Circular Path Following for the Spherical Pendulum on a Cart

Luca Greco * **Paolo Mason** * **Manfredi Maggiore** **

* *Laboratoire des Signaux et Systèmes (L2S, UMR 8506), CNRS - CentraleSupélec - Univ. Paris-Sud - Université Paris Saclay, 3, rue Joliot Curie, 91192, Gif-sur-Yvette, France,* luca.greco@l2s.centralesupelec.fr, paolo.mason@l2s.centralesupelec.fr
** *Department of Electrical and Computer Engineering, University of Toronto, 10 King's College rd., Toronto, ON, M5S3G4, Canada,* maggiore@ece.utoronto.ca

Abstract: This paper investigates the problem of making the cart of a spherical pendulum follow a circular path with constant speed, while guaranteeing that the pendulum does not fall over. A solution methodology is presented which is hierarchical in nature. First, one of the two control inputs is used to stabilize the angle of the pendulum from the vertical axis at a constant value. The remaining control input is used to make the cart converge to the circular path with a desired speed.

1. INTRODUCTION

The spherical pendulum is a benchmark mechanical system with two degrees-of-freedom and two controls. It may be viewed as a generalization of the simple pendulum, and as a simplified model of human stance. The typical control specifications investigated in the literature are the stabilization of the unstable equilibrium and the swing-up. We refer the reader to Shen et al. (2004); Chaturvedi and McClamroch (2007); Shiriaev et al. (2004) for more details.

The spherical pendulum *on a cart* has received much less attention in the literature. Unlike the standard spherical pendulum, the spherical pendulum on a cart is an underactuated mechanical system with four degrees-of-freedom (the position of the cart on the plane and the angles of the pendulum) and two controls (the planar force applied to the cart). Being underactuated, the spherical pendulum on a cart constitutes a challenging benchmark system. In Bloch et al. (1998), the authors use the method of controlled Lagrangians to stabilize the inverted equilibrium. In Jankuloski et al. (2012) the authors find a controlled invariant manifold on which the pendulum remains in the upper half-plane and the cart appears to move along bounded orbits.

In this paper, we address a motion control problem which involves making the cart follow a circular path on the horizontal plane, while guaranteeing that the pendulum maintains a constant angle with respect to the vertical axis. We refer to this as the *circular path following problem.*

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To the best of our knowledge, this problem as not been investigated before, but it bears some similarity to the problem of making Getz's bicycle model follow a circular path with constant roll angle, a problem which was solved in Consolini and Maggiore (2013). Indeed, Getz's bicycle model is mechanically equivalent to a planar pendulum on a cart, with the plane of the pendulum orthogonal to the direction of movement. The problem investigated in this paper is harder because the pendulum is not constrained to lie on a specific plane.

Our strategy to solve the circular path following problem is to use one of the control inputs to stabilize the angle of the pendulum from the vertical axis at a desired constant value. We use the other control input to make the cart converge to a circular path, while making the projection of the pendulum on the horizontal plane point towards the origin of the circle.

Notation. Throughout this paper we denote by $\mathbb{S}^1$ the set of real numbers modulo 2π, diffeomorphic to the unit circle. Whenever convenient, we identify n-tuples $(a_1, \ldots, a_n)$ with columns vectors $[a_1 \ \cdots \ a_n]^\top$. If $(\mathcal{X}, d)$ is a complete metric space and $\Gamma \subset \mathcal{X}$ is a closed set, the point-to-set distance of a point $x \in \mathcal{X}$ to the set Γ, denoted by $\|x\|_\Gamma$, is defined as $\|x\|_\Gamma := \inf_{y\in\Gamma} d(x, y)$. If $\delta > 0$, we denote $B_\delta(\Gamma) := \{x \in \mathcal{X} : \|x\|_\Gamma < \delta\}$. When convenient, we use the shorthand notation c_α and s_α for $\cos(\alpha)$ and $\sin(\alpha)$, respectively.

