Robust Output Feedback Tracking With a Matching Condition

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***Abstract*— We study the tracking problem in the presence of smooth, bounded uncertainty and find sufficient conditions so that, if the uncertainty satisfies a suitable matching condition, one can design a partial information controller (i.e., an output feedback controller) achieving arbitrarily small steady-state tracking error.**

I. Introduction

In [1] and [2] the notion of a practical internal model was introduced as a paradigm to solve the output feedback (or partial information) tracking problem for nonlinear systems. The word *practical internal model* was chosen to indicate the fact that this paradigm allows to solve the tracking problem *practically* (i.e., to an arbitrary degree of accuracy), rather than asymptotically, and that its solution relies on the existence of a compensator (the practical internal model) which has a conceptually similar role to a nonlinear internal model in output regulation theory (see, e.g., [3] for an introduction to the output regulation problem and the definition of nonlinear internal model). In [2] it was also showed that, when the tracking problem is posed within an output regulation framework with appropriate restrictions, the practical internal model can be replaced by an internal model and the paradigm can still be employed. As pointed out in [1] and [2], this theory is still far from being self-contained and leaves several open questions. One of them is the extension of the results in [1], [2] to the case when the system is affected by disturbances. In a previous work [4] we took a first step in this direction by investigating the output tracking problem for systems satisfying a suitable matching condition. Unfortunately the results in [4] are *incorrect* mainly because the control strategy proposed there yields a closed-loop system which is *not proper*. In Example 1 we clarify the problem with the approach in [4]. The present paper fixes this problem by developing a solution which is rather different than the one proposed in [4].

Assuming that the disturbances satisfy a matching condition, we derive a set of sufficient conditions on the existence of a dynamic extension leading to the solution of the practical tracking problem using certainty equivalence. We show that, under suitable conditions, two classes of compensators represent feasible dynamic extensions, namely chains of integrators and linearizing compensators, and for these we provide a constructive procedure to design feedback control laws. An extended version of this paper is found in [5].

Throughout this paper we use $\mathrm{col}(a, b)$ to indicate the vector $[a^\top, b^\top]^\top$. If v is a n-dimensional vector, v_i, $i = 1, \ldots, n$, are its components. Given real numbers a, b, c, $diag[a, b, c]$ denotes the matrix with a, b, c on the diagonal and zeros elsewhere. Given matrices A, B, C, we denote by $block\text{-}diag[A, B, C]$ the matrix formed by placing A, B, C on the diagonal and zeros elsewhere.

To illustrate the main ideas of this paper, we will resort to a simple example.

Example 1 Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + u_1 + \Delta_1(t) \\ \dot{x}_3 &= x_4 - x_1^2 - u_1 - \Delta_1(t) \\ \dot{x}_4 &= u_2 + \Delta_2(t) \\ y &= \mathrm{col}(x_1, x_3), \end{aligned} \tag{1}$$

where $\Delta(t) = \mathrm{col}(\Delta_1(t), \Delta_2(t))$ is an unknown smooth function of time which is bounded with bounded time derivatives, u is the control input, and y is the measurable output (x is not available for feedback). Given a smooth reference trajectory $r(t) = \mathrm{col}(r_1(t), r_2(t))$, we seek to find a *partial information controller* (i.e., an output feedback controller) using only the information given by y and r to make $y(t)$ track $r(t)$. We begin by noticing that Δ satisfies a matching condition in that letting

$$\tilde{u} = u + \Delta,$$

the plant can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + \tilde{u}_1 \\ \dot{x}_3 &= x_4 - x_1^2 - \tilde{u}_1 \\ \dot{x}_4 &= \tilde{u}_2 \\ y &= \mathrm{col}(x_1, x_3), \end{aligned} \tag{2}$$

which is a system free of disturbance where, however, $\tilde{u}$ can not be freely assigned because Δ is not known. Given

