theorem for UGAS (Theorem 18), replaces assumption (ii) in that theorem (Γ_2 is LUS-Γ_1) with the assumption that Γ_2 is t_0-US. Even though the latter is more conservative, it is generally easier to verify than assumption (ii) in Theorem 18.

Proposition 23. *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. *If*

(i) Γ_1 *is* UAS *relative to* Γ_2, *and*
(ii) Γ_2 *is* t_0-UAS,

then Γ_1 *is* UAS. *Moreover,* Γ_1 *is* UGAS *if*

(iii) Γ_1 *is* UGAS *relative to* Γ_2,
(iv) Γ_2 *is* t_0-UGAS,
(v) all solutions are t_0*-uniformly bounded, i.e.,* $\mathbb{BS} = \mathbb{R}^n$.

Finally, if assumptions (iii) and (iv) hold, then Γ_1 *is* UAS *and all initial states giving rise to* t_0*-uniformly bounded solutions are contained in the basin of* t_0*-uniform attraction of* Γ, *i.e.,* $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$.

PROOF. Assumption (ii) implies that Γ_2 is t_0-US and t_0-UA, while if Γ_2 is t_0-UA then it is also t_0-UA near Γ_1. Therefore, conditions (i) and (iii) of Theorem 18 hold. That is, in order to prove the first statement, that Γ_1 is UAS, it suffices to establish the implication

$$(\Gamma_2 \text{ is } t_0\text{-US}) \implies (\Gamma_2 \text{ is LUS-}\Gamma_1). \tag{9}$$

Furthermore, assumption (iv) implies that Γ_2 is t_0-US and t_0-UGA. Therefore, the remaining statements in the proposition also follow directly from Theorem 18, provided that the implication (9) holds. Thus, to show that this is the case, assume Γ_2 is t_0-US. Then for each $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ such that $\Gamma_2 \subset U$ and for each $(t_0, x_0) \in \mathbb{R} \times U$ and each $t \in T^+_{t_0,x_0}$, $x(t,t_0,x_0) \in B_\varepsilon(\Gamma_2)$. By the compactness of Γ_1 and the fact that $\Gamma_1 \subset \Gamma_2$, there exists $\delta > 0$ such that $B_\delta(\Gamma_1) \subset U$. Then we have

$$(\forall x_0 \in B_\delta(\Gamma_1))(\forall t_0 \in \mathbb{R})(\forall t \in T^+_{t_0,x_0})\quad x([t_0,t],t_0,x_0) \subset B_\varepsilon(\Gamma_2). \tag{10}$$

Comparing with (5) in the definition of local uniform stability, we see that (10) implies (5) for arbitrary $r > 0$, and thus Γ_2 is LUS-Γ_1. □

From Proposition 23 we recover a well-known result concerning the stability of equilibria for cascade-connected systems (see [32, Theorem 1.1] for the time-invariant case, and Lemma 2 in [25] for the time-varying case).

Corollary 24 (Cascade-connected systems)**.** *Consider the cascade-connected system*

$$\begin{aligned} \dot{x}_1 &= f_1(t,x_1,x_2) \\ \dot{x}_2 &= f_2(t,x_2) \end{aligned} \tag{11}$$

where $f_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $f_2 : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfy the Basic Assumption and* $f_1(\cdot, 0_n, 0_m) \equiv 0_n$, $f_2(\cdot, 0_m) \equiv 0_m$. *Then the equilibrium* $(x_1, x_2) = (0_n, 0_m)$ *is* UGAS *for* (11) *if and only if*

(i) the equilibrium $x_1 = 0_n$ *is UGAS for* $\dot{x}_1 = f_1(t,x_1,0_m)$,
(ii) the equilibrium $x_2 = 0_m$ *is UGAS for* $\dot{x}_2 = f_2(t,x_2)$, *and*
(iii) all solutions of (11) *are* t_0*-uniformly bounded, i.e.,* $\mathbb{BS} = \mathbb{R}^n \times \mathbb{R}^m$.

On the other hand, if only assumptions (i) and (ii) hold and the set $\mathbb{BS}$ *of* t_0*-uniformly bounded solutions is only a subset of* $\mathbb{R}^n \times \mathbb{R}^m$, *then the equilibrium* $(x_1, x_2) = (0_n, 0_m)$ *is* UAS *and the set* $\mathbb{BS}$ *is contained in the basin of* t_0*-uniform attraction of the equilibrium* $(0_n, 0_m)$, *i.e.,* $\mathbb{BS} \subset \mathbb{B}(0_n, 0_m)$.

The proof of sufficiency of Corollary 24 follows directly from Proposition 23 by setting $\Gamma_1 := \{(0_n, 0_m) \in \mathbb{R}^n \times \mathbb{R}^m\}$ and $\Gamma_2 := \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m : x_2 = 0_m\}$. Then, assumption (i) implies that Γ_1 is UGAS relative to Γ_2, while assumption (ii) implies that Γ_2 is t_0-UGAS. The proof of necessity is straightforward and is omitted.

4.3 Lyapunov characterizations

The reduction theorems in Section 4, as well as Proposition 23, rely on assumptions that are somewhat unusual in the literature on stability theory:

- that Γ_2 is LUS-Γ_1. This is used in Theorems 14 and 18;
- that Γ_2 is either t_0-UA near Γ_1 or t_0-UGA. This is used in Theorems 16 and 18;
- and, that Γ_2 is either t_0-UAS or t_0-UGAS. This is used in Proposition 23.

In this section we give Lyapunov characterizations of the properties listed above. Even though these characterizations are more conservative in general, they may result easier to verify in concrete cases. An example that illustrates this assertion is given in Section 5.1.

Proposition 25 (Lyapunov characterization of $\mathsf{LUS}\text{-}\Gamma_1$ property)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Suppose there exist $r, s > 0$ and a C^1 nonnegative function $V : \mathbb{R} \times B_r(\Gamma_1) \to \mathbb{R}$ such that*

$$\alpha(\|x\|_{\Gamma_2}) \leq V(t,x) \leq \beta(\|x\|_{\Gamma_1}) \tag{12}$$

$$\partial_t V(t,x) + \partial_x V(t,x) f(t,x) \leq 0, \tag{13}$$

for all $(t,x) \in \mathbb{R} \times B_r(\Gamma_1)$, where $\alpha : [0,s) \to \mathbb{R}$ and $\beta : [0,r) \to \mathbb{R}$ are two class $\mathcal{K}$ functions. Then, Γ_2 is $\mathsf{LUS}\text{-}\Gamma_1$.

Remark 26. The proof of Proposition 25 is reminiscent of that in [13, Theorem 1], addressing the particular case in which Γ_1 is an equilibrium, and is omitted. △

Next, we provide a Lyapunov characterization of the t_0-UA, t_0-UGA, and t_0-UGAS properties for closed, but not necessarily compact sets.

Proposition 27 (Lyapunov characterization of t_0-UA, t_0-UGA, and t_0-UGAS properties)**.** *Consider the time-varying system* (3) *under the Basic Assumption. Let $\Gamma \subset \mathbb{R}^n$ be a closed, positively-invariant set, and $U \subset \mathbb{R}^n$ be an open set such that $\Gamma \cap U \neq \emptyset$. Let $V : \mathbb{R} \times U \to \mathbb{R}$ be a C^1 nonnegative function such that*

$$W_1(x) \leq V(t,x) \leq W_2(x) \tag{14}$$

$$\partial_t V(t,x) + \partial_x V(t,x) f(t,x) \leq -W_3(x), \tag{15}$$

for all $(t,x) \in \mathbb{R} \times U$, where $W_1, W_2, W_3 : U \to \mathbb{R}$ are continuous nonnegative functions such that $W_1^{-1}(0) = W_2^{-1}(0) = W_3^{-1}(0) = \Gamma \cap U$. Let $U^\star \subset U$ be defined as[5] *$U^\star := \{x_0 \in U : (\forall t_0 \in \mathbb{R})\, x(T^+_{t_0,x_0}, t_0, x_0) \subset U\}$. Then, the following implications hold:*

(a) All initial states in $U^\star$ giving rise to solutions that are t_0-uniformly bounded are contained in the basin of t_0-uniform attraction of Γ, i.e.,

$$\mathbb{BS} \cap U^\star \subset \mathbb{B}(\Gamma).$$

(b) If $U = \mathbb{R}^n$ and $\Gamma \subset \mathrm{int}(\mathbb{BS})$, then Γ is t_0-UA.

