Passivity-based Stabilization of Non-Compact Sets

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***Abstract*— We investigate the stabilization of closed sets for passive nonlinear systems which are contained in the zero level set of the storage function.**

I. INTRODUCTION

Equilibrium stabilization is one of the basic control specifications. It is one of the important research problems in nonlinear control theory which continues to receive much attention. The more general problem of stabilizing *sets* has received comparatively less attention. This problem has intrinsic interest because many control specifications can be naturally formulated as set stabilization requirements. The synchronization or state agreement problem, which entails making the states of two or more dynamical systems converge to each other, can be formulated as the problem of stabilizing the diagonal subspace in the state space of the coupled system. The observer design problem can be viewed in the same manner. The control of oscillations in a dynamical system can be thought of as the stabilization of a set homeomorphic to the unit circle or, more generally, to the k-torus. The maneuver regulation or path following problem, which entails making the output of a dynamical system approach and follow a specified path in the output space of a control system, can be thought of as the stabilization of a certain invariant subset of the state space compatible with the motion on the path.

Recently, significant progress has been made toward a Lyapunov characterization of set stabilizability. Albertini and Sontag showed in [1] that uniform asymptotic controllability to a closed, possibly non-compact set is equivalent to the existence of a continuous control-Lyapunov function. Kellet and Teel in [2], [3], and [4], proved that for a locally Lipschitz control system, uniform global asymptotic controllability to a closed, possibly non-compact set is equivalent to the existence of a *locally Lipschitz* control Lyapunov function. Moreover, they were able to use this result to construct a semiglobal practical asymptotic stabilizing feedback.

A geometric approach to a specific set stabilization problem for single-input systems was taken by Banaszuk and Hauser in [5]. There, the authors characterized conditions for dynamics transversal to an open-loop invariant periodic orbit to be feedback linearizable. In [6], Nielsen and one of the authors generalized Banaszuk and Hauser's results to more general controlled-invariant sets. In [7], the same authors extended the theory to multi-input systems.

In a series of papers, [8], [9], [10], Shiriaev and co-workers addressed the problem of stabilizing compact invariant sets for passive nonlinear systems. Their work can be seen as a direct extension of the equilibrium stabilization results for passive systems by Byrnes, Isidori, and Willems in [11]. One of the key ingredients is the notion of V-detectability which generalizes zero-state detectability introduced in [11].

This paper follows Shiriaev's line of work by investigating the passivity-based stabilization of closed, possibly non-compact sets. Our setting is more general than Shiriaev's in that, rather than requiring the goal set to be the zero level set of the storage function, we only require it to be a subset thereof. Moreover, the goal set may be unbounded.

Our main contributions are as follows.

- We introduce a more general notion of detectability with respect to sets, called Γ-detectability (Definition III.1), which is closer in spirit to the original notion of zero-state detectability. We provide geometric sufficient conditions to characterize Γ-detectability.
- We show (Theorem III.1) that Γ-detectability, among other conditions, guarantees that a passivity-based feedback renders the goal set attractive in the special case when all trajectories are bounded.
- In the case of unbounded trajectories, we provide sufficient conditions for a passivity-based feedback to render the goal set attractive or asymptotically stable (Theorems V.1 and V.2).
- We present necessary and sufficient conditions for the equivalence between output convergence and set attractivity (Theorem IV.1).

II. PRELIMINARIES AND PROBLEM STATEMENT

In the sequel, we denote by $\phi(t, x_0, u(t))$ the unique solution to a smooth differential equation $\dot{x} = f(x) + g(x)u(t)$, with initial condition x_0 and piecewise continuous control input signal $u(t)$. Given a control input signal $u(t)$, we let $L^+(x_0, u(t))$ and $L^-(x_0, u(t))$ denote the positive and negative limit sets (ω and α limit sets) of $\phi(t, x_0, u(t))$. Given a closed nonempty set $\Gamma \subset \mathcal{X}$, where $\mathcal{X}$ is a vector space, a point $\xi \in \mathcal{X}$, and a vector norm $\|\cdot\| : \mathcal{X} \to \mathbb{R}$, the point-to-set distance $\|\xi\|_\Gamma$ is defined as $\|\xi\|_\Gamma := \inf_{\eta \in \Gamma} \|\xi - \eta\|$. We use the standard notation $L_f V$ to denote the Lie derivative of a C^1 function V along a vector field f, and $dV(x)$ to denote the differential map of V. We denote by $ad_f g$ the Lie bracket of two vector fields f and g, and by $ad_f^k g$ its k-th iteration.

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In this paper, we consider the control-affine system,

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned} \tag{1}$$

with state space $\mathcal{X} = \mathbb{R}^n$, set of input values $\mathcal{U} = \mathbb{R}^m$ and set of output values $\mathcal{Y} = \mathbb{R}^m$. Throughout this paper, it is assumed that f and the m columns of g are smooth vector fields, and that h is a smooth mapping. It is further assumed that (1) is passive with smooth storage function $V : \mathcal{X} \to \mathbb{R}$, i.e., V is a C^r ($r \geq 1$) nonnegative function such that, for all piecewise-continuous functions $u : [0, \infty) \to \mathcal{U}$, for all $x_0 \in \mathcal{X}$, and for all t in the maximal interval of existence of $\phi(\cdot, x_0, u)$,

$$V(x(t)) - V(x_0) \leq \int_0^t u(\tau)^\top y(\tau) d\tau, \tag{2}$$

where $x(t) = \phi(t, x_0, u(t))$ and $y(t) = h(x(t))$. It is well-known that the passivity property is equivalent to the two conditions

$$\begin{aligned}(\forall x \in \mathcal{X})\ & L_f V(x) \leq 0 \\ (\forall x \in \mathcal{X})\ & L_g V(x) = h(x)^\top.\end{aligned}$$

We next present stability definitions used in this paper. Let $\Gamma \subset \mathcal{X}$ be a closed invariant set for a system $\Sigma : \dot{x} = f(x)$, $x \in \mathcal{X}$.