2. PROBLEM FORMULATION

Consider the spherical pendulum depicted in Figure 1. This is a mechanical system comprised of a cart of mass M moving on the horizontal plane, and a pendulum of length l and mass m concentrated at the tip. The pivot point of the pendulum on the cart is a spherical joint. We assume that the cart is fully actuated by two control forces, while the pendulum has no control.

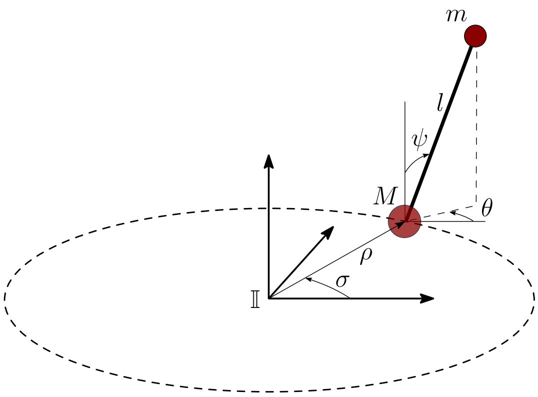

Fig. 1. Schematic representation of a spherical pendulum on a planar cart.

Referring to Figure 1, we denote by (ρ, σ) the polar coordinates of the cart with respect to an inertial frame $\mathbb{I}$, and by (θ, ψ) the spherical coordinates of the pendulum tip with respect to a parallel translation of frame $\mathbb{I}$ with origin at the pivot point. The configuration vector of the spherical pendulum on a cart is thus $q = (\sigma, \rho, \theta, \psi)$, and its configuration space is the set $\mathcal{Q} := \mathbb{S}^1 \times \mathbb{R}_{>0} \times \mathbb{S}^1 \times (0, \pi)$. The state is $(q, \dot{q})$, and the state space is the tangent bundle of $\mathcal{Q}$, namely the set $T\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^4$.

As we mentioned earlier, the control input is the force vector applied to the cart on the horizontal plane. For convenience, and without loss of generality, we assume that a feedback transformation has been applied so that the control inputs are the torque f_σ acting on the angle σ and the force f_ρ in the radial direction, as illustrated in Figure 2. We let $f = (f_\sigma, f_\rho)$.

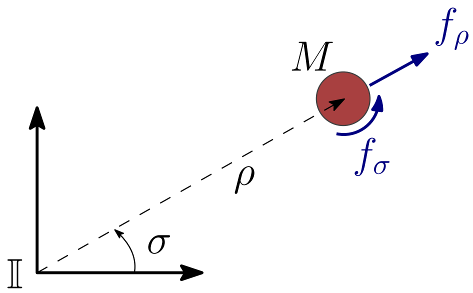

Fig. 2. The control inputs of the spherical pendulum on a cart.

Before stating the problem investigated in this paper, we need some notions of set stability. In what follows, we consider a dynamical system Σ generating a local phase flow ϕ on a complete metric space $(\mathcal{X}, d)$. For each $(t, x_0) \in \mathbb{R} \times \mathcal{X}$ in the domain of ϕ, $\phi(t, x_0)$ denotes the solution of Σ at time t with initial condition x_0.

Definition 1. (Asymptotic stability of sets). A compact set $\Gamma \subset \mathcal{X}$ is **stable** for Σ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma)$, $\phi(t, x_0) \in B_\varepsilon(\Gamma)$ for all $t \geq 0$.

The set Γ is **attractive** for Σ if there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma)$, $\|\phi(t, x_0)\|_\Gamma \to 0$ as $t \to \infty$.

Finally, Γ is **asymptotically stable** for Σ if it is stable and attractive.

Definition 2. (Stability near a set). Let $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ with Γ_2 closed, and Γ_1 compact. The set Γ_2 is **stable near** Γ_1 for Σ if there exists $c > 0$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma_2) \cap B_c(\Gamma_1)$, $\phi(t, x_0) \in B_\varepsilon(\Gamma_2)$ for all $t \geq 0$. The set Γ_2 is **asymptotically stable near** Γ_1 if it is stable near Γ_1 and there exists $c > 0$ such that for all $x_0 \in B_c(\Gamma_1)$, $\|\phi(t, x_0)\|_{\Gamma_2} \to 0$ as $t \to \infty$.