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a smooth reference trajectory $r(t)$, the smooth functions of time $x^r(t) = \text{col}\left(r_1(t), \dot{r}_1(t), r_2(t), \ddot{r}_1(t) + \dot{r}_2(t)\right)$, $u^r(t) = \text{col}\left(\ddot{r}_1(t) - r_1^2(t), \ddot{r}_2(t) + \dddot{r}_1(t)\right)$, are feasible state and input trajectories for (2) since

$$\begin{aligned} \dot{x}_1^r &= x_2^r \\ \dot{x}_2^r &= (x_1^r)^2 + u_1^r \\ \dot{x}_3^r &= x_4^r - (x_1^r)^2 - u_1^r \\ \dot{x}_4 &= u_2^r. \end{aligned} \tag{3}$$

Further, it is readily seen that the output of (3) is precisely r. Thus the problem of tracking can be converted to one of stabilization by setting $\tilde{x} = x - x^r$ and computing the associated error dynamics. By doing so, one finds that the controller

$$\begin{aligned} \tilde{u} &= \bar{\bar{u}}(x, x^r, u^r) \\ &= \text{col}(u_1^r + (x_1^r)^2 - x_1^2 + K_1^\top(x - x^r), u_2^r + K_2^\top(x - x^r)) \end{aligned} \tag{4}$$

globally uniformly asymptotically stabilizes the equilibrium $\tilde{x} = 0$ of the error dynamics and hence solves the tracking problem globally. This solution, however, presents some problems. Firstly, the feedback controller $\bar{\bar{u}}$ is not implementable because Δ is unknown. Secondly, the pair (x^r, u^r), called the stable inverse of (2) (see [6]), may in general be difficult or even impossible to exactly compute and therefore it would be desirable to develop a solution that does not rely on its knowledge. Finally, $\bar{\bar{u}}$ depends on x which is not available for feedback.

View (3) as a copy of the disturbance-free plant (2) with unknown state x^r, unknown input u^r, known output r, and augment it with the following compensator

$$\begin{aligned} \dot{\zeta}_1^r &= \zeta_2^r + \zeta_3^r \\ \dot{\zeta}_2^r &= v_1^r \\ \dot{\zeta}_3^r &= v_2^r \\ u^r &= \text{col}(\zeta_1^r, \zeta_2^r). \end{aligned} \tag{5}$$

Define $X_1 \overset{\triangle}{=} \text{col}(x^r, \zeta^r)$ and form the mapping $\mathcal{H}_X(X_1) : (x^r, \zeta^r) \mapsto (r_1, \dot{r}_1, \ddot{r}_1, \dddot{r}_1, r_2, \dot{r}_2, \ddot{r}_2)$, where the time derivatives are calculated along the vector fields (3), (5). Since $\mathcal{H}_X(X_1)$ is everywhere smooth and bijective (a diffeomorphism), from y_{X_1} one can calculate $X_1 = \text{col}(x^r, \zeta^r) = \mathcal{H}_X^{-1}(y_{X_1})$ from which one gets the stable inverse $(x^r, u^r) = (x^r, (\zeta_1^r, \zeta_2^r))$. In conclusion, through the compensator (5), which we call a *practical internal model*, one can formulate the problem of calculating the stable inverse (x^r, u^r) as that of estimating some time derivatives of r (the vector y_{X_1}) and then inverting the mapping $\mathcal{H}_X(X_1)$. Notice that the practical internal model is not *directly* implemented, as it is only used for estimation purposes.

We now turn our attention to the disturbance-free plant (2) and augment it with a compensator with identical structure to (5)

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + \zeta_3 \\ \dot{\zeta}_2 &= v_1 \\ \dot{\zeta}_3 &= v_2 \\ \tilde{u} &= \text{col}(\zeta_1, \zeta_2). \end{aligned} \tag{6}$$