Definition II.1 (Set Stability, [12]) *(i)* Γ *is* stable *with respect to* Σ *if for all* $\varepsilon > 0$ *there exists a neighbourhood* $\mathcal{N}(\Gamma)$ *of* Γ*, such that* $x_0 \in \mathcal{N}(\Gamma) \Rightarrow \|x(t)\|_\Gamma < \varepsilon$ *for all* $t \geq 0$.

(ii) Γ *is* uniformly stable *with respect to* Σ *if* $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x_0 \in \mathcal{X})$ $(\|x_0\|_\Gamma < \delta \Rightarrow (\forall t \geq 0)\|x(t)\|_\Gamma < \varepsilon)$.

(iii) Γ *is a* semi-attractor *of* Σ *if there exists a neighbourhood* $\mathcal{N}(\Gamma)$ *such that* $x_0 \in \mathcal{N}(\Gamma) \Rightarrow \lim_{t\to\infty} \|x(t)\|_\Gamma = 0$.

(iv) Γ *is an* attractor *of* Σ *if* $(\exists \delta > 0)$ $(\forall x_0 \in \mathcal{X})$ $\|x_0\|_\Gamma < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\|_\Gamma = 0$.

(v) Γ *is a* global attractor *of* Σ *if it is an attractor with* $\delta = \infty$ *or a semi-attractor with* $\mathcal{N}(\Gamma) = \mathcal{X}$.

(vi) Γ *is a* stable semi-attractor *of* Σ *if it is stable and semi-attractive with respect to* Σ.

(vii) Γ *is* asymptotically stable *with respect to* Σ *if it is a uniformly stable attractor of* Σ.

(viii) Γ *is* globally asymptotically stable *with respect to* Σ *if it is a uniformly stable global attractor of* Σ.

Remark II.1 *If* Γ *is compact, then* Γ *is stable if, and only if, it is uniformly stable. Moreover,* Γ *is a semi-attractor if, and only if, it is an attractor.*

Definition II.2 (Relative Set Stability) *Let* $\Xi \subset \mathcal{X}$ *be such that* $\Xi \cap \Gamma \neq \emptyset$. Γ *is* stable with respect to Σ relative to Ξ *if, for any* $\varepsilon > 0$*, there exists a neighbourhood* $\mathcal{N}(\Gamma)$ *such that* $x_0 \in \mathcal{N}(\Gamma) \cap \Xi \Rightarrow \|x(t)\|_\Gamma < \varepsilon$ *for all* $t \geq 0$*. Similarly, one modifies all other notions in Definition II.1 by restricting initial conditions to lie in* Ξ.

Problem Statement Given a closed set $\Gamma \subseteq V^{-1}(0) = \{x \in \mathcal{X} : V(x) = 0\}$ which is open-loop invariant for (1), we seek to find conditions under which a passivity-based feedback $u = -\varphi(y)$ makes Γ either a stable semi-attractor, a (global) attractor, or a (globally) asymptotically stable set for the closed-loop system. We refer to Γ as the *goal set.*

This problem was investigated by Shiriaev and co-workers in [8], [9], [10] in the case when Γ is compact and $\Gamma = V^{-1}(0)$. Their work extended the landmark results by Byrnes, Isidori, and Willems in [11] developed for the case when $\Gamma = V^{-1}(0) = \{0\}$. We next review Shiriaev's main results.

Definition II.3 (V-detectability [8]-[9]) *System* (1) *is* locally V-detectable *if there exists a constant* $c > 0$ *such that for all* $x_0 \in V_c = \{x \in \mathcal{X} : V(x) \leq c\}$*, the solution* $x(t) = \phi(t, x_0, 0)$ *of the open-loop system satisfies* $y(t) = 0\ \forall t \geq 0 \Rightarrow \lim_{t\to\infty} V(x(t)) = 0$*. If* $c = \infty$*, then the system is* V-detectable.

Theorem II.1 ((Global) Asymptotic Stability [9]) *Let* $\varphi : \mathcal{Y} \to \mathcal{U}$ *be any smooth function such that* $\varphi(0) = 0$ *and* $y^\top \varphi(y) > 0$ *for* $y \neq 0$*. Consider the feedback* $u = -\varphi(y)$.

(i) If the set $V^{-1}(0)$ *is compact and the system is locally* V*-detectable, then* $V^{-1}(0)$ *is asymptotically stable for* (1).

(ii) If the function V *is proper and system* (1) *is* V*-detectable, then* $V^{-1}(0)$ *is globally asymptotically stable for* (1).

Shiriaev gave a sufficient condition for V-detectability which extends that of Proposition 3.4 in [11] for zero-state detectability.