(c) If $U = \mathbb{R}^n$ and all solutions are t_0-uniformly bounded, i.e., $U = \mathbb{BS} = \mathbb{R}^n$, then Γ is t_0-UGA.

(d) If $U = \mathbb{BS} = \mathbb{R}^n$, and there exist $r > 0$ and a class $\mathcal{K}$ function $\alpha_1 : [0,r) \to \mathbb{R}$ such that $\alpha_1(\|x\|_\Gamma) \leq W_1(x)$ for all $x \in B_r(\Gamma)$, then Γ is t_0-UGAS.

Remark 28. The inequalities (14) and (15) are reminiscent of the ones imposed for uniform asymptotic stability of equilibria in [14, Theorems 4.8, 4.9]. Proposition 27, however, deals with closed and not necessarily compact sets. If Γ were compact and U were a neighbourhood of Γ, inequalities (14) and (15) would imply the existence of class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1(\|x\|_\Gamma) \leq V(t,x) \leq \alpha_2(\|x\|_\Gamma) \tag{16}$$

$$\partial_t V(t,x) + \partial_x V(t,x) f(t,x) \leq -\alpha_3(\|x\|_\Gamma), \tag{17}$$

and these are exactly the bounds used in the proofs of [14, Theorems 4.8, 4.9]. The inequalities in (16)-(17) are also used more generally in Lyapunov characterizations of the UGAS property for compact sets (see, e.g., [17]). When Γ is not compact, however, inequalities (14)-(15) no longer imply, but are obviously implied by (16)-(17), so (14)-(15) are less restrictive. As a result, in contrast with the results in [14,17], parts (b) and (c) of Proposition 27 only establish local and global uniform attractivity of Γ, which does not imply uniform stability when Γ is unbounded. △

PROOF. Part (a). Letting $x_0 \in \mathbb{BS} \cap U^\star$ be arbitrarily fixed, we want to show that $x_0 \in \mathbb{B}(\Gamma)$, that is

$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})$$
$$t_0 + T \in T^+_{t_0,x_0} \text{ and } x(\mathbb{R}_{\geq t_0+T} \cap T^+_{t_0,x_0}, t_0, x_0) \subset B_\varepsilon(\Gamma). \tag{18}$$

If $x_0 \in \Gamma$, then $x_0 \in \mathbb{B}(\Gamma)$ because Γ is positively invariant. Suppose $x_0 \in U^\star \setminus \Gamma$. Since $x_0 \in \mathbb{BS}$, by definition there exists $c > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)$ for all $t_0 \in \mathbb{R}$, which implies that $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$ for all $t_0 \in \mathbb{R}$. Letting $K := \overline{B_c(0)}$, a compact set and using the fact that $x_0 \in \mathbb{BS} \cap U^\star$, we have

$$(\forall t_0 \in \mathbb{R})\; x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset K \cap U. \tag{19}$$

Since $x_0 \in U^\star \setminus \Gamma$, $\|x_0\|_\Gamma > 0$. Let $\varepsilon \in (0, \|x_0\|_\Gamma)$ and define

$$\delta_1 := \min_{x \in K, \|x\|_\Gamma \geq \varepsilon/2} W_1(x), \quad \hat{\delta}_2 := \min_{x \in K, W_2(x) \geq \delta_1} W_1(x),$$
$$\delta_2 := \min\{\hat{\delta}_2, W_2(x_0)\}.$$

Since $x_0 \in K$ and $\|x_0\|_\Gamma > \varepsilon$, the set $\{x \in K : \|x\|_\Gamma \geq \varepsilon/2\}$ is nonempty. It is also compact because K is. Since W_1 is continuous and nonnegative, δ_1 is well-defined and $\delta_1 > 0$ because $W_1 > 0$ on the set $\{x \in K : \|x\|_\Gamma \geq \varepsilon/2\}$. Since $W_1(x) \leq W_2(x)$, we have $\{x \in K : W_1(x) \geq \delta_1\} \subset \{x \in K : W_2(x) \geq \delta_1\}$. Noticing that $x_0 \in \{x \in K : W_1(x) \geq \delta_1\}$, the set $\{x \in K : W_2(x) \geq \delta_1\}$ is nonempty and compact, so $\hat{\delta}_2$ is well-defined. Further, $\hat{\delta}_2 > 0$ because $W_2(x) \geq \delta_1$ implies that $x \notin \Gamma$ and thus $W_1(x) > 0$. Finally, since $x_0 \notin \Gamma$, $W_2(x_0) > 0$, so $\delta_2 > 0$ as well. Next, let

$$k := \min_{x \in K, W_2(x) \geq \delta_2/2} W_3(x). \tag{20}$$

Since $x_0 \in K$ and, by the definition of δ_2, $W_2(x_0) \geq \delta_2$, the set $\{x \in K : W_2(x) \geq \delta_2/2\}$ is nonempty and compact so k is well defined and $k > 0$ because W_3 is positive on this set. In view of the definition of δ_1 and δ_2, we have

$$\{x \in K : W_1(x) \leq \delta_1\} \subset \overline{B_{\varepsilon/2}(\Gamma)} \subset B_\varepsilon(\Gamma), \tag{21}$$

$$\{x \in K : W_1(x) \leq \delta_2\} \subset \{x \in K : W_2(x) \leq \delta_1\}. \tag{22}$$

We claim that

$$(\exists T > 0)(\forall t_0 \in \mathbb{R})\; W_1(x(t_0 + T, t_0, x_0)) \leq \delta_2. \tag{23}$$

By way of contradiction, suppose that

$$(\forall T > 0)(\exists t_0 \in \mathbb{R})\; W_1(x(t_0 + T, t_0, x_0)) > \delta_2. \tag{24}$$

By (15) and (19), for any $t_0 \in \mathbb{R}$ the function $t \mapsto V(t, x(t, t_0, x_0))$ is nonincreasing. Then using (14), (24) implies that $x([t_0, t_0 + T], t_0, x_0) \subset K \cap \{x \in \mathbb{R}^n : W_2(x) \geq \delta_2/2\}$.

[5] The set $U^\star$ may be empty. If $U = \mathbb{R}^n$, then $U^\star = \mathbb{R}^n$.

Let $T{:=}(W_2(x_0)-\delta_2)/k \geq 0$, and let $t_0 \in \mathbb{R}$ be such that (24) holds. Using (15), (19), and the definition of k in (20), we have

$$V(t_0+T, x(t_0+T, t_0, x_0)) \leq V(t_0, x_0) - kT \leq W_2(x_0) - kT = \delta_2.$$

By the first inequality in (14), $W_1(x(t_0+T, t_0, x_0)) \leq \delta_2$, contradicting (24). Thus (23) holds. Henceforth, fix $T \geq 0$ such that (23) holds. Since $W_1(x(t_0+T, t_0, x_0)) \leq \delta_2$, by (19) and (22) we have that $W_2(x(t_0+T, t_0, x_0)) \leq \delta_1$ for any $t_0 \in \mathbb{R}$. Since for any $t_0 \in \mathbb{R}$ the function $t \mapsto V(t, x(t, t_0, x_0))$ is nonincreasing, and since $V(t_0+T, x(t_0+T, t_0, x_0)) \leq W_2(x(t_0+T, t_0, x_0)) \leq \delta_1$, we have that $V(t, x(t, t_0, x_0)) \leq \delta_1$ for all $t_0 \in \mathbb{R}$ and all $t \geq t_0+T$. Using the first inequality in (14), we deduce that $W_1(x(t, t_0, x_0)) \leq \delta_1$ for all $t_0 \in \mathbb{R}$ and all $t \geq t_0+T$. By (19) and (21), we conclude that for all $t_0 \in \mathbb{R}$, $x(\mathbb{R}_{\geq t_0+T}, t_0, x_0) \subset B_\varepsilon(\Gamma)$, and thus (18) holds. We have thus shown that for each $x_0 \in \mathbb{BS} \cap U^\star$, $x_0 \in \mathbb{B}(\Gamma)$. This concludes the proof of part (a).