Roughly speaking, Γ_2 is (asymptotically) stable near Γ_1 if it is (asymptotically) stable with the restriction that initial conditions be sufficiently close to Γ_1.

We are ready to state the problem investigated in this paper.

Circular Path Following Problem. Given $\bar{\psi} \in (0, \frac{\pi}{2})$ and $\bar{\rho} > l \sin \bar{\psi}$, find a smooth feedback $f(q, \dot{q})$ meeting the following control specifications:

(i) There exist a constant $\bar{\omega} > 0$ and a smooth function $\mu : \mathbb{S}^1 \to \mathbb{S}^1$ such that the compact set

$$\begin{aligned}\Gamma_1 = \{(q, \dot{q}) \in T\mathcal{Q} :& (\rho, \psi, \dot{\rho}, \dot{\sigma}, \dot{\psi}) = (\bar{\rho}, \bar{\psi}, 0, \bar{\omega}, 0), \\ & (\theta, \dot{\theta}) = (\mu(\sigma), \mu'(\sigma)\dot{\sigma})\},\end{aligned} \tag{1}$$

is asymptotically stable for the closed-loop system.

(ii) The set

$$\Gamma_2 = \{(q, \dot{q}) \in T\mathcal{Q} : \psi = \bar{\psi}, \dot{\psi} = 0\} \tag{2}$$

is stable near Γ_1 for the closed-loop system.

Remark 3. Specification (i) roughly translates to requiring the cart to follow a circular path of radius $\bar{\rho}$ with speed $\bar{\omega} > 0$, while keeping the pendulum at a constant angle $\bar{\psi}$ from the vertical axis. Specification (ii), on the other hand, requires that if the pendulum is initially close to the configuration $(\psi, \dot{\psi}) = (\bar{\psi}, 0)$, and if the initial condition of the system is not too far from Γ_1, then during transient the pendulum remains close to the above configuration. Roughly speaking, this guarantees that if during transient a trajectory of the system "overshoots" the set Γ_1, then it does so without making the pendulum sway too far from the desired inclination $\psi = \bar{\psi}$.

3. MODELING

In this section we derive the Lagrangian of the spherical pendulum on a cart depicted in Figure 1. Letting $x \in \mathbb{R}^2$ denote the position of the cart on the horizontal plane measured in the coordinates of frame $\mathbb{I}$, we have

$$x = \rho \begin{bmatrix} \cos \sigma \\ \sin \sigma \end{bmatrix}.$$

Letting ζ denote the unit vector in $\mathbb{R}^3$ representing the direction of the pendulum rod, we have

$$\zeta = \begin{bmatrix} \sin \psi \cos \theta \\ \sin \psi \sin \theta \\ \cos \psi \end{bmatrix}.$$

Denoting

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

the tip of the pendulum has coordinates in frame $\mathbb{I}$ given by $P^\top x + l\zeta$. The kinetic energy of the system is

$$
\begin{aligned}
K(q,\dot q) &= \frac{1}{2}M\dot x^\top \dot x + \frac{1}{2}m\left(P^\top \dot x + l\dot\zeta\right)^\top\left(P^\top \dot x + l\dot\zeta\right)\\
&= \frac{1}{2}\dot q^\top D(q)\dot q,
\end{aligned}
$$

with

$$
D = \begin{bmatrix}
(M+m)\rho^2 & 0 & lm\rho \mathrm{c}_{\sigma-\theta}\mathrm{s}_\psi & -lm\rho \mathrm{s}_{\sigma-\theta}\mathrm{c}_\psi\\
0 & M+m & lm\mathrm{s}_{\sigma-\theta}\mathrm{s}_\psi & lm\mathrm{c}_{\sigma-\theta}\mathrm{c}_\psi\\
lm\rho \mathrm{c}_{\sigma-\theta}\mathrm{s}_\psi & lm\mathrm{s}_{\sigma-\theta}\mathrm{s}_\psi & l^2m\mathrm{s}_\psi^2 & 0\\
-lm\rho \mathrm{s}_{\sigma-\theta}\mathrm{c}_\psi & lm\mathrm{c}_{\sigma-\theta}\mathrm{c}_\psi & 0 & l^2 m
\end{bmatrix}.
$$