Proposition II.1 (Condition for V-detectability [8]-[9]) *Let* $S = \{x \in \mathcal{X} : L_f^m L_\tau V(x) = 0,\ \forall \tau \in D,\ 0 \leq m < r\}$*, where* D *is the distribution* $D = span\{ad_f^k g_i : 0 \leq k \leq n-1, 1 \leq i \leq m\}$*. Further, define* $\Omega = \bigcup_{x_0 \in \mathcal{X}} L^+(x_0, 0)$*. If* $S \cap \Omega \subset V^{-1}(0)$ *then system* (1) *is* V*-detectable.*

III. Set Stabilization - Case of Bounded Trajectories

The notion of V-detectability in Definition II.3 is only applicable to the situation when $\Gamma = V^{-1}(0)$. If $\Gamma \subsetneq V^{-1}(0)$, then the property $x(t) \to V^{-1}(0)$ does not imply that $x(t) \to \Gamma$. For this reason, we introduce a *different* generalization of the zero-state detectability notion which is independent of the storage function and, as such, is closer in spirit to the original definition of zero-state detectability in [11].

Definition III.1 (Γ-Detectability) *System* (1) *is* locally Γ-detectable *if there exists a neighborhood* $\mathcal{N}(\Gamma)$ *of* Γ *such that, for all* $x_0 \in \mathcal{N}(\Gamma)$*, the solution* $x(t) = \phi(t, x_0, 0)$ *satisfies* $y(t) = 0\quad \forall t \geq 0 \Rightarrow \lim_{t\to\infty} \|x(t)\|_\Gamma = 0$*. The system is* Γ-detectable *if it is locally* Γ*-detectable with* $\mathcal{N}(\Gamma) = \mathcal{X}$.

Remark III.1 *If $\Gamma = V^{-1}(0)$ and V is proper (and therefore Γ is compact), then system* (1) *is (locally) Γ-detectable if, and only if, it is (locally) V-detectable.*

The sufficient condition for zero-state detectability of Proposition 3.4 in [11] is easily extended to get a condition for Γ-detectability as follows.

Proposition III.1 (Condition for Γ-detectability) *Let S and D be defined as in Proposition II.1. Suppose that all trajectories on the maximal open-loop invariant subset of $h^{-1}(0)$ are bounded. If $S \cap \Omega \subset \Gamma$ then system* (1) *is Γ-detectable.*

The proof of this proposition is a straightforward extention of that of Proposition 3.4 in [11], and is therefore omitted.

Remark III.2 *The assumption, in [11] and [8]-[9], that V is proper implies that* all *trajectories of the open-loop system are bounded.*

The next, equivalent, condition for Γ-detectability does not require the knowledge of the storage function V.

Proposition III.2 *Let S and Ω be defined as in Proposition II.1 and define $S' = \{x \in \mathcal{X} : L_f^m h(x) = 0, 0 \leq m \leq r+n-2\}$. Then, $S \cap \Omega = S' \cap \Omega$ and therefore under the assumption of Proposition III.1 system* (1) *is Γ-detectable if $S' \cap \Omega \subset \Gamma$.*

The proof of the this proposition is omitted for brevity.

Shiriaev's result in Theorem II.1 states that V-detectability, and hence Γ-detectability, is a sufficient condition to asymptotically stabilize Γ in the case when $\Gamma = V^{-1}(0)$ and Γ is compact. In the more general setting under investigation, that is, when $\Gamma \subset V^{-1}(0)$ and Γ is not necessarily compact, Γ-detectability is no longer a sufficient condition for asymptotic stability of Γ. We show this by means of the following counterexample.

Example III.1 Consider the following system: $\dot{x}_1 = (x_2^2 + x_3^2)(-x_2)$, $\dot{x}_2 = (x_2^2 + x_3^2)(x_1)$, $\dot{x}_3 = -x_3^3 + u$, with output function $h(x_1, x_2, x_3) = x_3^3$. This system is passive with storage $V = \frac{1}{4}x_3^4$. On the set $h^{-1}(0) = \{(x_1, x_2, x_3) : x_3 = 0\}$, the system dynamics take the form: $\dot{x}_1 = (x_2^2)(-x_2)$, $\dot{x}_2 = (x_2^2)(x_1)$. If the goal set is given as $\Gamma = \{(x_1, x_2, x_3) : x_2 = x_3 = 0\}$, then it can be seen from Figure 1 that all trajectories originating in $h^{-1}(0)$ approach Γ and hence the system is Γ-detectable. Using the control $u = -y$, all system trajectories are bounded for all $x_0 \in \mathbb{R}^3$. Thus, for all $x_0 \in \mathbb{R}^3$, the positive limit set $L^+(x_0, u)$ is nonempty, compact, and formed by trajectories on $h^{-1}(0) = \{x_3 = 0\}$. For $x_0 \notin (h^{-1}(0) \cup \{(x_1, x_2, x_3) : x_1 = x_2 = 0)\})$, it can be shown that the positive limit set $L^+(x_0, u)$ is a circle which intersects Γ at equilibrium points. Thus, $x(t) \not\to \Gamma$ because $L^+(x_0, 0) \not\subset \Gamma$. This is illustrated in Figure 1.