Part (b). If $U = \mathbb{R}^n$ and $\Gamma \subset \text{int}(\mathbb{BS})$, then $U^\star = \mathbb{R}^n$, and by part (a), $\Gamma \subset \text{int}(\mathbb{BS}) \subset \text{int}(\mathbb{B}(\Gamma))$, which implies that Γ is t_0-UA.

Part (c). If $U = \mathbb{BS} = \mathbb{R}^n$ then $U^\star = \mathbb{R}^n$, and by part (a), $\mathbb{B}(\Gamma) = \mathbb{R}^n$, which implies that Γ is t_0-UGA.

Part (d). Now suppose that $U = \mathbb{BS} = \mathbb{R}^n$ so that, by part (c), Γ is t_0-UGA, and there exist $r > 0$ and a class $\mathcal{K}$ function $\alpha_1 : [0, r) \to \mathbb{R}$ such that $\alpha_1(\|x\|_\Gamma) \leq W_1(x)$ for all $x \in B_r(\Gamma)$. We need to show that Γ is t_0-US. Let $\varepsilon > 0$ be arbitrary, without loss of generality $\varepsilon \in (0, r)$. Define the open set $\mathcal{W}{:=}\{x \in \mathbb{R}^n : W_2(x) < \alpha_1(\varepsilon)\}$. For any initial data $(t_0, x_0) \in \mathbb{R} \times \mathcal{W}$, we have $W_2(x_0) < \alpha_1(\varepsilon)$, and by (14) and (15) we have that $W_1(x(t, t_0, x_0)) < \alpha_1(\varepsilon)$ for all $t \in T^+_{t_0,x_0}$. Since $W_1(x) \geq \alpha_1(\|x\|_\Gamma)$, $\|x(t, t_0, x_0)\|_\Gamma < \varepsilon$ for all $t \in T^+_{t_0,x_0}$. This proves that Γ is t_0-US. In conclusion, we have shown that Γ is both t_0-UGA and t_0-US, which implies that Γ is t_0-UGAS. This concludes the proof of the proposition. □

Part (a) of Proposition 27 yields the next Lyapunov characterization of the property that a set Γ_2 is t_0-UA near Γ_1, used in Theorems 16 and 18.

Corollary 29 (Lyapunov characterization of the property that Γ_2 is t_0-UA near Γ_1). *Consider the time-varying system* (3) *under the Basic Assumption. Let* Γ_1 *be a compact set and* Γ_2 *be a closed set, both positively invariant and such that* $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. *Suppose that* Γ_1 *is* US, *and for some open set* $U \subset \mathbb{R}^n$ *such that* $\Gamma_1 \subset U$, *there exists a* C^1 *nonnegative function* $V : \mathbb{R} \times U \to \mathbb{R}$ *satisfying* (14) *and* (15), *where* $W_1, W_2, W_3 : U \to \mathbb{R}$ *are continuous nonnegative functions such that* $W_1^{-1}(0) = W_2^{-1}(0) = W_3^{-1}(0) = \Gamma \cap U$. *Then* Γ_2 *is* t_0-UA *near* Γ_1.

PROOF. Since Γ_1 is compact and contained in the open set U, there exists $\varepsilon > 0$ such that $B_\varepsilon(\Gamma_1) \subset U$. Since Γ_1 is US, there exists $\delta > 0$ such that

$$(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1) \subset U, \qquad (25)$$

which implies that $B_\delta(\Gamma_1) \subset U^\star$, with $U^\star$ defined in the statement of Proposition 27. Moreover, since Γ_1 is compact the set $B_\varepsilon(\Gamma_1)$ is bounded, and thus property (25) implies that $B_\delta(\Gamma_1) \subset \mathbb{BS}$. We have thus established that $B_\delta(\Gamma_1) \subset \mathbb{BS} \cap U^\star$. By part (a) of Proposition 27, $B_\delta(\Gamma_1) \subset \mathbb{BS} \cap U^\star \subset \mathbb{B}(\Gamma_2)$, and thus Γ_2 is t_0-UA near Γ_1. □

5 Examples

In this section we present three examples demonstrating the utility of the theoretical results in Section 4. In the first example we revisit the Slotine & Li controller mentioned in the introduction, considering (for simplicity) the special case of one degree-of-freedom mechanical systems, and we propose a reduction viewpoint to understand its operation. In particular, we show that its uniform global tracking properties can be derived using Propositions 25, 27, and Theorem 18. The second example illustrates the reduction theorem for t_0-uniform attractivity (Theorem 16). Finally, in the third example we use Proposition 23 to derive a global path following controller for a kinematic unicycle meeting a position tracking requirement on the path.

5.1 The Slotine & Li controller

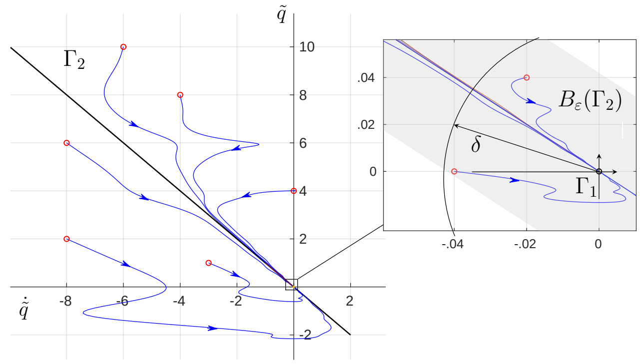

Fig. 3. Trajectories generated by the Slotine-and-Li controller represented on the plane

Consider the following Lagrangian control system

$$d(q)\ddot{q} + c(q)\dot{q}^2 + g(q) = u,$$

where, for simplicity of exposition, we assume that $q \in \mathbb{R}$. The function $q \mapsto d(q)$ denotes the system's inertia and it is bounded, smooth and bounded away from zero uniformly for all $q \in \mathbb{R}$, i.e., $0 < d_m \leq d(\cdot) \leq d_M$; the function $q \mapsto c(q)$ is uniformly bounded and satisfies $2c(q) := d'(q)$; the function $q \mapsto g(q)$ denotes forces stemming from potential energy and it is also uniformly bounded. Consider the problem of making the generalized positions and velocities q and $\dot{q}$ follow some given desired smooth bounded reference trajectories $q_d(t)$ and $\dot{q}_d(t)$. This problem was solved (for systems with $q \in \mathbb{R}^n$, $n \geq 1$) in [33], where the now well-known *Slotine & Li controller* was proposed. This is defined as follows. Let $\lambda, k_d > 0$ be two design parameters and let

$$u = d(q)\ddot{q}_r + c(q)\dot{q}\dot{q}_r + g(q) - k_d s \qquad \text{(26a)}$$

$$s := \dot q - \dot q_r \tag{26b}$$

$$\dot q_r := \dot q_d(t) - \lambda \tilde q, \quad \tilde q := q - q_d(t). \tag{26c}$$