Define $X_2 \overset{\triangle}{=} \text{col}(x, \zeta)$ and note that since the augmented system (2), (6) has the same structure as (3), (5), its observability mapping is given by $y_{X_2} \overset{\triangle}{=} \text{col}(y_1, \dot{y}_1, \ddot{y}_1, \dddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2) = \mathcal{H}_X(X_2)$. Since $\mathcal{H}_X(X_2)$ is a diffeomorphism, we conclude that from y and its time derivatives one gets $X_2 = \text{col}(x, \zeta) = \mathcal{H}_X^{-1}(y_{X_2})$ and hence also $\tilde{u} = \text{col}(\zeta_1, \zeta_2)$. Recalling that $\tilde{u} = u + \Delta$, estimating $\tilde{u}$ is equivalent to estimating Δ as $\Delta = \text{col}(\zeta_1, \zeta_2) - u$. Summarizing our observations so far, using two practical internal models and estimating the time derivatives of r (the vector y_{X_1}) and y (the vector y_{X_2}), one can estimate the stable inverse of the system, the state of the plant, and the disturbance. Unfortunately, however, such estimates cannot be employed in the feedback controller (4) because the vector relative degree of the disturbance-free system (2) is $\{2, 1\}$ while the number of time derivatives of its output that need to be estimated is $\{3, 2\}$, and thus the resulting closed-loop system would not be proper. Due to this observation, the methodology presented in [4] is *incorrect*. To address this problem in this paper we employ an input dynamic extension with the property that the relative degree of the extended system is equal to the relative degree of the system augmented with a practical internal model (in this example, $\{3, 2\}$). Such an input dynamic extension cannot always be found, however we show that under suitable conditions it can take the form of chains of integrators or linearizing compensators.

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II. Problem Statement and Assumptions

Given the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u, \Delta(t)) \\ y &= h(x), \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurable output, and $\Delta(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is an unknown smooth function of its arguments which is bounded with bounded derivatives, we seek to find a tracking controller solving the following problem

Problem 1 (Output Feedback Practical Tracking): *Given the dynamical system (7) and a sufficiently smooth reference trajectory $r(t) = col(r_1(t), \ldots, r_m(t))$, design a dynamic output feedback controller*

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r) \\ u &= h_c(x_c, y) \end{aligned} \tag{8}$$

where f_c and h_c are sufficiently smooth, such that the closed-loop system (7)-(8) has the property that there exists *a $T > 0$ such that $\|e(t)\| \leq e_0$ for all $t \geq T$, and such that the internal states x and x_c are bounded for all $t \geq 0$, and for all initial conditions $(x(0), x_c(0)) \in \mathcal{A}$, for some closed set $\mathcal{A}$.*

In [1], we have showed that, when no uncertainty affects the system, if there exists a practical internal model then Problem 1 has a solution. We start by assuming that the uncertainty $\Delta(t)$ satisfies a matching condition.

Assumption A1 (Matching Condition): There exists a smooth function $m(x, u, \Delta(t)) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that $m(x, u, 0) = u$ and, setting $\tilde{u} = m(x, u, \Delta(t))$, (7) can be rewritten as

$$\begin{aligned} \dot{x} &= f(x, \tilde{u}, 0) \\ y &= h(x), \end{aligned} \tag{9}$$

and the function $m(x, u, \Delta)$ is a diffeomorphism with respect to its second and third argument, i.e., there exist smooth functions $m_\Delta^{-1}(x, u, \tilde{u})$ and $m_u^{-1}(x, \tilde{u}, \Delta)$ such that

$$\Delta = m_\Delta^{-1}(x, u, \tilde{u}), \; u = m_u^{-1}(x, \tilde{u}, \Delta). \tag{10}$$

This assumption is rather restrictive. The new plant (9) obtained using A1 and letting $\tilde{u}$ be the new control input is free of disturbance (however, since $\Delta(t)$ is not known, $\tilde{u}$ cannot be freely assigned). In what follows we make additional assumptions allowing us to define a controller $\tilde{u}$ to solve Problem 1 for the disturbance-free plant (9). This, together with the estimation of $\Delta(t)$, will allow us to derive a controller for the original plant (7).

The following is a basic requirement for the solution of the tracking problem (see [6]).

Assumption A2 (Stable Inverse): Given $r(t)$, there exist sufficiently smooth and bounded functions $x^r(t)$ and $u^r(t)$ such that

$$\begin{aligned} \dot{x}^r(t) &= f(x^r(t), u^r(t), 0) \\ r(t) &= h(x^r(t)) \end{aligned} \tag{11}$$

for some initial condition $x^r(0), u^r(0)$, and all $t \geq 0$.