In this example the feedback $u = -y$ guarantees that all system trajectories are bounded for all initial conditions, and

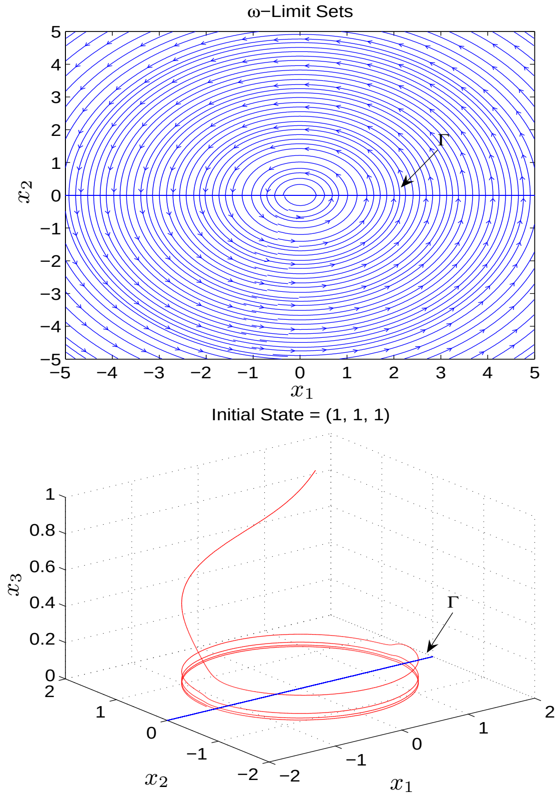


Fig. 1. Counterexample

$x(t) \to h^{-1}(0) = V^{-1}(0)$. The Γ-detectability property guarantees that on $V^{-1}(0)$, which is open-loop invariant, all trajectories approach Γ. However, these two facts do not imply that $x(t) \to \Gamma$. The problem lies in the fact that, while Γ is a global attractor relative to $V^{-1}(0)$, Γ is *not stable* relative to $V^{-1}(0)$.

We next introduce sufficient conditions to insure that Γ is a global attractor of the closed-loop system in the special case when all trajectories of the system are bounded. The condition, in the result below, dealing with relative stability of Γ is similar to a condition presented in [13] for equilibrium stabilization of positive systems with first integrals. Part of the proof of our result is inspired by the stability results using positive semi-definite Lyapunov functions presented in [14].

Theorem III.1 *Consider system* (1) *and a feedback $u = -\varphi(y)$, where $\varphi : \mathcal{Y} \to \mathcal{U}$ is a smooth function such that $\varphi(0) = 0$ and $y^\top \varphi(y) > 0$ for $y \neq 0$. Suppose that, for all $x_0 \in \mathcal{X}$ [resp., for all x_0 in a neighborhood of Γ], $x(t) = \phi(t, x_0, u)$ is bounded. If the system is Γ-detectable [resp., locally Γ-detectable] and Γ is stable with respect to the open-loop system relative to $V^{-1}(0)$, then Γ is a global attractor [resp., a semi-attractor] of the closed-loop system.*

Remark III.3 *When $\Gamma = V^{-1}(0)$, then the relative stability assumption of Theorem III.1 is trivially fulfilled.*

Proof: We prove the global result of the theorem. The proof of the local result is very similar *mutatis mutandis*.

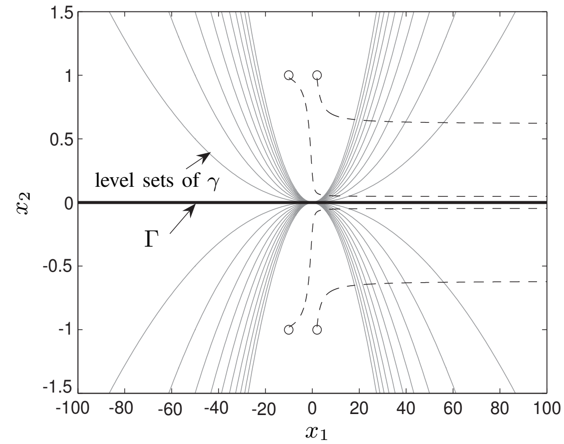
Since the trajectory $x(t) = \phi(t, x_0, u)$ is bounded, its positive limit set, $L^+(x_0, u)$, is nonempty, compact, and invariant for the closed-loop system. Along the trajectory $x(t)$ we have $V(x(t)) - V(x_0) \leq \int_0^t -y(\tau)^\top \varphi(y(\tau))d\tau \leq 0$. Since V is continuous and nonincreasing, $\lim_{t\to\infty} V(x(t)) = c \geq 0$. This implies that $V(x) = c$ on $L^+(x_0, u)$. Let $\bar{x}$ be a point of $L^+(x_0, u)$ and set $\bar{x}(t) = \phi(t, \bar{x}, u)$. Since $\bar{x}(t) \in L^+(x_0, u)$, then $V(\bar{x}(t)) = c$ for all $t \geq 0$ and $0 = V(\bar{x}(t)) - V(\bar{x}) \leq \int_0^t -y(\tau)^\top \varphi(y(\tau))d\tau \leq 0$, so that $y(t) = h(\bar{x}(t)) = 0, \forall t \geq 0$. By Γ-detectability, $\bar{x}(t) \to \Gamma \subset V^{-1}(0)$, and hence $c = 0$. We have thus shown that $L^+(x_0, u) \subset V^{-1}(0)$. In order to show that $x(t) \to \Gamma$ we will show that, by the assumption of stability of Γ relative to $V^{-1}(0)$, $L^+(x_0, u) \subset \Gamma$.

