Synthesis of virtual holonomic constraints for 3-DOF mechanical systems

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***Abstract*— The paper presents a new method for determining virtual holonomic constraints for a mechanical system. We focus on systems with 3 DOF and 2 controls. The main result of the paper shows that, if the distribution generated by the vector fields representing the accelerations due to the inputs is not involutive near a reference configuration, then it is possible to find a closed virtual holonomic constraint defined in an arbitrarily small neighborhood of the reference configuration, which is controlled invariant and stabilizable. We present a constructive proof that provides a method for the numerical computation of the constraint. As an application example, we present some families of virtual holonomic constraints for an underactuated double pendulum on a cart.**

I. INTRODUCTION

A virtual holonomic constraint (VHC) is a relation of the form $h(q) = 0$ that can be made invariant via feedback control. The early idea of VHCs appeared in the work of Nakanishi *et a.* [1] where the authors enforced the angle of a virtual pendulum on the configuration of a brachiating robot in order to follow the target dynamics of a harmonic oscillator and to imitate the pendulum-like motion of an ape's brachiation. In the past decade, VHCs have emerged as a useful tool for motion control in biped robots (see, e.g., [2], [3], [4], [5]), and as an approach to motion planning for general robotic systems (e.g., [6], [7], [8], [9]). In the context of motion control of biped robots, researchers encode a walking gait by imposing, through feedback control, relations between the joint angles of the robot, and they show that when the relations are satisfied, the reduced motion arising is a stable limit cycle corresponding to a periodic walking motion [2], [3]. In the context of motion planning, researchers use VHCs to aid the selection of closed orbits corresponding to desired repetitive behaviors, which can then be stabilized in a variety of ways [6], [7].

Despite the considerable progress in the investigation of virtual holonomic constraints, some problems remain unaddressed. In particular, the derivation of stabilizable constraints is often obtained with methods tailored for a specific system and a general procedure is still missing in literature. In our paper [10], we presented a systematic procedure for the determination of stabilizable virtual holonomic constraint for systems with n degrees of freedom and $n-1$ actuators. The method allows to find constraints expressed as a function of one cyclic configuration variable, in which the constraints for $n-2$ variables are chosen arbitrary, while the remaining constraint is obtained as the solution of a scalar differential equation. With the aim of providing new results on the problem of the synthesis of stabilizable virtual constraints, this paper develops a different approach for determining a family of virtual holonomic constraints that are contained in an arbitrarily small neighborhood of a point of the configuration space. In particular, we focus on Lagrangian control systems with 3 DOF and 2 actuators. The dynamics are modeled in the standard form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla P(q) = B(q)\tau\,,$$

where τ is the control vector and the input matrix $B(q)$ has two columns. In the expression of $\ddot{q}$, the control vector τ appears in term $D^{-1}(q)B(q)\tau$, we name $g_1(q)$, $g_2(q)$ the columns of $D^{-1}(q)B(q)$. The span of vector fields g_1, g_2 represent the possible accelerations that can be obtained with the control inputs. The main result of this paper shows that, if the distribution generated by $\operatorname{span}\{g_1, g_2\}$ is **not involutive** in a neighborhood of a configuration q_0, then it is possible to find a closed virtual holonomic constraint defined in an arbitrarily small neighborhood of q_0, diffeomorphic to $\mathbb{S}^1$, which is controlled invariant and stabilizable. We present a constructive proof that provides a method for the actual computation of the constraint. The fact that non involutivity guarantees the existence of a virtual constraint is not surprising. In fact, it is well known that involutivity properties plays a fundamental role in motion planning for nonlinear systems (see for instance [11], [12]). As an application example, we present some families of virtual holonomic constraints for an underactuated double pendulum on a cart.

Notation: Given a n-dimensional smooth manifold $\mathcal{Q}$, $T\mathcal{Q}$ denotes its tangent bundle $T\mathcal{Q} = \{(q, v_q) : p \in \mathcal{Q}, v_q \in T_q\mathcal{Q}\}$ and $\Lambda^k\mathcal{Q}$ is the k-th exterior power of the cotangent bundle of $\mathcal{Q}$. A differential k-form ω is a smooth mapping $\mathcal{Q} \to \Lambda^k\mathcal{Q}$ that associates to each point $q \in \mathcal{Q}$ an alternating multilinear map ω_q of order k. We denote by $\omega_q(x_1, x_2, \ldots, x_k)$ the scalar obtained by evaluating map ω_q on vectors $x_1, x_2, \ldots, x_k$. The exterior derivative of ω is denoted by $d\omega : \mathcal{Q} \to \Lambda^{k+1}\mathcal{Q}$. Given a vector field v on $\mathcal{Q}$, the differential 1-form v^* is defined as $v_q^*(x) = v_q \cdot x$, the operation that associates v^* to v is called the canonical isomorphism. Given two vector fields v, w on $\mathcal{Q}$, $[v, w]$ denotes the Lie bracket of v and w. Given a function $v : \mathbb{R}^n \to \mathcal{Q}$ and vectors $x, w \in \mathbb{R}^n$, $L_w v(x)$ represents the directional derivative of v along vector w, evaluated at x. We let $[x]_{2\pi} : x$ modulo 2π, and $[\mathbb{R}]_{2\pi} = \{[x]_{2\pi} : x \in \mathbb{R}\}$.