Consider the change of coordinates $\tilde{x} = x - x^r(t)$, rewrite (9) in new coordinates as

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, \tilde{u}), \tag{12}$$

and notice that the asymptotic stability of the origin of (12) is equivalent to the stability of the trajectory $x^r(t)$. We now introduce a condition to estimate the functions $x^r(t)$ and $u^r(t)$ on-line. It is useful to think of (11) as a copy of the plant with unknown state x^r, unknown input u^r, but a known output which is the reference trajectory $r(t)$. Consider a compensator of the type

$$\begin{aligned} \dot{\zeta}^r &= a(\zeta^r, x^r, v^r) \\ u^r &= b(\zeta^r, x^r), \end{aligned} \tag{13}$$

where $\zeta^r \in \mathbb{R}^q$ ($q \geq m$), $v^r \in \mathbb{R}^m$, a and b are sufficiently smooth, and v^r is the new input of the composite system (11),(13). Let $X_1 = \text{col}(x^r, \zeta^r)$ and rewrite (11), (13) as

$$\begin{aligned} \dot{X}_1 &= F(X_1, v^r) \\ r &= H(X_1) \end{aligned} \tag{14}$$

(with obvious definition of F and H). Define the observability mapping associated with X_1 in (14) as

$$\begin{aligned} y_{X_1} &\triangleq \text{col}\left(r_1, \ldots, r_1^{(\bar{k}_1-1)}, \ldots, r_m, \ldots, r_m^{(\bar{k}_m-1)}\right) \\ &\triangleq \mathcal{H}_X\left(X_1, v^r, \ldots, (v^r)^{(\bar{n}_u-1)}\right), \end{aligned}$$

where $\sum_{i=1}^p \bar{k}_i = n+q$, $0 \leq \bar{n}_u \leq \max\{\bar{k}_1, \ldots, \bar{k}_m\} - 1$.

Assumption A3 (Practical Internal Model [1]): There exists a compensator of the form (13), which we call a *practical internal model*, which is regular (i.e., for each $x(0)$ and $u(t)$ there exist $\zeta(0)$ and $v(t)$ such that $b(\zeta(t), x(t)) = u(t)$, for all $t \geq 0$) and such that the following two properties hold for the composite system (11), (13).

(i) $\mathcal{H}_X$ does not depend on v^r and its derivatives, i.e., $\mathcal{H}_X = \mathcal{H}_X(X_1)$.

(ii) There exists a set of indices $\{\bar{k}_1, \ldots, \bar{k}_m\}$ and a set $\mathcal{X} \subset \mathbb{R}^{n+q}$ such that the mapping $\mathcal{H}_X : \mathcal{X} \to \mathcal{H}_X(\mathcal{X})$ defined by $y_{X_1} = \mathcal{H}_X(X_1)$ is a diffeomorphism.

Notice that, by replacing x^r, ζ^r, u^r, and v^r in (11), (13) by x, ζ, $\tilde{u} = m(x, u, \Delta(t))$, and v, we get an observability assumption for (9) augmented with a practical internal model with state ζ and input v. Thus, letting $X_2 = \text{col}(x, \zeta)$, the dynamics of the two augmented systems can be written as

$$\begin{aligned} \dot{X}_i &= F(X_i, v^i) \\ y^i &= H(X_i), \; i = 1, 2, \end{aligned} \tag{15}$$

where $v^i = v^r$, $v^2 = v$, $y^1 = r = H(X_1)$, $y^2 = y = H(X_2)$. A3 guarantees that from y^i, $i = 1, 2$, and its time derivatives (i.e., the vectors $y_{X_1} = \text{col}\left(r_1, \ldots, r_1^{(\bar{k}_1-1)}, \ldots, r_m, \ldots, r_m^{(\bar{k}_m-1)}\right)$, $y_{X_2} = \text{col}\left(y_1, \ldots, y_1^{(\bar{k}_1-1)}, \ldots, y_m, \ldots, y_m^{(\bar{k}_m-1)}\right)$) one can get $X_1 = (x^r, \zeta^r)$ and $X_2 = (x, \zeta)$, respectively, and thus also $u^r = b(\zeta^r, x^r)$ and $\tilde{u} = b(\zeta, x)$. We will use this fact, together with A1, to estimate x and $\Delta(t)$. We stress that the two practical internal models with state ζ^r and ζ are not *directly* implemented. Rather, they are used to define estimators for x and x^r.