Then, the closed-loop nonlinear time-varying system is given by

$$d\big(\tilde q + q_d(t)\big)\dot s + c(\tilde q + q_d(t))\big(s + \dot q_d(t) + \lambda \tilde q\big)s + k_d s = 0 \tag{27a}$$

$$\dot{\tilde q} = -\lambda \tilde q + s. \tag{27b}$$

It is well known that for the system (27) the origin, $\{(\tilde q, s) = (0,0)\}$, is uniformly globally asymptotically stable; this may be established via various methods, including Lyapunov's first [36]. We revisit the analysis of this system because the rationale that leads to the design of this controller in [34] captures well the essence of the reduction theorems. Indeed, the controller is designed in a manner to steer the trajectory $\dot q(t)$ to the artificially-defined reference $\dot q_r$ generated by (26c). Given that $\dot q = \dot q_r$ is equivalent to $s = 0$, the controller is designed to steer the trajectories towards the set $\Gamma_2 := \{(\tilde q, s) : s = 0\}$ (see Figure 3 for an illustration with $\lambda = 1$ and $k_d = 3$) on which the dynamics is *reduced* to $\dot{\tilde q} = -\lambda \tilde q$. More precisely, the function

$$V(t, \tilde q, s) := \frac{1}{2} d\big(\tilde q + q_d(t)\big)s^2, \tag{28}$$

satisfies

$$\frac{1}{2} d_m s^2 \le V(t, \tilde q, s) \le d_M(s^2 + \tilde q^2) \tag{29a}$$

$$\dot V(t, \tilde q, s) \le -k_d s^2 \tag{29b}$$

in view of the assumption that $0 < d_m \le d(\cdot) \le d_M$ and $\dot d(q) = 2c(q)\dot q$. From these inequalities, it follows that $s \to 0$ for any $k_d > 0$. That is, the trajectories tend to the set Γ_2 on which they satisfy $\dot{\tilde q} = -\lambda \tilde q$, so $\tilde q \to 0$ for any $\lambda > 0$. It is important to stress that, although intuitive, this argument tacitly relies on the set Γ_2 being reached in finite time, which is not the case for this controller; the trajectories only *tend asymptotically* to Γ_2. A formal argument may be made using Theorem 18 even if the convergence to Γ_2 is only asymptotic. To this end, letting $\Gamma_1 := \{(\tilde q, s) = (0,0)\}$ and $\Gamma_2 := \{(\tilde q, s) : s = 0\}$, the following remarks are in order:

- The set Γ_1 is UGAS relative to Γ_2. This follows from the fact that, for the system (27b) with $s = 0$, $\{\tilde q = 0\}$ is UGAS.
- All solutions of (27) are t_0-uniformly bounded. This follows from (29) and from the fact that (27b) constitutes an exponentially stable linear time-invariant system with uniformly bounded input $s(t)$.
- The set Γ_2 is LUS-Γ_1. This follows from Proposition 25. Clearly, $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^2$. Also, Γ_2 is positively invariant since $s = 0$ is a solution of (27a). Finally, (12) and (13) hold in view of (29) with $\alpha(\|x\|_{\Gamma_2}) = (1/2)d_m s^2$, $\beta(\|x\|_{\Gamma_1}) = d_M(s^2 + \tilde q^2)$. The property is also illustrated in the zoomed plot in Figure 3: solutions that start in a neighbourhood of the origin [6], $B_\delta(\Gamma_1)$, remain in a neighbourhood of Γ_2, $B_\varepsilon(\Gamma_2)$, the gray band.
- The set Γ_2 is t_0-UGA. This follows from part (c) of Proposition 27, in view of the t_0-uniformly boundedness of all solutions, with V as in (28), $U = \mathbb{R}$, and $\Gamma = \Gamma_2$. The property is illustrated in Figure 3, where all solutions converge to Γ_2, the line on the plane $\{\dot{\tilde q} = -\tilde q\}$.
- By Theorem 18, we conclude that Γ_1 is UGAS.
- One can also use part (d) of Proposition 27 with V as in (28), $\Gamma = \Gamma_2$, $U = \mathbb{R}^2$, and $\alpha_1(\|x\|_{\Gamma_2}) = (1/2)d_m s^2$ to arrive at the conclusion that Γ_2 is t_0-UGAS, then use Proposition 23 to conclude that Γ_1 is UGAS.

[6] Strictly speaking, in Figure 3 Γ_1 is represented as the point $\{(\tilde q, \dot{\tilde q}) = (0,0)\}$ which is equivalent to $\{(\tilde q, s) = (0,0)\}$

Even though the Slotine-Li controller does not make trajectories $s(t)$ converge to zero in finite time, the reduction argument presented above captures the intuition behind the operation of the controller presented at the beginning of this discussion, namely the idea that the controller makes solutions approach the line $s = 0$, that on this line solutions converge exponentially to the origin, and that these two properties imply that solutions converge to the origin.

5.2 *Illustration of reduction theorem for t_0-uniform attractivity*

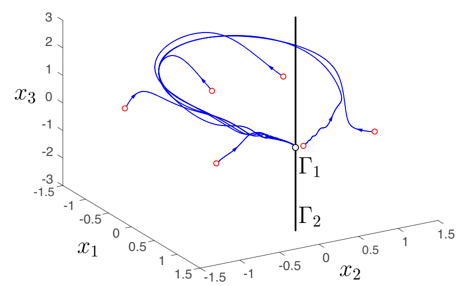

Fig. 4. A few solutions for the example in Section 5.2. The equilibrium Γ_1 is almost globally t_0-UA but unstable.

Consider the time-varying system

$$\dot x_1 = x_2(x_1 - 1) - x_1(x_1^2 + x_2^2 - 1) \tag{30a}$$

$$\dot x_2 = -x_1(x_1 - 1) - x_2(x_1^2 + x_2^2 - 1) \tag{30b}$$

$$\dot x_3 = -x_3^3 + (x_1 - 1)^2 + x_2^2 f(t), \tag{30c}$$

where $f(t)$ is a continuous bounded function. This system satisfies the Basic Assumption. Letting $\Gamma_2 = \{(x_1, x_2, x_3) : x_1 = 1, x_2 = 0\}$ and $\Gamma_1 = \{(x_1, x_2, x_3) = (1,0,0)\}$, we claim that Γ_1 is t_0-UA with basin of attraction given by the whole state space minus a set of measure zero (i.e., it is *almost globally* t_0-UA). On Γ_2 the dynamics are described by the differential equation $\dot x_3 = -x_3^3$, whose origin represents the set Γ_1, and therefore Γ_1 is UGAS relative to Γ_2. In Example 2 we showed that the equilibrium $(x_1, x_2) = (1, 0)$ of the subsystem

$$\dot x_1 = x_2(x_1 - 1) - x_1(x_1^2 + x_2^2 - 1) \tag{31a}$$

$$\dot x_2 = -x_1(x_1 - 1) - x_2(x_1^2 + x_2^2 - 1) \tag{31b}$$

is t_0-UA with basin of attraction given by $\mathbb{R}^2 \setminus \{(0,0)\}$. In particular, all its solutions are bounded, and in fact t_0-uniformly bounded because this system is time-invariant. Since the control system $\dot{x}_3 = -x_3^3 + u$ is input-to-state stable, all solutions of the subsystem

$$\dot{x}_3 = -x_3^3 + (x_1-1)^2 + x_2^2 f(t),$$

are also t_0-uniformly bounded because the pair $(x_1(t), x_2(t))$ and the function $f(t)$ are bounded. The considerations above show that all solutions of system (30) are t_0-uniformly bounded, i.e., $\mathbb{BS} = \mathbb{R}^3$. Letting $\mathcal{X}:=\mathbb{R}^3 \setminus \{(x_1,x_2,x_3) : x_1 = x_2 = 0\}$, $\mathcal{X}$ has full measure in $\mathbb{R}^3$, and is positively invariant because its complement, the set $\{(x_1,x_2,x_3) : x_1 = x_2 = 0\}$, is invariant. Since, for system (31), $\mathbb{B}(\{(1,0)\}) = \mathbb{R}^2 \setminus \{(0,0)\}$, we have that, for system (30), $\mathbb{B}(\Gamma_2) = \mathcal{X}$, i.e., Γ_2 is t_0-UGA relative to $\mathcal{X}$. To summarize, we have determined that (a) Γ_1 is UGAS relative to Γ_2, (b) Γ_2 is t_0-UGA relative to $\mathcal{X}$, and (c) $\mathbb{BS} = \mathcal{X}$. By Theorem 16, Γ_1 is t_0-UGA relative to $\mathcal{X}$ or, what is the same, Γ_1 is almost globally t_0-UA, as claimed. A few solutions of the system with $f(t) = \sin(t)^2$ and $t_0 = 0$ are depicted in Figure 4. Note that Γ_1 is unstable, and indeed Figure 4 shows an initial state very close to Γ_1 giving rise to an orbit with a large excursion away from Γ_1.