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II. Introduction on Virtual Holonomic Constraints

This section presents some results on VHCs taken from [13], to which we refer the reader for further details. Consider a Lagrangian control system with n DOF and $n-1$ actuators modeled as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = B(q)\tau.$$

In the above, $q = (q_1, \dots, q_n) \in \mathcal{Q}$ is the configuration vector. We assume that each component q_i, $i = 1, \dots, n$, is either a linear displacement in $\mathbb{R}$, or an angular displacement in $[\mathbb{R}]_{2\pi}$. With this assumption, the configuration manifold $\mathcal{Q}$ is a generalized cylinder, and $T\mathcal{Q}$ is the Cartesian product $T\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^n$. Hence, the inner product of two vectors v, w on the tangent space is given by the standard inner product on $\mathbb{R}^n$ denoted by $v \cdot w$. The term $B(q)\tau$ represents external forces produced by the control vector $\tau \in \mathbb{R}^{n-1}$. We assume that $B : \mathcal{Q} \to \mathbb{R}^{n\times(n-1)}$ is smooth and rank $B(q) = n-1$ for all $q \in \mathcal{Q}$. Further, the function $\mathcal{L} : T\mathcal{Q} \to \mathbb{R}$ is assumed to be smooth and to have the special form $\mathcal{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$, where $D(q)$, the generalized mass matrix, is symmetric and positive definite for all $q \in \mathcal{Q}$. We will assume that there exists a left annihilator of B on Q. That is to say, there exists a smooth function $B^\perp : \mathcal{Q} \to \mathbb{R}^{1\times n}$ which does not vanish and is such that $B^\perp(q)B(q) = 0$ on $\mathcal{Q}$. With the above mentioned assumptions, the Lagrangian control system takes on the following standard form

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla P(q) = B(q)\tau. \qquad (1)$$

Remark 1: Set $\mathbb{S}^1$ is diffeomorphic to the set of real numbers modulo 2π, via the diffeomorphism $\phi : [\mathbb{R}]_{2\pi} \to \mathbb{S}^1$, $\phi(t) = \begin{pmatrix} \cos t, \sin t \end{pmatrix}^T$. The map $\bar{\pi} : \mathbb{R} \to [\mathbb{R}]_{2\pi}$, defined as $\bar{\pi}(x) = [x]_{2\pi}$, is a covering map from $\mathbb{R}$ to $[\mathbb{R}]_{2\pi}$. We define a covering map $\pi : \mathbb{R}^n \to \mathcal{Q}$ by $\pi(x) = [\pi_1(x_1), \pi_2(x_2), \dots, \pi_n(x_n)]$, where $\pi_i(x_i) = x_i$, if $q_i \in \mathbb{R}$, and $\pi_i(x_i) = \bar{\pi}(x_i)$, if $q_i \in [\mathbb{R}]_{2\pi}$.

Definition 1 ([13]): A **virtual holonomic constraint (VHC)** of order $n-1$ for system (1) is a relation $h(q) = 0$, where $h : \mathcal{Q} \to \mathbb{R}^{n-1}$ is a smooth function which has a regular value at 0, i.e., rank$(dh_q) = n-1$ for all $q \in h^{-1}(0)$, and is such that the set

$$\Gamma = \{(q, \dot{q}) : h(q) = 0,\ dh_q\dot{q} = 0\} \qquad (2)$$

is controlled invariant. That is to say, there exists a smooth feedback $\tau : \Gamma \to \mathbb{R}^{n-1}$ such that Γ is positively invariant for the closed-loop system. The set Γ is called the **constraint manifold** associated with $h(q) = 0$. A VHC is said to be **stabilizable** if there exists a smooth feedback $\tau(q, \dot{q})$ that asymptotically stabilizes Γ. Such a stabilizing feedback is said to **enforce the VHC** $h(q) = 0$.

By the preimage theorem [14], if $h(q) = 0$ is a VHC of order $n-1$, then the set $h^{-1}(0)$ is a one-dimensional embedded submanifold of $\mathcal{Q}$. In other words, $h^{-1}(0)$ is a regular curve without self-intersections. As such, $h^{-1}(0)$ is diffeomorphic to either the real line or the unit circle $\mathbb{S}^1$.

Definition 2 ([13]): A relation $h(q) = 0$, where $h : \mathcal{Q} \to \mathbb{R}^{n-1}$ is a smooth function, is a **regular VHC** of order $n-1$ for (1) if system (1) with output function $e = h(q)$ has well-defined vector relative degree $\{2, \dots, 2\}$ everywhere on the constraint manifold given in (2).