Assumption A4 (Input Dynamic Extension): There exists a compensator

$$\begin{aligned} \dot{\xi} &= c(\xi, x, w) \\ u &= d(\xi, x) \end{aligned}, \quad \xi \in \mathbb{R}^{q'}, q' \geq q. \tag{16}$$

where $w \in \mathbb{R}^m$ is the new control input, such that

(i) *(Compensator Relative Degree).* The augmented system

$$\begin{aligned} \dot{x} &= f(x, d(\xi, x), \Delta) \\ \dot{\xi} &= c(\xi, x, w) \\ y &= h(x) \end{aligned} \tag{17}$$

has the property that y_{X_2}, calculated along the vector field (17), does not depend on w.

(ii) *(Information Vector).* For any $\vartheta \in (0, 1)$ there exist a smooth function $\bar{w}(x^r, \zeta^r, x, \zeta, \xi) = \bar{w}(X_1, X_2, \xi)$, a positive integer n_Δ, a smooth function $\gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)})$, a C^1 function $V(\tilde{x}, \tilde{\xi}) : \tilde{D} \to \mathbb{R}^+$, with $\tilde{\xi} = \xi - \gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)})$, and a real number $c^* \geq 1$ such that $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q'} \,|\, V(\tilde{x}, \tilde{\xi}) \leq c^*\}$ is a compact subset of $\tilde{D}$ and the time derivative of V along the trajectories of

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, m(x, d(\xi, x), \Delta)) \\ \dot{\tilde{\xi}} &= c(\xi, x, \bar{w}(X_1, X_2, \xi)) - \dot{\gamma}. \end{aligned} \tag{18}$$

satisfies $\dot{V} \leq -\Phi(\tilde{x}, \tilde{\xi})$, where $\Phi(\tilde{x}, \tilde{\xi})$ is continuous on $\tilde{D}$ and positive definite on the set $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q'} \,|\, \vartheta \leq V(\tilde{x}, \tilde{\xi}) \leq c^*\}$.

Part (ii) of this assumption, derived from Assumption ULP in [7], implies that the smooth feedback $\bar{w}(x^r, \zeta^r, x, \zeta, \xi)$ practically stabilizes the origin of (18) and the set $\{(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^{q'} \,|\, V(\tilde{x}, \tilde{\xi}) \leq c^*\}$ is included in its domain of attraction. Moreover, it requires that the information needed to do so is contained in the vector $(x^r, \zeta^r, x, \zeta, \xi) = (X_1, X_2, \xi)$. This is useful because from A3 one can estimate X_1 and X_2 from r and y, respectively, while ξ being the state of the controller is available for feedback. Thus, A3 and A4 allow to use a separation principle to solve Problem 1. The existence of $\gamma(X_1, \Delta, \ldots, \Delta^{(n_\Delta)})$ ensures that the boundedness of X_1 and $\tilde{\xi}$ implies the boundedness of ξ. In the next section we specify two classes of compensators satisfying A4.

Next, we need to guarantee that the reference trajectory is contained in within an observable region.

Assumption A5 (Reference Trajectory): The reference trajectory $r(t)$ is such that, for all $t \geq 0$, $y_{X_1} \in \mathcal{C}_1 \subset \mathcal{H}_X(\mathcal{X})$, for some convex compact set $\mathcal{C}_1$ with boundary $\partial\mathcal{C}_1 = \{X_1 \in \mathbb{R}^{n+q} \,|\, g^1(X_1) = 0\}$, where $g^1 : \mathbb{R}^{n+q} \to \mathbb{R}$ is a C^1 function for which 0 is a regular value, i.e., $\forall X_1 \in \partial\mathcal{C}_1$, $\partial g/\partial X_1 \neq 0$.