It is interesting to compare system (30) to system (6) in Example 2, both time-varying perturbations of the second-order dynamics in (31) for which the equilibrium $(x_1,x_2) = (1,0)$ is almost globally t_0-UA. While the perturbation in Example 2 destroys the t_0-UA property outside of the (x_1,x_2)-plane, the perturbation in (30) preserves it.

5.3 *Circular path following for a kinematic unicycle*

This example illustrates the use of reduction theorems in the context of hierarchies of control specifications that were mentioned in the introduction. Consider the kinematic unicycle

$$\dot{x}_1 = u_1 \cos(\theta) \tag{32a}$$
$$\dot{x}_2 = u_1 \sin(\theta) \tag{32b}$$
$$\dot{\theta} = u_2, \tag{32c}$$

where $x \in \mathbb{R}^2$ are the Cartesian coordinates of the unicycle in the plane, $\theta \in \mathbb{S}^1$ is the unicycle heading, and $(u_1,u_2) \in \mathbb{R}\times\mathbb{R}$, the linear and angular speeds of the unicycle, are the control inputs. We denote by $\chi:=(x,\theta)$ the state of the unicycle, and by $\mathcal{X}:=\mathbb{R}^2\times\mathbb{S}^1$ its state space. For a vector $x \in \mathbb{R}^2$, we denote by angle(x) the angle that the vector makes with the positive x_1 axis. Let $C_r:=\{x\in\mathbb{R}^2 : x^\top x = r^2\}$ denote the circle of radius $r>0$ centred at the origin, and consider the following list of control specifications:

(a) For each initial position $x(0) \in C_r$ and initial heading $\theta(0) = \text{angle}(x(0)) + \pi/2$ (i.e., heading tangent to C_r with counterclockwise orientation), $x(t)$ must remain on C_r for all $t \geq 0$ and follow C_r counterclockwise.

(b) For all other initial states, the unicycle position, $x(t)$, must asymptotically converge to C_r.

(c) For each initial state in some neighborhood of the reference signal
$$\chi_d(t) = \big(r\cos(\alpha_d(t)), r\sin(\alpha_d(t)), \alpha_d(t)+\pi/2\big),$$
where $\alpha_d : \mathbb{R}\to\mathbb{S}^1$ is a given C^1 function such that $\dot{\alpha}_d \geq 0$, the unicycle's state must asymptotically converge to $\chi_d(t)$.

In essence, for any initial state we want the unicycle to approach and follow the circle C_r counterclockwise, rendering the circle invariant for the position dynamics. Moreover, we want to ensure that, on C_r, the motion of the unicycle matches a prescribed reference signal. This latter specification is only required to be met locally.

A controller meeting specifications (a) and (b) was presented in [5]. Using Proposition 23, we now enhance the controller in [5] to meet also specification (c). Define the set

$$\begin{aligned}\Gamma &= \{\chi = (x_1,x_2,\theta)\in\mathcal{X} : x^\top x = r^2, \\ &\qquad\qquad \theta = \text{angle}(x) + \pi/2\} \\ &= \{\chi = (x_1,x_2,\theta)\in\mathcal{X} : x_1 = r\sin(\theta), \\ &\qquad\qquad x_2 = -r\cos(\theta)\}.\end{aligned}$$

The set Γ consists of the points in $\mathcal{X}$ corresponding to the unicycle's position being on C_r and its heading being tangent to C_r with counterclockwise orientation. It is clear that in order to meet specifications (a) and (b), we need to render Γ UGAS. The controller in [5, Proposition III.1] does just that. For any $v\in\mathbb{R}$, the smooth feedback

$$u_1 = v \tag{33a}$$
$$u_2 = \frac{u_1}{r} + r\left(x_1\cos(\theta) + x_2\sin(\theta)\right), \tag{33b}$$

renders Γ UGAS. One can replace $v\in\mathbb{R}$ by any smooth real-valued function without affecting the result. In order to meet specification (c), we assign v in the feedback (33a) so as to incorporate an additional stabilization mechanism. Define $\theta_d(t):=\alpha_d(t)+\pi/2$, and note that, having met specifications (a) and (b), specification (c) corresponds to making $\theta\to\theta_d(t)$ for suitable initial states. This control objective can be attained without affecting the UGAS property of Γ, as follows. On Γ, the feedback (33) reduces to $(u_1,u_2) = (v, v/r)$, and therefore the evolution of $\theta(t)$ is governed by $\dot{\theta} = v/r$. Letting $v = r\left[\dot{\theta}_d(t) - \sin(\theta-\theta_d(t))\right]$, i.e., letting

$$u_1 = r\left[\dot{\theta}_d - \sin(\theta-\theta_d(t))\right], \tag{34a}$$
$$u_2 = \frac{u_1}{r} + r\left[x_1\cos(\theta) + x_2\sin(\theta)\right], \tag{34b}$$

we obtain that $\theta(t)\to\theta_d(t)$ for almost all initial states on Γ. However, this does not yet imply that specification (c) is met, since initial states outside of Γ must also be considered, but Proposition 23 yields the desired result. In order to formulate a reduction problem, we define the error state $\tilde{\chi}\in\mathcal{X}$ as $\tilde{\chi}:=(\tilde{x},\tilde{\theta})$, with $\tilde{x}:=(x_1 - r\sin(\theta), x_2 + r\cos(\theta))$ and $\tilde{\theta} = \theta - \theta_d(t)$. The closed-loop system in error coordinates reads as

$$\begin{aligned}\dot{\tilde{x}}_1 = -r^2 \Big[&\tilde{x}_1\cos(\tilde{\theta}+\theta_d(t)) \\ &+ \tilde{x}_2 \sin(\tilde{\theta}+\theta_d(t))\cos(\tilde{\theta}+\theta_d(t))\Big]\end{aligned} \tag{35a}$$

$$\dot{\tilde{x}}_2 = -r^2 \Big[\tilde{x}_2\sin(\tilde{\theta}+\theta_d(t))$$

$$+\tilde{x}_1 \sin(\tilde{\theta}+\theta_d(t))\cos(\tilde{\theta}+\theta_d(t))\Big] \tag{35b}$$

$$\dot{\tilde{\theta}} = -\sin(\tilde{\theta}) + r\left[\tilde{x}_1 \cos(\tilde{\theta}+\theta_d(t)) + \tilde{x}_2 \sin(\tilde{\theta}+\theta_d(t))\right]. \tag{35c}$$

The system above satisfies the Basic Assumption (see Remark 1), and in $\tilde{\chi}$-coordinates the set Γ becomes

$$\Gamma_2 = \{\tilde{\chi} \in \mathcal{X} : \tilde{x} = 0\},$$

and meeting specification (c) corresponds to stabilizing the equilibrium $\Gamma_1 = \{0 \in \mathcal{X}\}$. Clearly, Γ_1 is compact, Γ_2 is closed, and $\Gamma_1 \subset \Gamma_2$. By [5, Proposition III.1], the feedback (34) renders Γ_2 UGAS, and to meet specification (c), we need to show that Γ_1 is UAS. On Γ_2, the dynamics are described by the differential equation $\dot{\tilde{\theta}} = -\sin(\tilde{\theta})$. The equilibrium $\tilde{\theta} = 0$ is asymptotically stable for the above differential equation, which means that Γ_1 is UAS relative to Γ_2. By Proposition 23, Γ_1 is UAS. In conclusion, the feedback (34) simultaneously renders Γ_2 UGAS and Γ_1 UAS, thereby meeting specifications (a)-(c). Figure 5 shows simulation results for

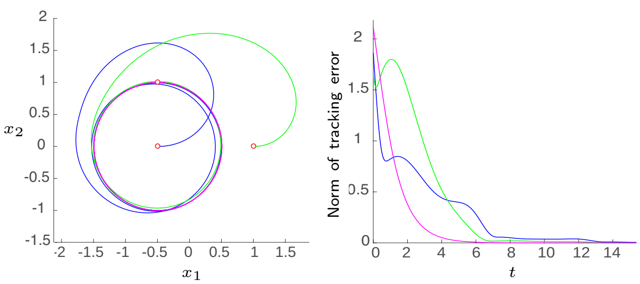

Fig. 5. Unicycle path following with simultaneous trajectory tracking. On the left-hand side, behaviour of the unicycle on the plane for three different initial conditions, including one of the circle. On the right-hand side, the corresponding norm of the tracking error for each solution.