A regular VHC is a VHC. Regular VHCs enjoy two important properties. First, under mild assumptions (see [13]), regular VHCs are stabilizable by input-output feedback linearizing feedback. Second, regular VHCs induce well-defined reduced dynamics. Specifically, the dynamics on Γ (i.e., the zero dynamics associated with the output $e = h(q)$) are given by a second-order unforced system. In order to find the reduced dynamics, we follow a procedure presented in [15]. We first pick a regular parametrization $\gamma : \Theta \to \mathcal{Q}$ of the curve $h^{-1}(0)$, where $\Theta = \mathbb{R}$ if $h^{-1}(0) \simeq \mathbb{R}$, while $\Theta = [\mathbb{R}]_{2\pi}$, if $h^{-1}(0) \simeq \mathbb{S}^1$. Next, multiplying (1) on the left by $B^\perp(q)$ we obtain

$$B^\perp D\ddot{q} + B^\perp(C\dot{q} + \nabla P) = 0.$$

The dynamics on Γ are found by restricting the equation above on Γ. To this end, we let $q = \gamma(\lambda)$, $\dot{q} = \gamma'(\lambda)\dot{\lambda}$, and $\ddot{q} = \gamma'(\lambda)\ddot{\lambda} + \gamma''(\lambda)\dot{\lambda}^2$. By so doing, we obtain

$$\ddot{\lambda} = \Psi_1(\lambda) + \Psi_2(\lambda)\dot{\lambda}^2, \qquad (3)$$

where

$$\Psi_1(\lambda) = -\left.\frac{B^\perp \nabla P}{B^\perp D\gamma'}\right|_{q=\gamma(\lambda)},$$

$$\Psi_2(\lambda) = -\left.\frac{B^\perp D\gamma'' + \sum_{i=1}^n B_i^\perp \gamma'^T Q_i \gamma'}{B^\perp D\gamma'}\right|_{q=\gamma(\lambda)},$$

and where $B_i^\perp$ is the i^{th} component of $B^\perp$ and $(Q_i)_{jk} = 1/2(\partial_{q_k} D_{ij} + \partial_{q_j} D_{ik} - \partial_{q_i} D_{kj})$.

The unforced autonomous system (3) represents the reduced dynamics of system (1) when the regular VHC of order $n-1$, $h(q) = 0$, is enforced.

The following condition for regularity is a direct consequence of Proposition 3.2 of [13].

Proposition 1: The set $\gamma([0, 2\pi])$, where $\gamma : [\mathbb{R}]_{2\pi} \to \mathcal{Q}$ is a smooth function, is a regular VHC of order $n-1$ if and only if Im $(D^{-1}B(\gamma(\lambda)))$ and Im $(\gamma'(\lambda))$ are independent subspaces, $\forall\lambda \in [0, 2\pi]$.

III. Problem formulation

In this paper, we consider a configuration manifold with dimension $n = 3$ and VHCs of order 2 in which $h^{-1}(0)$ is a closed curve, with the interpretation that the constraint corresponds to a desired repetitive behavior. It is convenient to adopt a parametric description of the VHC, in which the 3 configuration variables are expressed as smooth functions $\gamma(\lambda) : [\mathbb{R}]_{2\pi} \to \mathcal{Q}$ and the constraint manifold is defined as $h^{-1}(0) = \gamma([0, 2\pi])$.

In these hypotheses, Im $(D^{-1}(q)B(q))$ has rank 2 and it is possible to define vector fields g_1, g_2 on $\mathcal{Q}$ such that

$$\text{Im } (D^{-1}(q)B(q)) = \text{Im } (g_1(q), g_2(q)),$$

we set $G(q) = (g_1(q), g_2(q))$.

We consider the problem of generating closed VHCs defined in a neighborhood of an assigned configuration $q_0 \in \mathcal{M}$. In this conference paper, we consider the case in which span$\{g_1(q), g_2(q)\}$ **is not locally involutive** in a neighborhood of q_0.

IV. Method for generation of VHCs

In this section, we report the method for the generation of a VHC in the neighborhood of an assigned configuration $q_0 \in \mathcal{Q}$ and present the main result of the paper, that shows that if the distribution span$\{g_1, g_2\}$ is not locally involutive at q_0, there exists a VHC, diffeomorphic to $\mathbb{S}^1$, contained in an arbitrarily small neighborhood of q_0. The method is summarized in the following steps.

***Procedure** 1: To find a regular VHC in a neighborhood of a configuration q_0.*

1) Set $V = \text{Im}\,(g_1(q_0), g_2(q_0)) \subset \mathbb{R}^3 = T_{q_0}\mathcal{Q}$ and choose any Jordan curve $\gamma : [\mathbb{R}]_{2\pi} \to V$ that surrounds the origin of V. Let $\eta(q)$ be a unitary vector field normal to span$\{g_1(q), g_2(q)\}$, (i.e. set $\eta(q) = \frac{g_1(q)\times g_2(q)}{\|g_1(q)\times g_2(q)\|}$).