Notice that A3 and A5 imply A2 which, therefore, is redundant and is introduced solely for the sake of illustration.

We now use V to characterize a set which is positively invariant and is contained in within the observable set $\mathcal{X}$ of X_2. This puts a constraint on the topology of the set $\mathcal{X}$. First recall that, from A2 and A3, $x^r(t)$ and $\zeta^r(t)$, and thus $X_1(t)$, are bounded functions of time. For any positive real number $c \leq c^*$, let $\Omega_c = \{(x, \xi) \in \mathbb{R}^{n+q'} \,|\, V(\tilde{x}, \tilde{\xi}) \leq c\}$. Since $\Delta(t)$ and its time derivatives are uniformly bounded, $\gamma(X_1(t), \Delta(t), \ldots, \Delta^{(n_\Delta)}(t))$ is also uniformly bounded, and thus, by the definition of $\tilde{\xi}$ in A4, Ω_c is a bounded set. From the definition of $n_1, \ldots, n_m$, we have that

$$(x, \xi) \text{ bounded } \overset{\text{A4}}{\Longrightarrow} y_{X_2} \text{ bounded } \overset{\text{A3}}{\Longleftrightarrow} X_2 \text{ bounded}$$

that is, there exists a bounded set $\Sigma_c \subset \mathbb{R}^{n+q}$ such that $(x, \xi) \in \Omega_c \quad \Rightarrow \quad X_2 \in \Sigma_c$.

Assumption A6 (Topology of $\mathcal{X}$): There exists a positive scalar $\bar{c} \leq c^*$ such that $\mathcal{H}_X(\Sigma_{\bar{c}}) \subset \mathcal{C}_2 \subset \mathcal{H}_X(\mathcal{X})$, for some convex compact $\mathcal{C}_2$ with boundary $\partial\mathcal{C}_2 = \{X_2 \in \mathbb{R}^{n+q} \,|\, g^2(X_2) = 0\}$, where $g^2 : \mathbb{R}^{n+q} \to \mathbb{R}$ is a C^1 function for which 0 is a regular value.

III. Compensator Choice

In this section we focus our attention on two classes of compensators, namely chains of integrators and linearizing compensators. In both cases we provide sufficient conditions for A4 to be satisfied and a constructive procedure to find the feedback controller $\bar{w}(X_1, X_2, \xi)$.

A. Chains of Integrators

The main idea in this section is illustrated in the following example.

Example 2 Go back to Example 1 and recall that, setting $\tilde{u} = \bar{\tilde{u}}(x, x^r, u^r)$ (with $\bar{\tilde{u}}(x, x^r, u^r)$ defined in (4)), we have that the origin $\tilde{x} = 0$ of the error dynamics is globally uniformly asymptotically stable. Recall further that if we express the stable inverse (x^r, u^r), the disturbance $\Delta(t)$, and the state x by means of y_{X_1} and y_{X_2}, such expressions cannot be used to control (1) because the vector relative degree of (1) is $\{2, 1\}$, while the number of time derivatives of its output that need to be estimated is $\{3, 2\}$. As we argued in Example 1, this would result in a non-proper closed-loop system. To address this problem the most obvious choice for dynamic extension is two chains of integrators of length $\{2, 1\}$, yielding the extended system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + \xi_1^1 + \Delta_1(t) \\ \dot{x}_3 &= x_4 - x_1^2 - \xi_1^1 - \Delta_1(t) \\ \dot{x}_4 &= \xi_1^2 + \Delta_2(t) \\ \dot{\xi}_1^1 &= \xi_2^1, \qquad \dot{\xi}_2^1 = w_1 \\ \dot{\xi}_1^2 &= w_2 \\ y &= \mathrm{col}(x_1, x_3), \end{aligned} \tag{19}$$

where $\xi = \mathrm{col}(\xi_1^1, \xi_2^1, \xi_1^2)$ is the state of the dynamic extension and w is the new control input. Indeed, notice that y_{X_2} calculated along the vector field (19) is independent of the control input w,

$$\begin{aligned} y_{X_2} &= \mathrm{col}(y_1, \dot{y}_1, \ddot{y}_1, \dddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2) \\ &= \mathrm{col}\Big(x_1, x_2, x_1^2 + \xi_1^1 + \Delta_1, 2x_1x_2 + \xi_2^1 + \dot{\Delta}_1, x_3, \\ &\quad x_4 - x_1^2 - \xi_1^1 - \Delta_1, \xi_1^2 + \Delta_2 - 2x_1x_2 - \xi_2^1 - \dot{\Delta}_1\Big). \end{aligned}$$