$r = 1$ and $\alpha_d(t) = t + \sin(t)$. As expected, all solutions converge to the circle, and move counterclockwise around it. The corresponding tracking errors (specification (c)) converge to zero. One of the displayed solutions (the one in magenta) corresponds to the unicycle being initialized on the circle, with heading tangent to it. In accordance with specification (a), the unicycle remains on the circle even though its initial tracking error is not zero. In accordance with specification (c), the unicycle adjusts its linear speed to synchronize with the reference signal without leaving the circle.

Finally, we remark that the closed-loop system (35) does not have a cascade-connected structure, because the $\tilde{x}$ dynamics depends on $\tilde{\theta}$. Therefore, in this example one cannot use the cascade systems theory of [24,25] nor Corollary 24 on p. 8 paper.

6 Proofs of Theorems 16 and 18

The proof of Theorem 16 relies on the following two lemmas whose proofs are omitted. The proof of the first lemma is standard, while the proof of the second one follows the same lines as for [13, Lemma 1].

Lemma 30. *Assume the differential equation (3) satisfies the Basic Assumption. Then for each compact set $K \subset \mathbb{R}^n$, each $\varepsilon > 0$, and each $T > 0$, there exists $\delta > 0$ such that for any initial data $(t_0, x_0) \in \mathbb{R} \times K$ such that $x([t_0, t_0 + T], t_0, x_0) \subset K$, the property $\|x(t, t_0, x_0) - x(t, t_0, x_1)\| < \varepsilon$ holds for all $x_1 \in B_\delta(x_0)$ and for all $t \in [t_0, t_0 + T]$.*

Lemma 31. *Consider the time-varying system (3) under the Basic Assumption. Let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. If Γ_1 is UA relative to Γ_2, then the threshold property ([31]) holds:*

$$\begin{aligned}&(\forall \varepsilon > 0)(\exists \delta, \eta > 0)(\forall x_0 \in B_\delta(\Gamma_1))(\forall t_0 \in \mathbb{R})(\forall t \geq t_0)\\ &x([t_0, t], t_0, x_0) \subset B_\eta(\Gamma_2) \implies x([t_0, t], t_0, x_0) \subset B_\varepsilon(\Gamma_1).\end{aligned} \tag{36}$$

6.1 Proof of Theorem 16

Suppose assumptions (i)-(iii) in Theorem 16 hold and let $x_0 \in B_\delta(\Gamma_1)$ be arbitrarily fixed. We need to show that $x_0 \in \mathbb{B}(\Gamma_1)$, or

$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0+T}, t_0, x_0) \subset B_\varepsilon(\Gamma_1). \tag{37}$$

By assumption (ii), the basin of attraction $\mathbb{B}(\Gamma_2)$ contains a neighbourhood of Γ_1. Therefore, without loss of generality we may assume that δ in assumption (iii) is small enough so that

$$B_\delta(\Gamma_1) \subset \mathbb{B}(\Gamma_2). \tag{38}$$

By the definition of K_δ in the theorem statement, we have that for each $t_0 \in \mathbb{R}$, $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset K_\delta$. Since, by assumption, K_δ is compact, for each $t_0 \in \mathbb{R}$, $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$.

Let $\varepsilon > 0$ be arbitrary, and pick $\varepsilon' \in (0, \varepsilon)$. By assumption (i), Γ_1 is UA relative to Γ_2, so by Lemma 31 the threshold property (36) holds, i.e.,

$$\begin{aligned}&(\exists \delta', \eta_1 > 0)(\forall x_0 \in B_{\delta'}(\Gamma_1))(\forall t_0 \in \mathbb{R})(\forall t \geq t_0)\\ &\text{if } x([t_0, t], t_0, x_0) \subset B_{\eta_1}(\Gamma_2) \text{ then } x([t_0, t], t_0, x_0) \subset B_{\varepsilon'}(\Gamma_1).\end{aligned} \tag{39}$$

By assumption (i), Γ_1 is UAS relative to Γ_2, and by assumption (iii), $K_\delta \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. Since K_δ is compact, by Corollary 33 in Appendix A the set $K_\delta \cap \Gamma_2$ enjoys the uniform attraction property (A.5) in the Appendix, and thus

$$(\exists T_2 > 0)(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0+T_2}, t_0, K_\delta \cap \Gamma_2) \subset B_{\delta'/2}(\Gamma_1). \tag{40}$$

By the Basic Assumption, and using Lemma 30 with K replaced by K_δ and $T_2 > 0$ given as in (40), we have:

$$\begin{aligned}&(\exists \eta_2 > 0)(\forall t_0 \in \mathbb{R})(\forall z_0 \in B_{\eta_2}(x_0))(\forall t \in [t_0, t_0 + T_2])\\ &\qquad \|x(t, t_0, x_0) - x(t, t_0, z_0)\| < \delta'/2.\end{aligned} \tag{41}$$

Let $\eta := \min\{\eta_1, \eta_2\}$. From (38) we get

$$(\exists T_1 > 0)(\forall t_0 \in \mathbb{R})\ x(\mathbb{R}_{\geq t_0+T_1}, t_0, x_0) \subset B_\eta(\Gamma_2). \tag{42}$$

Let $t_0 \in \mathbb{R}$ be arbitrary. By (42), and since $x(t_0 + T_1, t_0, x_0) \in K_\delta$, there exists $z_0 \in K_\delta \cap \Gamma_2$ such that $\|x(t_0 + T_1, t_0, x_0) - z_0\| < \eta$. By (41),

$$\begin{aligned}&\|x(t_0 + T_1 + T_2, t_0 + T_1, x(t_0 + T_1, t_0, x_0))\\ &\qquad - x(t_0 + T_1 + T_2, t_0 + T_1, z_0)\| < \delta'/2\end{aligned} \tag{43}$$

and, since $z_0 \in K_\delta \cap \Gamma_2$, by (40) it follows that
$$x(t_0 + T_1 + T_2, t_0 + T_1, z_0) \in B_{\delta'/2}(\Gamma_1). \tag{44}$$
Next, combining (43) and (44) we get
$$x(t_0 + T_1 + T_2, t_0, x_0) = x(t_0 + T_1 + T_2, t_0 + T_1, x(t_0 + T_1, t_0, x_0)) \in B_{\delta'}(\Gamma_1) \tag{45}$$
and from (42) we have
$$x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0, x_0) = x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0 + T_1 + T_2, x(t_0 + T_1 + T_2, t_0, x_0)) \subset B_\eta(\Gamma_2). \tag{46}$$
By the threshold property in (39), (45) and (46) imply that
$$x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0, x_0) = x(\mathbb{R}_{\geq t_0+T_1+T_2}, t_0 + T_1 + T_2, x(t_0 + T_1 + T_2, t_0, x_0)) \subset \overline{B_{\varepsilon'}(\Gamma_1)} \subset B_\varepsilon(\Gamma_1). \tag{47}$$
Setting $T{:=}T_1 + T_2$, (47) implies that property (37) holds. Hence, $B_\delta(\Gamma_1) \subset \mathbb{B}(\Gamma_1)$ so we conclude that Γ_1 is t_0-UA.