The potential energy is $V(q) = mgl\cos\psi$. We now define the Lagrangian $L : TQ \to \mathbb{R}$ as $L(q,\dot q) = K(q,\dot q) - V(q)$. Using the definition of the control vector $f = (f_\sigma, f_\rho)$ in Figure 2, the Euler-Lagrange equation gives

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = Bf,
$$

with

$$
B := \begin{bmatrix} I_2 \\ 0_{2\times 2}\end{bmatrix}.
$$

The equations of motion have the standard form

$$
D(q)\ddot q + C(q,\dot q)\dot q + \nabla V(q) = Bf, \tag{3}
$$

where C is a matrix formed using the Christoffel coefficients associated with the mass matrix $D(q)$ (see Spong et al. (2006)).

4. SOLUTION STRATEGY

Our solution to the circular path following problem unfolds in three steps.

(1) Find $\mu : \mathbb{S}^1 \to \mathbb{S}^1$ and $\bar\omega > 0$ such that the set Γ_1 in (1) is controlled invariant for (3) (i.e., there exists a smooth feedback f making Γ_1 invariant).
(2) Use the radial control input f_ρ to make the set Γ_2 in (2) asymptotically stable near Γ_1. In particular, the feedback will asymptotically enforce the relations $(\psi,\dot\psi) = (\bar\psi, 0)$ near Γ_1.
(3) We investigate a reduced-order control system representing the dynamics on Γ_2, and we design the control input f_σ to asymptotically stabilize Γ_1. This is achieved by noticing that Γ_1 is a closed orbit, linearizing the dynamics along it, and designing a linear feedback stabilizing the portion of the dynamics transverse to the orbit.

5. SOLUTION OF THE CIRCULAR PATH FOLLOWING PROBLEM.

In this section we develop a solution to the circular path following problem following the steps outlined in the previous section.

5.1 Step 1: Determination of μ and $\bar\omega$

Specification (i) requires the set Γ_1 in (3) to be asymptotically stable. A necessary condition for a set to be stable for the closed-loop system is that the set be positively invariant for the closed-loop system or, equivalently, that there exists a smooth feedback making Γ_1 invariant. The set Γ_1 is a smooth closed curve in TQ described by the smooth map $X : \mathbb{S}^1 \to TQ$, $X(\sigma) := (\bar q(\sigma), \dot{\bar q}(\sigma))$, where

$$
\begin{aligned}
\bar q(\sigma) &:= (\sigma, \bar\rho, \mu(\sigma), \bar\psi)\\
\dot{\bar q}(\sigma) &:= (\bar\omega, 0, \mu'(\sigma)\bar\omega, 0).
\end{aligned} \tag{4}
$$

The set Γ_1 is controlled invariant if and only if

$$
[0_{2\times 2}\ I_2]\left(D(\bar q)\ddot{\bar q} + C(\bar q,\dot{\bar q})\dot{\bar q} + \nabla V(\bar q)\right) = 0, \tag{5}
$$

where $\ddot{\bar q} = (\partial \dot{\bar q}/\partial\sigma)\bar\omega$.

The above two equations are parametrized by $\mu(\sigma)$ and $\bar\omega$. One can show they hold if and only if

$$
\begin{aligned}
\bar\omega &= \sqrt{\frac{g\tan\bar\psi}{\bar\rho - l\sin\bar\psi}}\\
\mu(\sigma) &= \sigma + \pi.
\end{aligned} \tag{6}
$$

Remark 4. The constant $\bar\omega$ is well-defined because in the statement of the circular path following problem we assumed that $\bar\rho > l\sin\bar\psi$. With the definition of $\mu(\sigma)$ in (6), we may now give a precise physical interpretation of the target set Γ_1. Namely, Γ_1 is a closed orbit along which the cart moves around a circle of radius $\bar\rho$ counterclockwise with constant speed $\bar\omega$. The pendulum has a constant inclination angle $\bar\psi$ from the vertical, and its projection on the horizontal plane is a vector pointing towards the centre of the circle. This configuration is depicted in Figure 3.