2) Choose a parameter $\epsilon > 0$ and let $x_0 \in \mathbb{R}^3$ such that $\pi(x_0) = q_0$. Define function $\gamma_\epsilon : [\mathbb{R}]_{2\pi} \times \mathbb{R} \to \mathcal{Q}$ as

$$\gamma_\epsilon(\lambda, z) = \pi(x_0 + \epsilon\gamma(\lambda) + \epsilon^2\eta(q_0)z)\ . \tag{4}$$

3) Find a constant $\delta \neq 0$ such that the solution of the following differential equation in $\mathbb{R}$ exists and is 2π-periodic

$$\begin{cases} z'_{\epsilon,\delta}(\lambda) = f_\epsilon(\lambda, z_{\epsilon,\delta}, \delta) = \frac{-\gamma'(\lambda)\cdot\eta(\gamma_\epsilon(\lambda, z_{\epsilon,\delta}))+\epsilon\delta}{\epsilon\eta(q_0)\cdot\eta(\gamma_\epsilon(\lambda,z_{\epsilon,\delta}))} \\ z_{\epsilon,\delta}(0) = 0\,, \end{cases} \tag{5}$$

4) The regular VHC is given by

$$\Gamma = \{\gamma_\epsilon(\lambda, z_{\epsilon,\delta}(\lambda)) : \lambda \in [0, 2\pi]\}\ . \tag{6}$$

***Remark** 2:* As we will show in proposition 3, if the distribution span$\{g_1, g_2\}$ is not involutive in a neighborhood of q_0, it is always possible to find ϵ, sufficiently small, and $\delta \neq 0$ such that the solution of (5) is 2π-periodic. Hence, if it is not possible to find $\delta \neq 0$ for a given ϵ in the solution of step 3), it is sufficient to repeat the procedure with a smaller value of ϵ.

Figure 1 shows a graphical representation of the construction of the constraint γ_ϵ.

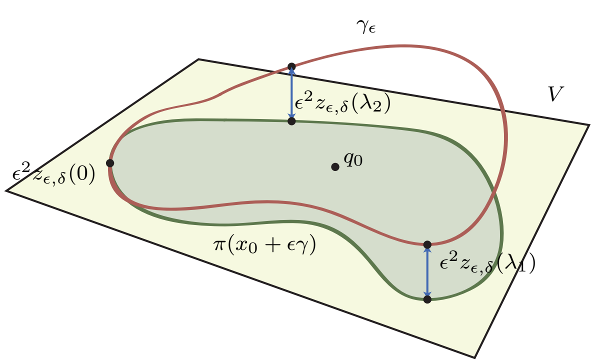

Fig. 1: Method for finding of a VHC in a neighborhood of a configuration q_0.

The following proposition shows that the constraint obtained with this procedure is a regular VHC.

***Proposition** 2:* If the solution of (5), for nonzero constants ϵ, δ, is well-defined and 2π-periodic, then the set Γ, defined by (6), is a regular VHC.

Proof: If $z = z_{\epsilon,\delta}$ is a solution of (5) for $\delta \neq 0$, it follows that

$$\gamma'_\epsilon(\lambda, z)\cdot\eta(\gamma(\lambda)) = \epsilon\gamma'(\lambda)\cdot$$
$$\eta(\gamma_\epsilon(\lambda, z)) + \epsilon^2 z'(\lambda)\eta(q_0)\cdot\eta(\gamma_\epsilon(\lambda, z)) = \epsilon^2\delta \neq 0\ .$$

Since $\eta(q)$ is orthogonal to Im $(D^{-1}B(q))$, this identity implies that the hypothesis of proposition 1 is satisfied. Hence, the closed curve $\gamma_\epsilon([0, 2\pi])$ is a regular VHC. ■

The following proposition, that will be proved in the next section, guarantees that it possible to find a 2π-periodic solution for $z_{\epsilon,\delta}$ if ϵ is sufficiently small and if span$\{g_1, g_2\}$ is not involutive near q_0.

***Proposition** 3:* Assume that the distribution span$\{g_1, g_2\}$ is not involutive in a neighborhood of $q_0 \in \mathcal{Q}$. Then, there exists a positive real constant $\bar{\epsilon}$ such that, for any $0 < \epsilon < \bar{\epsilon}$, there exists a value of $\delta \neq 0$ such that the solution of (5) is well-defined on $\mathbb{R}$ and 2π-periodic.

The following theorem is a direct consequence of propositions 2 and 3 and is the main result of this paper.

***Theorem** 1:* Let $q_0 \in \mathcal{Q}$, assume that the distribution span$\{g_1, g_2\}$ is not involutive in a neighborhood of q_0. Then, there exists a regular VHC, contained in an arbitrarily small neighborhood of q_0, diffeomorphic to S^1. This VHC can be found using procedure 1.

***Remark** 3:* The distribution span$\{g_1, g_2\}$ is not involutive in a neighborhood of q_0 if and only if

$$\det([g_1, g_2](q_0), g_1(q_0), g_2(q_0)) \neq 0\ . \tag{7}$$

V. Proof of proposition 3

We first prove that the family of functions f_ϵ, defined in (5) for $\epsilon \neq 0$, can be extended with continuity to $\epsilon = 0$. Note that, $\forall\lambda \in [0, 2\pi]$, $\forall z \in \mathbb{R}$:

$$\lim_{\epsilon\to 0}\gamma_\epsilon(\lambda, z) = \pi(x_0) = q_0$$

$$\lim_{\epsilon\to 0}\gamma'(\lambda)\cdot\eta(\gamma_\epsilon(\lambda, z)) = \gamma'(\lambda)\cdot\eta(q_0) = 0\,,$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\epsilon^{-1}\gamma'(\lambda)\cdot\eta(\gamma_\epsilon(\lambda, z))\\ &= \lim_{\epsilon\to 0}\epsilon^{-1}\gamma'(\lambda)\cdot\eta(\pi(x_0) + \epsilon\gamma(\lambda) + \epsilon^2 z\eta(q_0)))\\ &= \gamma'(\lambda)\cdot[L_{\gamma(\lambda)}(\eta\circ\pi)](x_0)\,,\end{aligned}$$

$$\lim_{\epsilon\to 0}\eta(q_0)\cdot\eta(\gamma_\epsilon(\lambda, z)) = \eta(q_0)\cdot\eta(q_0) = 1$$

which imply, $\forall\lambda \in [0, 2\pi]$, $\forall z \in \mathbb{R}$:

$$\begin{aligned}\lim_{\epsilon\to 0} f_\epsilon(\lambda, z, \delta) &= \lim_{\epsilon\to 0}\frac{-\epsilon\gamma'(\lambda)\cdot[L_{\gamma(\lambda)}(\eta\circ\pi)](x_0) + \epsilon\delta}{\epsilon\eta(q_0)\cdot\eta(\gamma_\epsilon(\lambda, z_{\epsilon,\delta}))}\\ &= -\gamma'(\lambda)\cdot[L_{\gamma(\lambda)}(\eta\circ\pi)](x_0) + \delta\,.\end{aligned}$$

Hence, setting $\bar{f}(\lambda) = -\gamma'(\lambda)\cdot L_{\gamma(\lambda)}(\eta\circ\pi)(x_0)$ and defining

$$f_0(\lambda, z, \delta) = \bar{f}(\lambda) + \delta\,,$$

the family of functions f_ϵ becomes continuous at $\epsilon = 0$. Consider the differential equation

$$\begin{cases} \dot{z}_{0,\delta}(\lambda) = f_0(\lambda, z, \delta) = \bar{f}(\lambda) + \delta \\ z_{0,\delta}(0) = 0\,, \end{cases}$$

where the dot indicates differentiation with respect to λ. The solution is well-defined for $\lambda \in \mathbb{R}$, it is given by

$$z_{0,\delta}(\lambda) = \int_0^{\lambda} \bar{f}(\lambda) d\lambda + \lambda\delta \,, \tag{8}$$

and it is 2π-periodic if $\delta = \delta_0 = -\frac{\int_0^{2\pi} \bar{f}(\lambda) d\lambda}{2\pi}$.

By the continuity of $f_{\epsilon,\delta}$ with respect to ϵ and δ, there exists a neighborhood of $(\epsilon, \delta) = (0, \delta_0)$ on which the solution $z_{\epsilon,\delta}$ of (5) exists on the interval $[0, 2\pi]$ and function $T(\epsilon, \delta) = z_{\epsilon,\delta}(2\pi)$ is well-defined. Note that $T(0, \delta_0) = 0$, $T(\epsilon, \delta)$ is differentiable at $(0, \delta_0)$ and $\partial_\delta T(\delta, \epsilon)|_{(0,\delta_0)} = 2\pi \neq 0$ (obtained by differentiating (8) with respect to δ, with $\lambda = 2\pi$). Then, by the Implicit Function Theorem, there exists an interval $[0, \bar{\epsilon})$ and a function $\delta(\epsilon)$ such that $T(\epsilon, \delta(\epsilon)) = 0$, $\forall \epsilon \in [0, \bar{\epsilon}]$ (which implies that $z_{\epsilon,\delta(\epsilon)}$ is 2π-periodic) and $\delta(\epsilon) \neq 0$. It remains to prove that $\delta_0 \neq 0$. To this end, note that, setting $\omega_q = (\eta(q_0) + [L_q(\eta \circ \pi)](x_0))^*$, the integral of the differential form ω on γ is given by $-2\pi\delta_0$. Indeed, since $\gamma'(\lambda) \cdot \eta(q_0) = 0$, $\forall \lambda \in [0, 2\pi]$,

$$-2\pi\delta_0 = \int_0^{2\pi} \bar{f}(\lambda) d\lambda = \int_0^{2\pi} \gamma'(\lambda) \cdot [L_{\gamma(\lambda)}(\eta \circ \pi)](x_0) d\lambda$$
$$= \int_0^{2\pi} \gamma'(\lambda) \cdot (\eta(q_0) + [L_{\gamma(\lambda)}(\eta \circ \pi)](x_0)) d\lambda = \int_\gamma \omega \,.$$

The differential form ω is linear with respect to q and therefore its exterior derivative $d\omega$ is a constant 2-form by property 1 (see the Appendix). By Stokes' theorem and Lemma 1 in the Appendix, calling Γ the interior of γ on V and $|\Gamma|$ its area,

$$-2\pi\delta = \int_\Gamma d\omega = C\, d(\omega)_0(g_1(q_0), g_2(q_0))|\Gamma| \,,$$

where C is a nonzero constant.