Assume, by way of contradiction, that $L^+(x_0, u) \not\subset \Gamma$, so that we can find a point $p \in L^+(x_0, u)$ and $p \notin \Gamma$. Since V is positive semi-definite, it follows that for all $x \in V^{-1}(0)$, $dV(x) = 0$, and hence $L_g V(x) = 0$, proving that $V^{-1}(0) \subset h^{-1}(0)$. This fact, the invariance of $L^+(x_0, u)$, and the fact that $L^+(x_0, u) \subset V^{-1}(0)$ together imply that $\phi(t, p, u) = \phi(t, p, 0)$ for all $t \in \mathbb{R}$, and therefore $\phi(t, p, 0) \in L^+(x_0, u)$ for all $t \in \mathbb{R}$. The invariance and closedness of $L^+(x_0, u)$ also imply that $L^-(p, 0) \subset L^+(x_0, u) \subset V^{-1}(0)$. Since $L^-(p, 0) \subset V^{-1}(0)$ is open-loop invariant, by the Γ-detectability property we have that open-loop trajectories originating in $L^-(p, 0)$ are contained in $L^-(p, 0)$ and approach Γ in positive time; this fact and the closedness of $L^-(p, 0)$ imply that $L^-(p, 0) \cap \Gamma \neq \emptyset$. Let $\bar{x}$ be a point in $L^-(p, 0) \cap \Gamma$. Since Γ is stable with respect to the open-loop system relative to $V^{-1}(0)$, for any $\epsilon > 0$ there exists a neighborhood $\mathcal{N}(\Gamma)$ such that for all $z \in \mathcal{N}(\Gamma) \cap V^{-1}(0)$, $\|\phi(t, z, 0)\|_\Gamma < \epsilon$ for all $t \geq 0$. Pick $\epsilon > 0$ such that $\|p\|_\Gamma > \epsilon$. Since $\bar{x} \in \Gamma$ is a negative limit point of $\phi(t, p, 0)$, there exist a point $z \in \mathcal{N}(\Gamma)$ and a time $T > 0$ such that $p = \phi(T, z, 0)$, contradicting the relative stability of Γ. Hence, $L^+(x_0, u) \subset \Gamma$. Since $\phi(t, x_0, u) \to L^+(x_0, u)$, we conclude that $\phi(t, x_0, u) \to \Gamma$. ■

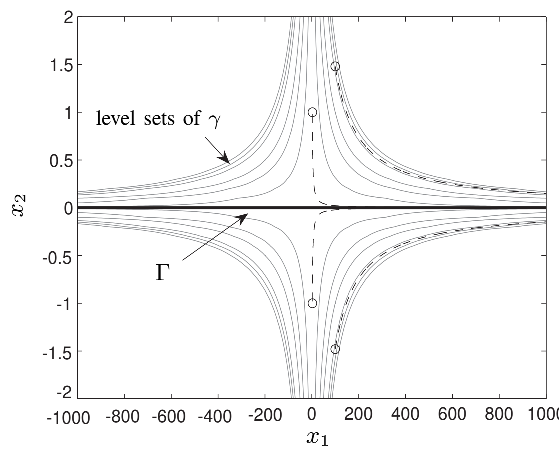
Example III.2 Consider the following system: $\dot{x}_1 = x_2 x_3$, $\dot{x}_2 = x_1 x_3 - x_2^3$, $\dot{x}_3 = -x_3 x_2^4 + u$, with output $y = x_3$. This system is passive with positive semi-definite storage $V(x) = \frac{1}{2}x_3^2$. Indeed, $L_g V(x) = x_3$ and $L_f V(x) = -x_3 x_2^2 \leq 0$. Consider the goal set $\Gamma = \{x : \; x_3 = x_2 = 0\}$. The set $V^{-1}(0) = h^{-1}(0) = \{x : x_3 = 0\}$ is open-loop invariant and the dynamics of the system on the set are given by $\dot{x}_1 = 0$, $\dot{x}_2 = -x_2^3$. Clearly, the system is Γ detectable and Γ is stable with respect to the open-loop system relative to $V^{-1}(0)$. The feedback $u = -y$ globally stabilizes Γ.

IV. Output Convergence vs. Set Attractivity

When the goal set Γ can be expressed as $\Gamma = \gamma^{-1}(0) = \{x \in \mathcal{X} : \gamma(x) = 0\}$, where γ is a continuous function $\mathcal{X} \to \mathbb{R}^p$, one approach to stabilizing Γ is to view $\gamma(x)$ as the output of the system and pose the problem of stabilizing Γ as that of stabilizing the output. In general, however, the convergence of the output to zero is not equivalent to the attractivity of the goal set. This is illustrated in the next two counterexamples.



(a) $\gamma(x(t)) \to 0 \not\Rightarrow x(t) \to \gamma^{-1}(0)$



(b) $x(t) \to \gamma^{-1}(0) \not\Rightarrow \gamma(x(t)) \to 0$

Fig. 2. Counterexamples showing that output convergence is not equivalent to set attractivity.

Example IV.1 ($\gamma(x(t)) \to 0 \not\Rightarrow x(t) \to \gamma^{-1}(0)$) Consider the system $\dot{x}_1 = 1$, $\dot{x}_2 = \frac{-x_2}{1+x_1^2}$ with output function $\gamma(x_1, x_2) = \frac{x_2}{1+x_1^2}$. The zero level set of this function is given by $\gamma^{-1}(0) = \{x : x_2 = 0\}$. The solution with initial condition (x_{10}, x_{20}) is $x_1(t) = x_{10} + t$, $x_2(t) = x_{20}\exp(-\arctan(x_{10}+t) + \arctan x_{10})$. It is clear that if $x_{20} \neq 0$, then $x(t) \not\to \gamma^{-1}(0)$. The output signal is $\gamma(x(t)) = [x_{20}\exp(-\arctan(x_{10}+t) + \arctan x_{10})]/[1 + (x_{10}+t)^2]$. Clearly, $\gamma(x(t)) \to 0$. The problem here is that for all $c \neq 0$, the distance between points in $\gamma^{-1}(c)$ and the set $\gamma^{-1}(0)$ is not uniformly bounded; see Figure 2(a).