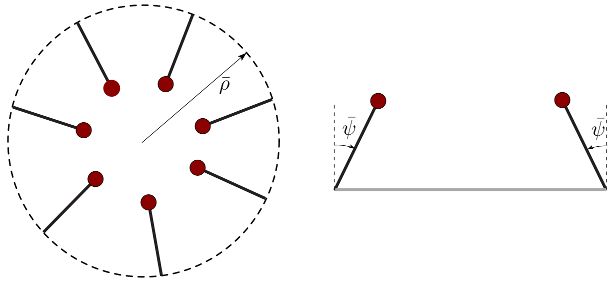

Fig. 3. The configurations of the spherical pendulum on the set Γ_1, as seen from above and from the side.

5.2 Step 2: Asymptotic stabilization of Γ_2 near Γ_1

We now seek to stabilize $\Gamma_2 = \{(q,\dot q) \in TQ : \psi = \bar\psi, \dot\psi = 0\}$. From the model (3) we extract the equation for $\ddot\psi$, which has the form

$$
\ddot\psi = f_1(q,\dot q, f_\sigma) + d(q) f_\rho,
$$

where

$$
d(q) = \frac{-\cos(s-\theta)\cos\psi}{l(m\sin^2\psi + M)}.
$$

On Γ_1, using the expression for μ in (6), $d(q)$ is a positive constant,

$$
d(q)\big|_{\Gamma_1} = \frac{\cos\bar\psi}{l(m\sin^2\bar\psi + M)}.
$$

Since d is continuous and Γ_1 is compact, there exists a constant $C > 0$ such that $d(q) > 0$ on the set $B_C(\Gamma_1)$.

To asymptotically stabilize Γ_2 near Γ_1, we define the feedback linearizing controller

$$
f_\rho(q,\dot q) = \frac{1}{d(q)}\left(-f_1(q,\dot q, f_\sigma) - K_p(\psi - \bar\psi) - K_d\dot\psi\right), \tag{7}
$$

where $K_p, K_d > 0$ are design parameters and f_σ is defined in the next section.

The feedback (7) is well-defined on $B_C(\Gamma_1)$, and on this set is yields the error dynamics ($\tilde\psi = \psi - \bar\psi$)

$$
\ddot{\tilde\psi} + K_d\dot{\tilde\psi} + K_p\tilde\psi = 0.
$$

Thus, conditional to the fact that solutions remain in $B_C(\Gamma_1)$, the feedback (7) asymptotically stabilizes Γ_2. If we show that Γ_1 is asymptotically stable, then the stability of Γ_1 implies that there exists $c \in (0, C)$ such that for each initial condition in $B_c(\Gamma_1)$, the solution remains in $B_C(\Gamma_1)$. This fact, combined with the discussion above, would imply that Γ_2 is asymptotically stable near Γ_1. In the next section we design f_σ to guarantee that Γ_1 is asymptotically stable.

5.3 Step 3: Asymptotic stabilization of Γ_1

After substituting in the feedback (7), we are left with a control system of the form

$$\begin{aligned}\begin{bmatrix}\ddot\sigma\\ \ddot\rho\\ \ddot\theta\end{bmatrix} &= F(q,\dot q) + G(q,\dot q) f_\sigma \\ \ddot\psi &= -K_p(\psi - \bar\psi) - K_d\dot\psi.\end{aligned} \tag{8}$$

As pointed out in Section 5.1, the set Γ_1 is a smooth closed curve with parametrization (using (4) and (6)) $(\bar q(\sigma), \dot{\bar q}(\sigma))$, with

$$\begin{aligned}\bar q(\sigma) &= (\sigma, \bar\rho, \sigma + \pi, \bar\psi)\\ \dot{\bar q}(\sigma) &= (\bar\omega, 0, \bar\omega, 0),\end{aligned} \tag{9}$$

with $\bar\omega$ as in (6). By the construction in Section 5.1, Γ_1 is an orbit of (8), and one can show that Γ_1 is invariant when $f_\sigma = 0$.