Set $\bar{\omega}_q = \eta(\pi(q + x_0))^*$, then $d\omega_0 = d\bar{\omega}_0$. Indeed,

$$d\omega_0 - d\bar{\omega}_0 = d(\omega - \bar{\omega})_0 = 0 \,,$$

by property (9), since ω is the linearization of $\bar{\omega}$ at 0.

Set $\bar{g}_1(q) = g_1(q + q_0)$, $\bar{g}_2(q) = g_2(q + q_0)$. Since $w(q_0) \cdot g_1(q_0) = w(q_0) \cdot g_2(q_0) = 0$, $\bar{\omega}_0(g_1(0)) = \bar{\omega}_0(g_1(0)) = 0$, it follows from property 2 (see the Appendix) that

$$d\omega_0(\bar{g}_1(0), \bar{g}_2(0)) = d\bar{\omega}_0(\bar{g}_1(0), \bar{g}_2(0))$$
$$= -\bar{\omega}_0(\bar{g}_1(0), \bar{g}_2(0)) = -\eta(q_0) \cdot ([g_1, g_2])(q_0) \,,$$

which is not null, since the distribution $\text{span}\{g_1, g_2\}$ is not involutive at q_0.

VI. Application to a "pendubot on a cart"

As an application example, we consider a double pendulum on a cart, shown in figure 2. The generalized coordinates are given by $q = (x, q_1, q_2) \in \mathcal{Q} = \mathbb{R} \times S^1 \times S^1$, in which x is the position of the cart, q_1 is the angle between that the first pendulum and the vertical axis and q_2 is the angle between the second pendulum and the vertical axis. Variables x and q_1 are actuated, while q_2 is not. Due to its actuator configuration, the system can be considered a "pendubot on a cart". We assume that the length of the two links and the mass of the cart are unitary. The mass of each link is unitary and concentrated at its final point. Under these hypotheses, the inertia matrix and potential energy are given by

$$D(q) = \begin{pmatrix} 3 & -2\cos q_1 & -\cos q_2 \\ -2\cos q_1 & 2 & \cos(q_1 - q_2) \\ -\cos q_2 & \cos(q_1 - q_2) & 1 \end{pmatrix},$$

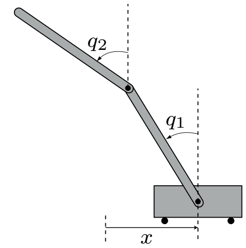

Fig. 2: Double pendulum on a cart.

$$P(q) = g(2\cos(q_1) + \cos(q_2)) \,.$$

The input matrix is given by $B(q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, and

$$D^{-1}(q)B(q) = \frac{1}{1 + 2\sin(q_1)^2 + \sin(q_1 - q_2)}$$
$$\cdot \begin{pmatrix} 2 - \cos(q_1 - q_2)^2 & \frac{3}{2}\cos q_1 - \frac{1}{2}\cos(q_1 - 2q_2) \\ \frac{3}{2}\cos q_1 - \frac{1}{2}\cos(q_1 - 2q_2) & 3 - \cos(q_2)^2 \\ \cos q_2 - \cos(2q_1 - q_2) & \cos(q_1 + q_2) - 2\cos(q_1 - q_2) \end{pmatrix},$$

To find a simple matrix function $G(q)$, let $D_1(q)$ be the 2×2 matrix composed of the first two rows of $D^{-1}(q)B(q)$. Since $D_1(q)$ is invertible, Im $D^{-1}(q)B(q) =$ Im $D^{-1}(q)B(q)D_1^{-1}(q)$ and we can set $G(q) = D^{-1}(q)B(q)D_1^{-1}(q) = [g_1(q), g_2(q)]$, where $g_1(q) = \begin{pmatrix} 1, 0, \cos q_2 \end{pmatrix}^T$, $g_2(q) = \begin{pmatrix} 0, 1, -\cos(q_1 - q_2) \end{pmatrix}^T$. Since $[g_1, g_2](q) = \begin{pmatrix} 0, 0, -\sin(q_1) \end{pmatrix}^T$, the non involutivity condition (7) is given by $\sin q_1 \neq 0$. By theorem 1, this implies that, if q_1 is not a multiple of π, a closed VHC of order 2 is defined in any sufficiently small neighborhood of q. To apply the method presented in section IV, we chose the curve $\gamma(\lambda) = G(q_0)\begin{pmatrix} \cos(\lambda), \sin(\lambda) \end{pmatrix}$, which, in general, represents an ellipse on the plane Im $(G(q_0))$. Proposition 3 guarantees that there exists a sufficiently small ϵ and a constant $\delta \neq 0$ such that the solution $z_{\epsilon,\delta}(\lambda)$ of (5) is 2π-periodic and, $\gamma_\epsilon(\lambda) = \pi(x_0 + \epsilon\gamma(\lambda) + \epsilon^2 w(q_0) z_{\epsilon,\delta}(\lambda))$ is a feasible VHC. We have considered the following two cases.