We can now seek a controller for the extended system that employs y_{X_1}, y_{X_2}, and ξ as feedback variables, resulting in a proper closed-loop system. This is desirable in our framework because y_{X_1} and y_{X_2} can be easily estimated from r and y, respectively, while ξ being the state of the dynamic extension is available for feedback. Since $X_1 = \mathcal{H}_X^{-1}(y_{X_1})$ and $X_2 = \mathcal{H}_X^{-1}(y_{X_2})$ ($X_1 = \mathrm{col}(x^r, \zeta^r)$ and $X_2 = \mathrm{col}(x, \zeta)$), where $\mathcal{H}_X$ is a diffeomorphism, we equivalently seek a controller that is a function of (X_1, X_2, ξ). Later, in Theorem 1 we estimate X_1 and X_2 from r and y *without using the inverse* $\mathcal{H}_X^{-1}$. Let $\bar{u}(x, x^r, u^r, \Delta) = \bar{\tilde{u}}(x, x^r, u^r) - \Delta(t) =$

$\mathrm{col}\left(u_1^r+(x_1^r)^2-x_1^2+K_1{}^\top(x-x^r), u_2^r+K_2{}^\top(x-x^r)\right)$ $-\Delta$, so that, if $u=\bar{u}$ in (1), $\tilde{x}=0$ is globally uniformly asymptotically stable. Use the fact that $u^r=\mathrm{col}(\zeta_1^r,\zeta_2^r)$, $\Delta=\mathrm{col}(\zeta_1,\zeta_2)-u=\mathrm{col}(\zeta_1,\zeta_2)-\mathrm{col}(\xi_1^1,\xi_1^2)$, to get

$$\begin{aligned}\bar{u}=\mathrm{col}\big(&\zeta_1^r+(x_1^r)^2-x_1^2+K_1{}^\top(x-x^r)-\zeta_1+\xi_1^1,\\ &\zeta_2^r+K_2{}^\top(x-x^r)-\zeta_2+\xi_1^2\big)\,.\end{aligned} \tag{20}$$

Clearly, if $\dot{\bar{u}}_1,\ \ddot{\bar{u}}_1,\ \dot{\bar{u}}_2$, calculated along the vector fields (3), (5), (2), (6), and (19), could also be expressed as functions of (X_1,X_2,ξ), then by using integrator backstepping one could derive a feedback controller $\bar{w}(X_1,X_2,\xi)$ that globally uniformly asymptotically stabilizes the equilibrium $(x,\xi)=(x^r,\bar{u}_1,\dot{\bar{u}}_1,\bar{u}_2)$ of the extended dynamics (19). However, this is not the case, as it is easily seen that $\ddot{\bar{u}}_1,\ \dot{\bar{u}}_2$ depend on the inputs v^r, v, and w. Integrator backstepping can thus be applied only to the first integrator of the first chain, ξ_1^1, since $\dot{\bar{u}}_1$ *can* be expressed as a function of (X_1,X_2,ξ). For the remaining two integrators at the end of each chain, ξ_2^1 and ξ_1^2, one can resort to high-gain feedback to get a controller $\bar{w}(X_1,X_2,\xi)$ at the expense of loosing asymptotic stability of the equilibrium $(x,\xi)=(x^r,\bar{u}_1,\dot{\bar{u}}_1,\bar{u}_2)$ and achieving instead *practical stability*, i.e., regulation to an arbitrarily small residual set around $(x,\xi)=(x^r,\bar{u}_1,\dot{\bar{u}}_1,\bar{u}_2)$. This idea, which is the basis of the design developed in this section, is formalized in Lemma 1.