The problem now is to design f_σ so as to asymptotically stabilize the closed orbit Γ_1. For this, we appeal to the classical theory of stability of closed orbits found in (Hale, 1980, Chapter VI). In general terms, the idea is as follows.

Consider a smooth dynamical system $\Sigma : \dot x = f(x)$, with $x \in \mathbb{R}^n$, and let $\bar x(t)$ be a twice-differentiable closed orbit of Σ. Let $T > 0$ be the minimal period of $\bar x(t)$. The linearization of Σ along $\bar x$ is

$$\dot z = \big(df_{\bar x(t)}\big) z = A(t) z. \tag{10}$$

This is a linear T-periodic differential equation. The eigenvalues of the monodromy matrix of (10) are called the **characteristic multipliers** of (10). We have the following result.

Theorem 5. (Hale (1980)). The characteristic multipliers of the linear T-periodic system (10) are $\{1, \mu_1, \dots, \mu_{n-1}\} \subset \mathbb{C}$. If $|\mu_i| < 1$, $i = 1, \dots, n-1$, then the closed orbit $\mathrm{Range}(\bar x(\cdot))$ is asymptotically stable.

In the special case when A is independent of t, one has the following.

Corollary 6. If the matrix A in (10) is constant, then A has eigenvalues $\{0, \lambda_1, \dots, \lambda_{n-1}\}$, and the closed orbit $\mathrm{Range}(\bar x(\cdot))$ is asymptotically stable provided that $\mathrm{Re}(\lambda_i) < 0$, $i = 1, \dots, n-1$.

The eigenspace associated with the eigenvalue at 0 is the tangent space to the closed orbit.

We now apply these ideas in our context. Referring to system (8), define the state

$$x = (\sigma, \dot\sigma, \rho, \dot\rho, \theta, \dot\theta, \psi, \dot\psi)$$

and the error from the parametrization of Γ_1 in (9),

$$\tilde x = x - (\sigma, \bar\omega, \bar\rho, 0, \sigma + \pi, \bar\omega, \bar\psi, 0).$$

Since Theorem 5 concerns dynamical systems without control, we define a linear feedback $f_\sigma(x) = K\tilde x$. We linearize (8) along Γ_1. The result has the form

$$\dot{\tilde x} = \begin{bmatrix} A & \star \\ 0 & H\end{bmatrix}\tilde x + \begin{bmatrix} B \\ 0_{2\times 1}\end{bmatrix} K\tilde x, \tag{11}$$

where H is the Hurwitz matrix

$$H = \begin{bmatrix} 0 & 1 \\ -K_p & -K_d\end{bmatrix},$$

and the pair (A, B) can be shown to be controllable for all $\bar\psi \in (0, \pi/2)$ and all $\bar\rho > l\sin\bar\psi$. In light of the block-upper triangular structure of the system matrix and the fact that H is already Hurwitz, appealing to Corollary 6 we need to design K such that the $(A + BK)$ has $n-1$ eigenvalues with negative real part, and an eigenvalue at 0 with eigenvector $v = (1, 0, 0, 0, 1, 0)$. This is a subspace stabilization problem for a linear time-invariant system. The problem is solvable because (A, B) is controllable. Define an invertible matrix $P \in \mathbb{R}^{6\times 6}$ whose first column is the vector v just defined. The isomorphism $z = P^{-1}\tilde x$ transforms (A, B) to $(\hat A, \hat B)$ with the form

$$\hat A = \begin{bmatrix} 0 & A_{12} \\ 0_{5\times 1} & A_{22}\end{bmatrix},\ \hat B = \begin{bmatrix} B_1 \\ B_2\end{bmatrix}.$$