A. Example 1

We have set $q_0 = (0, -\frac{\pi}{8}, \frac{\pi}{8})$, configuration shown in figure 3. We have numerically found a function $\delta(\epsilon)$, for $\epsilon \in [0.01, 0.8]$, for which the solution $z_{\epsilon,\delta}$ of (5) is well-defined on interval $[0, 2\pi]$ and 2π-periodic. The function $\delta(\epsilon)$ is represented in the left-hand plot of figure 4. The black curves in Figure 5 represent the regular constraints obtained for different values of ϵ from 0.01 to 0.8, while the red curve corresponds to $\epsilon = \bar{\epsilon} = 0.3$. Note that theorem 1 guarantees that the curves obtained for sufficiently small values of ϵ are regular VHC, however, as this example shows, a 2π-periodic solution of (5) may also exist for larger values of ϵ. The right-hand plot in Figure 4 shows the periodic solution of (5), obtained with $\bar{\epsilon}$ and $\delta = \bar{\delta} \simeq -0.1353$. Figure 6 shows the dynamics (3) on the constraint manifold for $\bar{\epsilon}$ and $\bar{\delta}$, the red curve is a particular solution that converges to an asymptotically stable limit cycle. Figure 7 shows the configurations of the system corresponding to the VHC obtained for $\bar{\epsilon}$ and $\bar{\delta}$.

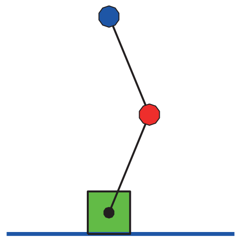

Fig. 3: Example 1: reference configuration q_0.

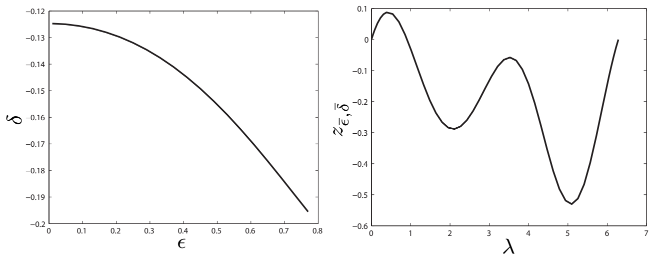

Fig. 4: Example 1: (left) function $\delta(\epsilon)$, (right) Periodic solution of equation (5) for $\bar{\epsilon}$, obtained with $\bar{\epsilon} = 0.3$, $\bar{\delta} \simeq -0.1353$.

B. Example 2

We have considered the configuration $q_0 = (0, \frac{\pi}{2}, 0)$, depicted in figure 8. Figures 9, 10, 12 presents the obtained results (see the captions). Figure 11 shows the dynamics on the constraint manifold, for $\bar{\epsilon}$ and $\bar{\delta}$, the red curve is a particular solution that corresponds to a closed orbit.

VII. Conclusions

We have presented an approach for determining a family of virtual holonomic constraints that are contained in an arbitrarily small neighborhood of a point of the configuration space. The provided results apply to mechanical systems with 3 DOF and 2 controls under the hypotheses that the distribution generated by the vector fields representing the accelerations achievable with the inputs is not involutive.

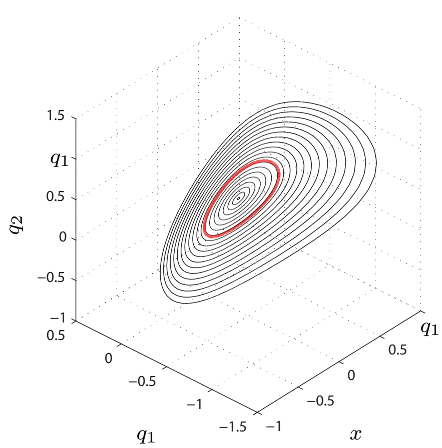

Fig. 5: Example 1: virtual holonomic constraints obtained for different values of ϵ, from 0.01 (smallest curve) to 0.8 (largest one). The red curve corresponds to the chosen values of $\bar{\epsilon} = 0.3$.

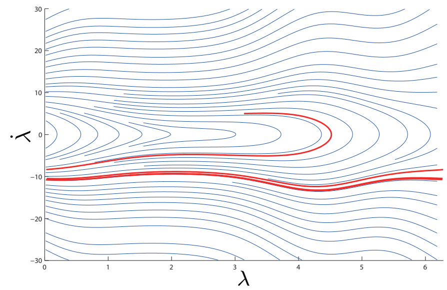

Fig. 6: Example 1: phase portrait of the internal dynamics.

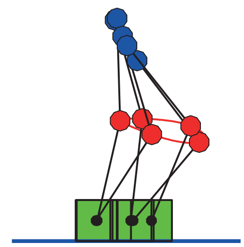

Fig. 7: Example 1: systems configurations corresponding to the VHC.

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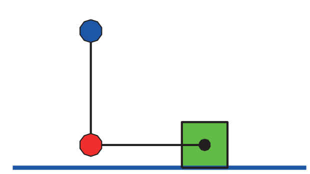

Fig. 8: Example 2: reference configuration q_0.