Now suppose that conditions (i)'-(ii)' hold and let $x_0 \in \mathbb{BS}$ be arbitrary, so that the set $K{:=}\overline{\bigcup_{t_0\in\mathbb{R}} x(\mathbb{R}_{\geq t_0}, t_0, x_0)}$ is compact. By (i)', $\Gamma_2 \subset \mathbb{B}(\Gamma_1)$, and therefore $K \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. Now repeating the proof above with K_δ replaced by K we reach the conclusion that (37) holds, thereby implying that $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$. This concludes the proof of Theorem 16. □

6.2 *Proof of Theorem 18*

($\Longrightarrow$) Suppose that Γ_1 is UAS. Since $\Gamma_1 \subset \Gamma_2$, Γ_1 is UAS relative to Γ_2, hence condition (i) holds. Since Γ_1 is UA, there exists $r > 0$ such that $B_r(\Gamma_1) \subset \mathbb{B}(\Gamma_1)$. Since $\Gamma_1 \subset \Gamma_2$, $\mathbb{B}(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$, and thus $B_r(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$, implying that condition (iii) holds. Since Γ_1 is US, we have
$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R})\, x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1).$$
Hence condition (5) in the definition of LUS-Γ_1 holds for arbitrary $r > 0$, so condition (ii) holds.

Next, suppose Γ_1 is UGAS. Then it is UGAS relative to Γ_2, so condition (i)' holds. Since $\mathbb{B}(\Gamma_1) = \mathbb{R}^n$ and since $\mathbb{B}(\Gamma_1) \subset \mathbb{B}(\Gamma_2)$, Γ_2 is t_0-UGA and hence condition (iii)' holds. As for condition (iv), let $x_0 \in \mathbb{R}^n$ be arbitrary and define $\delta{:=}2\|x_0\|_{\Gamma_1}$, so $x_0 \in B_\delta(\Gamma_1)$. Since Γ_1 is UGS, there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1)$ for all $t_0 \in \mathbb{R}$. Since Γ_1 is compact, there exists $c > 0$ such that $B_\varepsilon(\Gamma_1) \subset B_c(0)$. Thus for each $t_0 \in \mathbb{R}$, $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_c(0)$, implying that $x_0 \in \mathbb{BS}$. Since x_0 is arbitrary, $\mathbb{BS} = \mathbb{R}^n$ and condition (iv) holds.

($\Longleftarrow$) Suppose conditions (i)-(iii) hold. By Theorem 14, conditions (i) and (ii) imply that Γ_1 is US. To prove that Γ_1 is UAS, in view of item (iv) of Proposition 9 it suffices to show that Γ_1 is t_0-UA. To this end, we invoke Theorem 16. Conditions (i) and (ii) of Theorem 16 correspond to conditions (i) and (iii) of Theorem 18, which hold by assumption. It is only left to show that there exists $\delta > 0$ such that the set
$$K_\delta{:=}\overline{\bigcup_{t_0\in\mathbb{R}} x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1))}$$
is compact and $K_\delta \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. By assumption (i), there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(\Gamma_1)} \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1)$. Since Γ_1 is US, there exists $\delta > 0$ such that
$$(\forall t_0 \in \mathbb{R})\; x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma_1)) \subset B_\varepsilon(\Gamma_1).$$
The above implies that for the value of δ just discussed, $K_\delta \subset \overline{B_\varepsilon(\Gamma_1)}$, and therefore
$$K_\delta \cap \Gamma_2 \subset \overline{B_\varepsilon(\Gamma_1)} \cap \Gamma_2 \subset \mathbb{B}(\Gamma_1).$$
Moreover, since Γ_1 is compact and $K_\delta \subset \overline{B_\varepsilon(\Gamma_1)}$, K_δ is compact too. Thus assumption (iii) of Theorem 16 holds, and Γ_1 is t_0-UA. By part (iv) of Proposition 9, Γ_1 is UAS.

Now suppose that assumptions (i)', (ii), (iii)', and (iv) hold. By Theorem 14, Γ_1 is US, and by Theorem 16 it is t_0-UGA. Part (v) of Proposition 9 implies that Γ_1 is UGAS.

Finally, suppose that assumptions (i)', (ii), and (iii)' hold. By the first part of Theorem 18, Γ_1 is UAS. By Theorem 16, assumptions (i)' and (iii)' imply that all initial states giving rise to t_0-uniformly bounded solutions are contained in the basin of t_0-uniform attraction of Γ, i.e., $\mathbb{BS} \subset \mathbb{B}(\Gamma_1)$. This concludes the proof of the theorem. □

7 Conclusion

In this paper we presented reduction theorems for uniform stability, t_0-uniform attractivity, and uniform asymptotic stability of compact sets, as well as a number of consequences. We also presented Lyapunov characterizations of the properties of local uniform stability near a set and t_0-uniform attractivity. Further research on Lyapunov characterizations might provide useful extensions and new stability results. In an example we illustrated how in certain simple cases, reduction theorems can be used to assess the property of almost global t_0-uniform attractivity. The development of general reduction theorems for almost global uniform asymptotic stability remains an open problem.

A Proof of Proposition 9

Part (i). ($\Longrightarrow$) If Γ is US then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$. Letting $U = B_\delta(\Gamma)$, the above property implies that Γ is t_0-US.

($\Longleftarrow$) If Γ is t_0-US then for each $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $\Gamma \subset U$, and for each $x_0 \in U$, for each $t_0 \in \mathbb{R}$, and each $t \in T^+_{t_0,x_0}$, it holds that $x(t, t_0, x_0) \subset B_\varepsilon(\Gamma)$. Since Γ is compact, $B_\varepsilon(\Gamma)$ is bounded, and hence $T^+_{t_0,x_0} = \mathbb{R}_{\geq t_0}$ for all $(t_0, x_0) \in \mathbb{R} \times U$. By the compactness of Γ and the fact that $\Gamma \subset U$ with U open, there exists $\delta > 0$ such that $B_\delta(\Gamma) \subset U$. It then follows that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$, proving that Γ is US.

Part (ii). ($\Longrightarrow$) By definition, if Γ is UAS then it is UA. ($\Longleftarrow$) Assume Γ is UA. We will show that Γ is US. Let $\varepsilon > 0$ be arbitrary. By definition of UA, we have
$$(\exists r > 0)(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\; x(\mathbb{R}_{\geq t_0+T}, t_0, B_r(\Gamma)) \subset B_\varepsilon(\Gamma). \tag{A.1}$$

Let $\varepsilon > 0$ be arbitrary, and let $T > 0$ be such that (A.1) holds. By the positive invariance of Γ, for all $x_0 \in \Gamma$ we have that $x([t_0, t_0 + T], t_0, x_0) \subset \Gamma$. By the Basic Assumption and Lemma 30, and since Γ is compact, there exists $\delta_1 > 0$ such that $\|x(t, t_0, x_0) - x(t, t_0, x_1)\| < \varepsilon$ for all $x_0 \in \Gamma$, all $x_1 \in B_{\delta_1}(x_0)$, all $t_0 \in \mathbb{R}$, and all $t \in [t_0, t_0 + T]$. Therefore, $x([t_0, t_0 + T], t_0, B_{\delta_1}(x_0)) \subset B_\varepsilon(\Gamma)$ for all $x_0 \in \Gamma$ and all $t_0 \in \mathbb{R}$. Since Γ is compact, there exists $\delta_2 > 0$ such that $B_{\delta_2}(\Gamma) \subset \bigcup_{x_0 \in \Gamma} B_{\delta_1}(x_0)$, using which we obtain that

$$x([t_0, t_0 + T], t_0, B_{\delta_2}(\Gamma)) \subset B_\varepsilon(\Gamma), \tag{A.2}$$

for all $t_0 \in \mathbb{R}$. Picking $\delta = \min\{r, \delta_2\}$, (A.1) and (A.2) imply that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$, so that Γ is US. This proves that UA implies UAS.

Part (iii). ($\Longrightarrow$) If Γ is UGAS then by definition it is UGA and UGS. This latter property implies that all solutions are t_0-uniformly bounded, so that $\mathbb{BS} = \mathbb{R}^n$.