Example IV.2 ($x(t) \to \gamma^{-1}(0) \not\Rightarrow \gamma(x(t)) \to 0$) Consider the system $\dot{x}_1 = 1$, $\dot{x}_2 = \frac{-x_1 x_2}{1+x_1^2}$ with output function $\gamma(x_1(t), x_2(t)) = x_2^2(1 + x_1^2)$. The zero level set of this function is the set $\gamma^{-1}(0) = \{x : x_2 = 0\}$. The solution with initial condition (x_{10}, x_{20}) is $x_1(t) = x_{10} + t$,

$x_2(t) = x_{20}\sqrt{(1+x_{10}^2)/(t^2+2x_{10}t+1+x_{10}^2)}$, so for all initial conditions, $x(t) \to \gamma^{-1}(0)$. On the other hand, $\gamma(x_1(t), x_2(t)) = x_{20}^2[(1+x_{10}^2)[(x_{10}+t)^2+1]]/[t^2+2x_{10}t+1+x_{10}^2]$, so if $x_{20} \neq 0$, $\gamma(x_1(t), x_2(t)) \not\to 0$. The problem here is that for all $c \neq 0$, the distance between the sets $\gamma^{-1}(c)$ and $\gamma^{-1}(0)$ is not bounded away from zero; see Figure 2(b).

We next present necessary and sufficient conditions to establish the equivalence between output convergence and the attractivity of $\gamma^{-1}(0)$. The proof is omitted due to space limitations.

Theorem IV.1 *Let $\gamma : \mathcal{X} \to \mathbb{R}^p$ be a continuous function, and let $\Gamma = \{x \in \mathcal{X} : \gamma(x) = 0\}$.*

(i) (Class $\mathcal{K}$-lower bound) *A necessary and sufficient condition such that*

$$(\forall \{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X})\ (\gamma(x_n) \to 0 \implies x_n \to \Gamma),$$

is that there exists $r > 0$ and a class-$\mathcal{K}$ function $\alpha : [0, r) \to \mathbb{R}_+$ such that

$$(\forall \|x\|_\Gamma < r)\ \alpha(\|x\|_\Gamma) \leq \|\gamma(x)\|.$$

(ii) (Class $\mathcal{K}$-upper bound) *A necessary and sufficient condition such that*

$$(\forall \{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X})\ (x_n \to \Gamma \implies \gamma(x_n) \to 0),$$

is that there exists $r > 0$ and a class-$\mathcal{K}$ function $\beta : [0, r) \to \mathbb{R}_+$ such that

$$(\forall \|x\|_\Gamma < r)\ \|\gamma(x)\| \leq \beta(\|x\|_\Gamma).$$

(iii) (Bounded Sequences) *For all bounded sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$, $\gamma(x_n) \to 0 \iff x_n \to \Gamma$.*

V. Set Stabilization - Case of Unbounded Trajectories

We now turn our attention to situation when trajectories are unbounded and present two results concerning the (semi) attractivity and asymptotic stability of Γ. A key ingredient here is the result in Theorem IV.1.

Theorem V.1 *Consider system* (1) *and a feedback $u = -\varphi(x)$, where $\varphi : \mathcal{X} \to \mathcal{U}$ is a smooth function such that $h(x)^\top \varphi(x) \geq \psi(\|h(x)\|)$, and ψ is a class-$\mathcal{K}$ function. Define $\lambda(x) = h(x)^\top \varphi(x)$. Suppose that the closed-loop system has no finite escape times and that:*

(i) There exists $r > 0$ and a class-$\mathcal{K}$ function $\alpha : [0, r) \to \mathbb{R}_+$ such that $\alpha(\|x\|_\Gamma) \leq \|h(x)\|$ for all $\|x\|_\Gamma < r$.

(ii) There exists $s \in (0, r]$ such that $\inf_{\|x\|_\Gamma = s} V(x) > 0$.

(iii) For all $c > 0$ there exists $k \in \mathbb{R}$ such that whenever $V(x) \leq c$, either $L_f\lambda(x) - L_g\lambda(x)\varphi(x) \leq k$ or $L_f\lambda(x) - L_g\lambda(x)\varphi(x) \geq k$.

Then, Γ is a semi-attractor. If (i)-(iii) hold and

(iv) there exists a class-$\mathcal{K}$ function $\beta : [0, r) \to \mathbb{R}_+$ such that $V(x) \leq \beta(\|x\|_\Gamma)$ for all $\|x\|_\Gamma < r$,

then Γ is an attractor. Finally, if (i), (iii) hold and

(v) $\Gamma = V^{-1}(0) = h^{-1}(0)$,

then Γ is a global attractor.