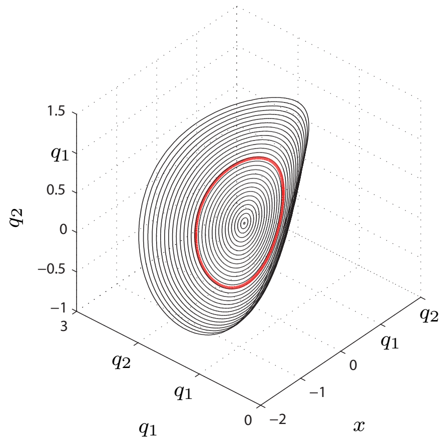

Fig. 9: Example 2: (left) virtual holonomic constraints obtained for different values of ϵ, from 0.01 (smallest curve) to 1.2 (largest one). The red curve corresponds to the chosen values of $\bar{\epsilon} = 0.6$.

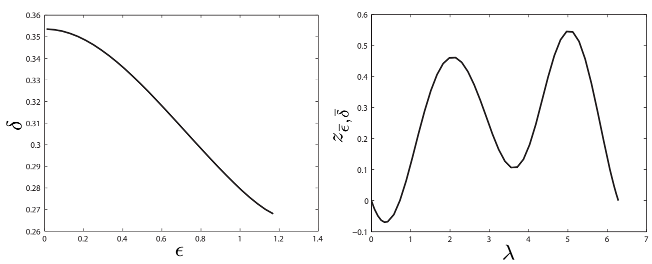

Fig. 10: Example 2: (left) function $\delta(\epsilon)$, (right) periodic solution of equation (5), obtained with $\bar{\epsilon} = 0.6, \bar{\delta} \simeq 0.3180$.

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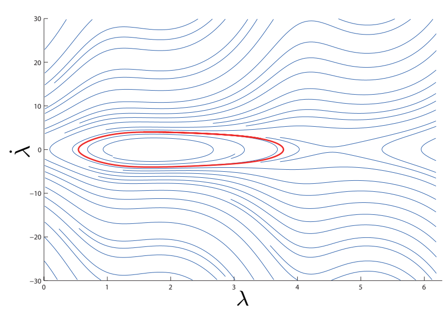

Fig. 11: Example 2: phase portrait of the internal dynamics.

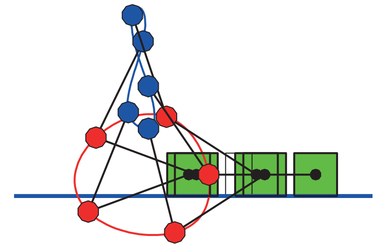

Fig. 12: Example 2: systems configurations corresponding to the VHC.

APPENDIX

The following are basic properties of 1-forms (see for instance chapter 12 of [16]).

***Property** 1:* Let v be a vector field defined on $\mathbb{R}^n$, then

$$(dv^*)_q = \sum_{0<i<j\leq n} (\frac{\partial f_j}{\partial x_i} - \frac{\partial f_j}{\partial x_i}) dx_j \wedge dx_i \, ,$$

where $q = (x_1, \dots, x_n)$.

***Property** 2:* The external derivative of a smooth 1-form ω on a manifold $\mathcal{Q}$, computed on smooth vector fields x, y defined on $\mathcal{Q}$, satisfies the property

$$d\omega(x, y) = x\omega(y) - y\omega(x) - \omega([x, y]) \, . \tag{9}$$

***Lemma** 1:* Let V be a 2-dimensional subspace of $\mathbb{R}^3$, let γ be a Jordan curve with values on V, positively oriented, whose interior Γ contains the origin. Let ω be a constant 2-form and let $\{v_1, v_2\}$ be an orthonormal basis of V. Then

$$\int_\gamma \omega = \omega(v_1, v_2)|\Gamma| \, , \tag{10}$$

where $|\Gamma|$ denotes the area of set Γ. Moreover, if $\{g_1, g_2\}$ is a basis of V, then

$$\int_\gamma \omega = \omega(g_1, g_2)|\Gamma| A \, , \tag{11}$$

where $A^2 = \det(g_i \cdot g_j)^{-1}$, $i, j = 1, 2$.

Proof: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T((u_1, u_2)^T) = u_1 v_1 + u_2 v_2$, then (10) holds since

$$\begin{aligned} \int_\Gamma \omega &= \int_{T^{-1}(\Gamma)} (T^*\omega)(e_1, e_2) du_1 du_2 \\ & \int_{T^{-1}(\Gamma)} d(\omega)(T(e_1), T(e_2)) du_1 du_2 \\ &= |T^{-1}(\Gamma)| d(\omega)(v_1, v_2) = |\Gamma| d(\omega)(v_1, v_2), \end{aligned}$$

where T^* is the pullback of the linear operator T, $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$ and $|T^{-1}\Gamma| = |\Gamma|$ since T is an orthonormal transformation. To prove (11), setting $g_1 = av_1 + bv_2$, $g_2 = cv_1 + dv_2$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} \omega(g_1, g_2)|\Gamma| &= \omega(av_1 + bv_2, cv_1 + dv_2)|\Gamma| \\ &= (ad - bc)\omega(v_1, v_2)|\Gamma| = \det M\omega(v_1, v_2) \end{aligned}$$

and, since $M = (g_1, g_2)(v_1, v_2)^{-1}$, $\det M^2 = A^2$.

■