Since (A, B) is controllable, so is $(\hat A, \hat B)$, which in turn implies that the same holds for (A_{22}, B_2). Let $\hat K$ be such that $(A_{22} + B_2\hat K)$ has all eigenvalues with negative real part, and set

$$K = \big[[0\ \ \hat K]\, P^{-1}\ 0\ 0\big].$$

By this construction, $A + BK$ has one eigenvalue at zero with eigenvector v and all other eigenvalues with negative real part. Returning to the linear time-invariant system (11) representing the linearization of (8) along the closed orbit Γ_1, this system has the same properties (the eigenvector associated with the eigenvalue at 0 is now $(v, 0, 0)$). By Corollary 6, Γ_1 is asymptotically stable for the closed-loop system formed by (8) with feedback

$$f_\sigma = K\left[0\ \ \dot\sigma - \bar\omega\ \ \rho - \bar\rho\ \ \dot\rho\ \ \theta - \sigma - \pi\ \ \dot\theta - \bar\omega\ \ 0\ \ 0\right]^\top. \tag{12}$$

5.4 Summary

Since Γ_1 is asymptotically stable, by the discussion in Section 5.2 we also have that Γ_2 is asymptotically stable near Γ_1. In conclusion, for any $\bar\psi \in (0, \pi/2)$ and any $\bar\rho > l\sin\bar\psi$, the feedback (f_ρ, f_σ), with f_ρ defined in (7) and f_σ defined in (12), with constant $\bar\omega$ defined in (6), solves the circular path following problem.

6. SIMULATION RESULTS

We consider a spherical pendulum with geometrical and mechanical parameters $l = 1\,\mathrm{m}$, $M = 1\,\mathrm{kg}$, $m = 0.1\,\mathrm{kg}$ and the circular path following problem with $\bar\rho = 10\,\mathrm{m}$, $\bar\psi = \frac{\pi}{4}\,\mathrm{rad}$. The corresponding value of the angular velocity is $\bar\omega = 1.0274\,\mathrm{rad/s}$. We show numerical simulations of the system with feedback f_ρ given in (7) with $K_d = 20$, $K_p = 100$ and f_σ given in (12) with $\hat K$ designed according to the LQR method. In particular, we take $K = [0\ 110.3\ 5.5078\ -4.92\ -275.421\ -68.0462\ 0\ 0]^\top$. The system is initialised at $\sigma = 0\,\mathrm{rad}$, $\rho = 12\,\mathrm{m}$, $\theta = \pi\,\mathrm{rad}$, $\psi = 10^{-3}\,\mathrm{rad}$, $\dot\sigma = 0.2\,\mathrm{rad/s}$, $\dot\rho = 0\,\mathrm{m/s}$, $\dot\theta = 0.5\,\mathrm{rad/s}$ and

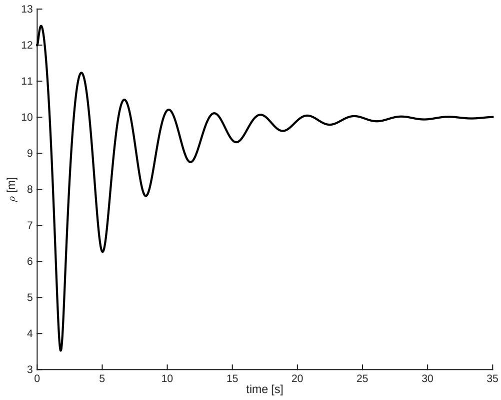

Fig. 4. Evolution of ρ.

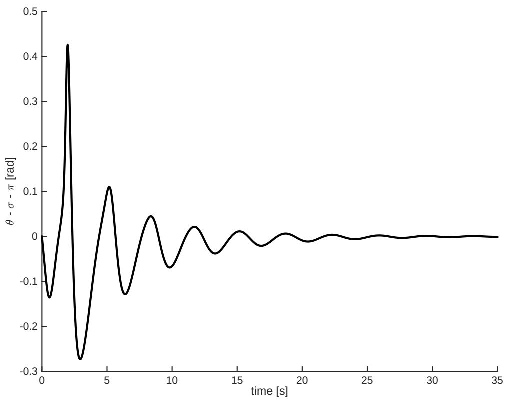

Fig. 5. Evolution of the error $\theta - \sigma - \pi$.