($\Longleftarrow$) Now suppose that Γ is UGA and $\mathbb{BS} = \mathbb{R}^n$. We need to show that Γ is UGS. We have already shown in part (ii) that Γ is US, so we need to show that for each $\delta > 0$ there exists $\varepsilon > 0$ such that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$. Let $\delta > 0$ be arbitrary, and pick $\varepsilon_1 > 0$. Since Γ is UGA, there exists $T > 0$ such that

$$x(\mathbb{R}_{\geq t_0+T}, t_0, B_\delta(\Gamma)) \subset B_{\varepsilon_1}(\Gamma) \tag{A.3}$$

for all $t_0 \in \mathbb{R}$. Since $\mathbb{BS} = \mathbb{R}^n$, for each $x_0 \in \overline{B_\delta(\Gamma)}$ there exists a constant $c(x_0)$ such that $x(\mathbb{R}_{\geq t_0}, t_0, x_0) \subset B_{c(x_0)}(0)$ for all $t_0 \in \mathbb{R}$. By continuous dependence on initial data, there exists a constant $\mu(x_0) > 0$ such that $x([t_0, t_0 + T], t_0, B_{\mu(x_0)}(x_0)) \subset B_{2c(x_0)}(0)$ for all $t_0 \in \mathbb{R}$. The collection of open balls $\{B_{\mu(x_0)}(x_0) : x_0 \in \overline{B_\delta(\Gamma)}\}$ is an open cover of $\overline{B_\delta(\Gamma)}$, and since this latter set is compact, it has a finite subcover, so that there exists a finite collection of points $x_i \in \overline{B_\delta(\Gamma)}$, $i \in \mathbf{k}$, such that $B_\delta(\Gamma) \subset \bigcup_{i \in \mathbf{k}} B_{\mu(x_i)}(x_i)$. Let $M = \max_{i \in \mathbf{k}} 2c(x_i)$. Then for each $t_0 \in \mathbb{R}$, $x([t_0, t_0+T], t_0, B_\delta(\Gamma)) \subset B_M(0)$. Letting $\varepsilon_2 > 0$ be such that $B_M(0) \subset B_{\varepsilon_2}(\Gamma)$, we get

$$x([t_0, t_0 + T], t_0, B_\delta(\Gamma)) \subset B_{\varepsilon_2}(\Gamma) \tag{A.4}$$

for all $t_0 \in \mathbb{R}$. Setting $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, by (A.3) and (A.4) we conclude that $x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$ for all $t_0 \in \mathbb{R}$. Thus Γ is UGS, and therefore also UGAS.

Part (iv). ($\Longrightarrow$) If Γ is UAS then by definition it is UA and US. The former property implies that Γ is t_0-UA and, by part (i), the latter property implies that Γ is t_0-US. Being t_0-US and t_0-UA, Γ is t_0-UAS.

($\Longleftarrow$) Assume Γ is t_0-UAS. Since Γ is t_0-UA, $\Gamma \subset \operatorname{int}(\mathbb{B}(\Gamma))$, where $\mathbb{B}(\Gamma)$ is the basin of t_0-uniform attraction of Γ. This fact and the assumption that Γ is compact imply that there exists $r > 0$ such that $B_r(\Gamma) \subset \operatorname{int}(\mathbb{B}(\Gamma))$. The set $K = \overline{B_r(\Gamma)} \subset \mathbb{B}(\Gamma)$ is compact, and by Lemma 32 below it enjoys the uniform attraction property (A.5):

$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\, x(\mathbb{R}_{\geq t_0+T}, t_0, \overline{B_r(\Gamma)}) \subset B_\varepsilon(\Gamma).$$

The above property implies that Γ is UA.

Part(v). ($\Longrightarrow$) If Γ is UGAS, then by definition it is UGA and UGS. By part (i), we deduce that Γ is t_0-US. Moreover, the UGS property implies that all solutions are t_0-uniformly bounded, so that $\mathbb{BS} = \mathbb{R}^n$. The UGA property implies that Γ is t_0-UGA. In conclusion, Γ is t_0-UGAS and $\mathbb{BS} = \mathbb{R}^n$.

($\Longleftarrow$) Assume Γ is t_0-US and t_0-UGA, and $\mathbb{BS} = \mathbb{R}^n$. Since Γ is compact and t_0-UGA, we have $\mathbb{B}(\Gamma) = \mathbb{R}^n$, so we may repeat the argument in the proof of part (iv) with arbitrary $r > 0$ to conclude that Γ is UGA. Then, in light of part (iii), Γ is UGAS. This concludes the proof of Proposition 9. $\square$

Lemma 32. *Consider the differential equation* (3)*, in which the vector field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Basic Assumption. Let $\Gamma \subset \mathbb{R}^n$ be a compact positively invariant set that is t_0-*UAS*. Then, for each compact set $K \subset \mathbb{B}(\Gamma)$ the following* uniform attraction property *holds:*

$$(\forall \varepsilon > 0)(\exists T > 0)(\forall t_0 \in \mathbb{R})\, x(\mathbb{R}_{\geq t_0+T}, t_0, K) \subset B_\varepsilon(\Gamma). \tag{A.5}$$

Proof. Let $\varepsilon > 0$ be arbitrarily fixed. By part (i) of Proposition 9, since Γ is t_0-US it is also US, and we have

$$(\exists \delta > 0)(\forall t_0 \in \mathbb{R})\, x(\mathbb{R}_{\geq t_0}, t_0, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma). \tag{A.6}$$

By t_0-uniform attractivity of Γ and the fact that $K \subset \mathbb{B}(\Gamma)$, we have

$$\begin{aligned} &(\forall x_0 \in K)(\exists T > 0)(\forall t_0 \in \mathbb{R}) \\ &\quad t_0 + T \in T^+_{t_0,x_0} \text{ and } x(t_0 + T, t_0, x_0) \subset B_{\delta/2}(\Gamma). \end{aligned} \tag{A.7}$$

Let $x_0 \in K$ be arbitrary, and let $T > 0$ be as in (A.7). Using Lemma 30 with K in the lemma given by the set $\{x([t_0, t_0 + T], t_0, x_0)\}$, we have that

$$\begin{aligned} &(\exists \delta' > 0)(\forall z_0 \in B_{\delta'}(x_0))(\forall t \in [t_0, t_0 + T]) \\ &\quad \|x(t, t_0, x_0) - x(t, t_0, z_0)\| < \delta/2. \end{aligned} \tag{A.8}$$

By (A.7) and (A.8) we have that $x(t_0 + T, t_0, B_{\delta'}(x_0)) \subset B_\delta(\Gamma)$, and by (A.6) we conclude that

$$x(\mathbb{R}_{\geq t_0+T}, t_0, B_{\delta'}(x_0)) \subset B_\varepsilon(\Gamma). \tag{A.9}$$

By property (A.9) and the fact that the set K is compact, there exists a finite cover of K by balls $B_{\delta_i}(x_i)$, $i \in \mathbf{k}$, where $x_i \in K$, and associated times $T_i > 0$, $i \in \mathbf{k}$, such that

$$(\forall t_0 \in \mathbb{R})\, x(\mathbb{R}_{\geq t_0+T_i}, t_0, B_{\delta_i}(x_i)) \subset B_\varepsilon(\Gamma). \tag{A.10}$$

Letting $T := \max\{T_1, \ldots, T_n\}$, we conclude that

$$(\forall t_0 \in \mathbb{R})\, x(\mathbb{R}_{\geq t_0+T}, t_0, K) \subset B_\varepsilon(\Gamma),$$

proving that (A.5) holds. $\square$

Corollary 33. *In the setup of Lemma 32, let Γ_1 be a compact set and Γ_2 be a closed set, both positively invariant and such that $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. If Γ_1 is* UAS *relative to Γ_2, then for each compact set $K \subset \mathbb{B}(\Gamma_1) \cap \Gamma_2$ the uniform attraction property* (A.5) *holds.*

The proof of this corollary follows by repeating the argument of the proof of Lemma 32, replacing $B_\delta(\Gamma)$ in (A.6) by $B_\delta(\Gamma) \cap \Gamma_2$, and making analogous changes in (A.9) and (A.10).

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