Remark V.1 *Assumption (i) implies that $\Gamma = V^{-1}(0) \cap \{x \in \mathcal{X} : \|x\|_\Gamma < r\} = h^{-1}(0) \cap \{x \in \mathcal{X} : \|x\|_\Gamma < r\}$. In other words, the assumption imposes that each connected component of Γ coincides with a connected component of $V^{-1}(0)$ and $h^{-1}(0)$.*

Proof: Let $x_0 \in \mathcal{X}$ be arbitrary and denote $x(t) = \phi(t, x_0, u)$. Since $V(\cdot)$ is nonnegative and $V(x(t)) - V(x(0)) \leq -\int_0^t \lambda(x(\tau))\,d\tau \leq 0$, this implies that $V(x(t)) \leq V(x(0))$ for all $t \geq 0$ and thus $\lim_{t\to\infty} \int_0^t \lambda(x(\tau))\,d\tau$ exists and is finite. By (iii), there exists $k \in \mathbb{R}$ such that $\dot{\lambda}(x(t)) = L_f\lambda(x(t)) + L_g\lambda(x(t))\varphi(x(t))$ is either $\leq k$ or $\geq k$. Take the case where $\dot{\lambda}(x(t)) \geq k$; we will show that $\lambda(x(t)) \to 0$. Suppose, by way of contradition, that $\lambda(x(t)) \not\to 0$. This implies that there exists $\epsilon > 0$ and a sequence $\{t_n\}$ such that $t_n \to \infty$ and $\lambda(x(t_n)) \geq \epsilon$ for all n. By the mean value theorem, for all n and for all $t \geq t_n$, there exists $s_n \in [t_n, t]$ such that $\lambda(x(t)) = \lambda(x(t_n)) + \dot{\lambda}(x(s_n))(t - t_n) \geq \epsilon + k(t - t_n) \geq \epsilon - |k|\,|t - t_n|$. Pick $\epsilon_0 \in (0, \epsilon)$ and let $\delta = (\epsilon - \epsilon_0)/|k|$. Then, for all n and for all $t \in [t_n, t_n + \delta]$, $\lambda(t) \geq \epsilon_0 > 0$. Take a subsequence $\{t_{n_k}\}$ such that $t_{n_{k+1}} - t_{n_k} \geq \delta$. Since, by assumption, $\lambda(x(t)) \geq 0$, we have

$$\lim_{t\to\infty} \int_0^t \lambda(x(\tau))d\tau \geq \sum_k \int_{t_{n_k}}^{t_{n_k}+\delta} \lambda(x(\tau))d\tau \geq \sum_k \delta\epsilon_0,$$

contradicting the fact that $\lim_{t\to\infty} \int_0^t \lambda(x(\tau))\,d\tau$ exists and is finite. Thus, $\lambda(x(t)) \to 0$ and since $\psi(\|h(x)\|) \leq \lambda(x(t))$, $h(x(t)) \to 0$. The proof for the case $\dot{\lambda}(x(t)) \leq k$ is almost identical and therefore omitted. Recall (see the proof of Theorem III.1) that $\Gamma \subset V^{-1}(0) \subset h^{-1}(0)$. By assumption (i), $\Gamma = V^{-1}(0) \cap \{\|x\|_\Gamma < r\} = h^{-1}(0) \cap \{\|x\|_\Gamma < r\}$. Moreover, for all $\|x\|_\Gamma < r$, $\|h(x)\| \geq \alpha(\|x\|_\Gamma) \geq \alpha(\|x\|_{h^{-1}(0)})$. By (ii), $c = \inf_{\|x\|_\Gamma = s} V(x)$ is positive. Let $\mathcal{N}(\Gamma) = \{x \in \mathcal{X} : \|x\| < s,\ V(x) < c\}$. It follows that $\mathcal{N}(\Gamma)$ is positively invariant and thus for all $x_0 \in \mathcal{N}(\Gamma)$, $h(x(t)) \to 0$ and $\|x(t)\|_\Gamma < r$. This, together with the fact that $\|h(x(t))\| \geq \alpha(\|x(t)\|_{h^{-1}(0)})$ implies that $x(t) \to h^{-1}(0) \cap \{\|x\|_\Gamma < r\} = \Gamma$, proving that Γ is a semi-attractor.

If, in addition to assumptions (i)-(iii), assumption (iv) holds then there exists $\delta > 0$ such that $\{x \in \mathcal{X} : \|x\|_\Gamma < \delta\} \subset \mathcal{N}(\Gamma)\}$, and thus Γ is an attractor.

If assumptions (i), (iii) and (v) hold, then we have shown that $(\forall x_0 \in \mathcal{X})\ h(\phi(t, x_0, u)) \to 0$ and by Theorem IV.1, $\phi(t, x_0, u) \to h^{-1}(0) = \Gamma$. Thus, Γ is a global attractor. ■

By strengthening conditions (i), (ii) and removing condition (iii) in Theorem V.1, we gain stability.

Theorem V.2 *Consider system* (1) *and a feedback $u = -\varphi(x)$, where $\varphi : \mathcal{X} \to \mathcal{U}$ is a smooth function such that $\varphi(x)^\top h(x) \geq \psi(\|h(x)\|)$, and ψ is a class-$\mathcal{K}$ function. Suppose that the closed-loop system has no finite escape times and that:*

(i) There exists $r > 0$ and a class-$\mathcal{K}$ function $\alpha : [0, r) \to \mathbb{R}_+$ such that $\alpha(\|x\|_\Gamma) \leq V(x)$ for all $\|x\|_\Gamma < r$.

(ii) There exists a class-$\mathcal{K}$ function $\mu : [0, +\infty) \to \mathbb{R}_+$ such that $\mu(V(x)) \leq \|h(x)\|$ for all $x \in \mathcal{X}$.