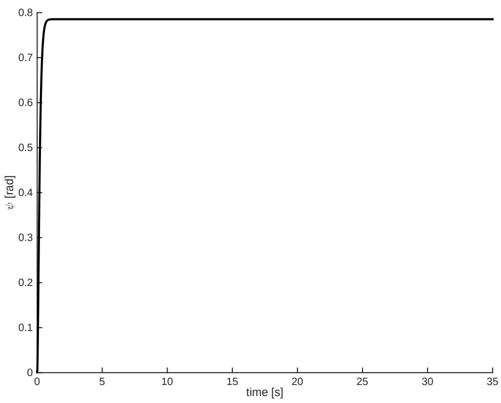

Fig. 6. Evolution of the polar angle ψ.

$\dot{\psi} = 0\,\mathrm{rad/s}$. Fig. 4 and 5 show the asymptotic convergence of ρ to $\bar{\rho}$ and of $\theta - \sigma$ to π respectively. The convergence of the polar angle ψ to its desired value $\bar{\psi}$ is much faster than the other state variables (see Fig. 6). In Fig. 7 the trajectory of the cart is depicted. It is apparent that the cart approaches a circle centered in the origin and of radius $\bar{\rho}$.

7. CONCLUSION

In this paper we considered a simple model of a spherical pendulum on a cart and we dealt with the problem of

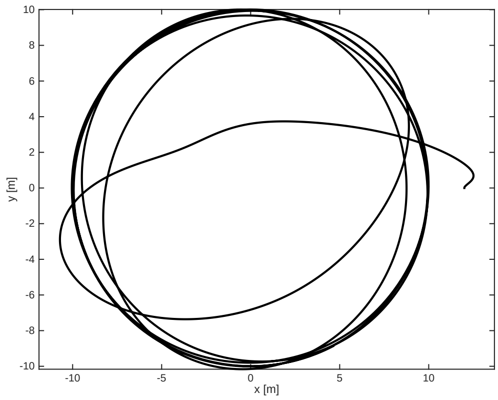

Fig. 7. Trajectory of the cart on the plane.

making the cart follow a circular path with constant speed, while guaranteeing that the pendulum does not fall over. The two-dimensional control input achieving this objective was constructed by means of a hierarchical approach. More precisely, one of the two components was designed as a function of the state and of the other component so that to stabilize the angle of the pendulum from the vertical axis at a constant value. The remaining control input was designed to make the cart converge to the circular path with a desired speed. Future work will focus partly on the stabilization of our model to closed paths other than the one considered here and partly on the adaptation of the method to a more realistic, possibly nonholonomic, model of the spherical pendulum on the cart.

REFERENCES

Bloch, A.M., Leonard, N.E., and Marsden, J.E. (1998). Matching and stabilization by the method of controlled lagrangians. In *Decision and Control, 1998. Proceedings of the 37th IEEE Conference on*, volume 2, 1446–1451. IEEE.

Chaturvedi, N.A. and McClamroch, N.H. (2007). Asymptotic stabilization of the hanging equilibrium manifold of the 3d pendulum. *Int. J. Robust Nonlinear Control*, 17, 1435–1454.

Consolini, L. and Maggiore, M. (2013). Control of a bicycle using virtual holonomic constraints. *Automatica*, 49(9), 2831–2839.

Hale, J.K. (1980). *Ordinary differential equations*. Robert E. Krieger Publishing Company, second edition.

Jankuloski, D., Maggiore, M., and Consolini, L. (2012). Further results on virtual holonomic constraints. *IFAC Proceedings Volumes*, 45(19), 84–89.

Shen, J., Sanyal, A., Chaturvedi, N., Bernstein, D., and McClamroch, H. (2004). Dynamics and control of a 3d pendulum. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 1, 323–328. IEEE.

Shiriaev, A.S., Ludvigsen, H., and Egeland, O. (2004). Swinging up the spherical pendulum via stabilization of its first integrals. *Automatica*, (40), 73–85.

Spong, M.W., Hutchinson, S., and Vidyasagar, M. (2006). *Robot modeling and control*, volume 3. Wiley New York.