Then, Γ is a stable semi-attractor of the closed-loop system. If (i), (ii) hold and

(iii) there exists a class-$\mathcal{K}$ function $\beta : [0, r) \to \mathbb{R}_+$ such that $V(x) \leq \beta(\|x\|_\Gamma)$ for all $\|x\|_\Gamma < r$,

then Γ is asymptotically stable. If (i), (ii) hold and

(iv) $\Gamma = V^{-1}(0) = h^{-1}(0)$,

then Γ is globally attractive. Finally, if (i)-(iv) hold, then Γ is globally asymptotically stable.

Remark V.2 *Assuptions (i), (ii) in Theorem V.2 imply assumptions (i), (ii) of Theorem V.1. Indeed, for all $\|x\|_\Gamma < r$, $\|h(x)\| \geq \mu(V(x)) \geq \mu \circ \alpha(\|x\|_\Gamma)$. Moreover, the inequality $V(x) \geq \alpha(\|x\|_\Gamma)$ implies $\inf_{\|x\|_\Gamma = s} V(x) \geq \alpha(s) > 0$ for all $s \in (0, r]$.*

Proof: By assumption (ii), for all $x_0 \in \mathcal{X}$,

$$\begin{aligned}\frac{dV(\phi(t, x_0, u))}{dt} &\leq -h(\phi(t, x_0, u))^\top \varphi(\phi(t, x_0, u)) \\ &\leq -\psi \circ \mu(V(\phi(t, x_0, u))).\end{aligned}$$

By the comparison lemma, it follows that $V(\phi(t, x_0, u)) \to 0$. Let $\epsilon \in (0, r)$, choose $c > 0$ such that $\alpha^{-1}(c) < \epsilon$, and define $\mathcal{N}(\Gamma) = \{x \in \mathcal{X} : V(x) < c, \|x\|_\Gamma < r\}$. Clearly, $\mathcal{N}(\Gamma)$ is a neigbourhood of Γ. By our choice of c and assumption (i), $\mathcal{N}(\Gamma) \subset \{x \in \mathcal{X} : \|x\|_\Gamma < \epsilon\}$ and the boundary of $\mathcal{N}(\Gamma)$ is $\partial\mathcal{N}(\Gamma) = \{x \in \mathcal{X} : V(x) = c, \|x\|_\Gamma < r\}$. Since V is nonincreasing, it follows that $\mathcal{N}(\Gamma)$ is positively invariant and hence Γ is stable. Since $V(\phi(t, x_0, u)) \to 0$ for all $x_0 \in \mathcal{N}(\Gamma)$, assumption (i) implies that $x(t) \to \Gamma$. Hence, Γ is a semi-attractor. If, in addition, assumption (iii) holds, then there exists $\delta > 0$ such that $\{x \in \mathcal{X} : \|x\|_\Gamma < \delta\} \subset \mathcal{N}(\Gamma)$, and thus Γ is asymptotically stable.

If assumptions (i), (ii), and (iv) hold, then we have shown that $(\forall x_0 \in \mathcal{X})\ V(\phi(t, x_0, u)) \to 0$. Hence, by Theorem IV.1, $\phi(t, x_0, u) \to V^{-1}(0) = \Gamma$. Thus, Γ is globally attractive.

If assumptions (i)-(iv) hold, then Γ is a uniformly stable global attractor, that is, Γ is globally asymptotically stable. ∎

Example V.1 Consider the system $\dot{x}_1 = \tan^{-1}(x_1)$, $\dot{x}_2 = x_2(1+x_1^2)u$, with output $y = x_2^2$. This system is passive with a storage function $V(x) = \frac{1}{2}\frac{x_2^2}{1+x_1^2}$. Let the goal set be $\Gamma = V^{-1}(0) = h^{-1}(0) = \{x : x_2 = 0\}$. It is seen that V does not have a class-$\mathcal{K}$ lower bound with respect to Γ and thus Theorem V.2 cannot be applied to this example. On the other hand, $h(x)$ does have a class-$\mathcal{K}$ lower bound with respect to Γ, as $\|h(x)\| = \|x\|_\Gamma^2$. Moreover, $\|h(x)\| = \|x\|^2_{h^{-1}(0)}$. Using the control $u = -\varphi(x)$, with $\varphi(x) = h(x)$, we have $L_f\lambda(x) - L_g\lambda(x)\varphi(x) = -4x_2^6(1 + x_1^2)$, which is bounded from above. Thus we have that conditions (i), (iii) and (v) of Theorem V.1 are satisfied and Γ is globally attractive.

Example V.2 Consider the system $\dot{x}_1 = 1$, $\dot{x}_2 = (1 + 0.5\cos(x_1^2))u$, with output $y = x_2(1+0.5\cos(x_1^2))$, which is passive with storage function $V(x) = \frac{1}{2}x_2^2$. Let the goal set be $\Gamma = \{x : x_2 = 0\}$ so that $V^{-1}(0) = h^{-1}(0) = \Gamma$. Moreover, $V(x) = 1/2\|x\|_\Gamma^2$ and $\|h(x)\| \geq 0.5|x_2| = 0.5\|x\|_\Gamma$, so by Theorem V.2 the feedback $u = -\varphi(x)$, with $\varphi(x) = h(x)$, renders Γ globally asymptotically stable. It is easily seen that Theorem V.1 cannot be applied to this example because assumption (iii) does not hold.

VI. Conclusions

This paper leaves several open questions that we will investigate in future work. In Theorem III.1, we assume that the trajectories of the closed-loop system are bounded. Boundedness is a property that one should seek to achieve by means of a suitable feedback. The same comment holds for Theorems V.1 and V.2, where we assume that the closed-loop system does not possess finite escape times.

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