$\triangle$

In formalizing this idea, we begin by assuming that, at least in the ideal case when x, x^r, and u^r are available for feedback, there exists a smooth controller that uniformly asymptotically stabilizes the origin of (12).

Assumption A7 (Stabilizability of the Trajectory $x^r(t)$): There exist a smooth function $\bar{\bar{u}}(x,x^r,u^r)$, a C^1 function $V'(\tilde{x})$, $V':\tilde{D}'\to\mathbb{R}^+$, and a real number $c'\geq 1$ such that $\bar{\bar{u}}(x^r,x^r,u^r)=u^r$, $\{\tilde{x}\in\mathbb{R}^n\,|\,V'(\tilde{x})\leq c'\}$ is a compact subset of $\tilde{D}'$, and the time derivative of V' along the trajectories of

$$\dot{\tilde{x}}=\tilde{f}(t,\tilde{x},\bar{\bar{u}}(x,x^r,u^r))$$

satisfies

$$\dot{V}'\leq-\Phi'(\tilde{x}),$$

where $\Phi'(\tilde{x})$ is continuous on $\tilde{D}'$ and positive definite on the set $\{\tilde{x}\in\mathbb{R}^n\,|\,V'(\tilde{x})\leq c'\}$.

Next, consider (7) and let $n_1,\ldots,n_m$ be the number of time derivatives of $u_1,\ldots,u_m$, respectively, appearing in $y_{X_2}=y_1^{(\bar{k}_1-1)},\ldots,y_m^{(\bar{k}_m-1)}$ (if u_j does not appear in $y_1^{(\bar{k}_1-1)},\ldots,y_m^{(\bar{k}_m-1)}$ we set $n_j=0$). Consider the following choice for the compensator (16)

$$\begin{aligned}&\dot{\xi}=A_c\xi+B_cw,\quad \xi\in\mathbb{R}^{q'},q'=n_1+\ldots,n_m\\ &u=C_c\xi\end{aligned} \tag{21}$$

where the triple (A_c,B_c,C_c) is in controllable/observable canonical form with eigenvalues at zero. The compensator (21) is given by m chains of integrators - one chain for every input channel u_i - of order $n_1,\ldots,n_m$, respectively.

Lemma 1 *Assume that A7 holds and that, for the system with outputs $z^1,z^2\in\mathbb{R}^m$*

$$\begin{aligned}&\dot{X}_1=F(X_1,v)\\ &\dot{X}_2=F(X_2,v^r)\\ &\dot{\xi}=A_c\xi+B_c\bar{w}\\ &z^1=\alpha_1(X_1,X_2,\xi),\quad z^2=\alpha_2(X_1,\Delta),\end{aligned} \tag{22}$$

where

$$\begin{aligned}\alpha_1(X_1,X_2,\xi)=m_u^{-1}\big(&x,\bar{\bar{u}}(x,x^r,b(x^r,\zeta^r)),\\ &m_\Delta^{-1}(x,C_c\xi,b(\zeta,x))\big)\\ \alpha_2(X_1,\Delta)=m_u^{-1}(&x^r,b(\zeta^r,x^r),\Delta),\end{aligned}$$

the output derivatives $[z_1^j,\ldots,(z_1^j)^{(n_1-1)},\ldots,z_m^j,\ldots,(z_m^j)^{(n_m-1)}]$, $j=1,2$ calculated along the vector field of (22), do not depend on v, v^r, and w. Then (21) satisfies A4.

B. Linearizing Compensators

Assume that (9) is affine in the input, i.e., it reads as

$$\begin{aligned}&\dot{x}=f_1(x)+f_2(x)\tilde{u}=f_1(x)+f_2(x)m(x,u,\Delta)\\ &y=h(x).\end{aligned} \tag{23}$$

Assume further that (23), viewed as a system with input $\tilde{u}$, is dynamic feedback linearizable (differentially flat), i.e., there exists a *linearizing* compensator

$$\begin{aligned}&\dot{\xi}=c'(\xi,x,w),\qquad \xi\in\mathbb{R}^r\\ &\tilde{u}=d'(\xi,x,w)\end{aligned} \tag{24}$$

such that the plant augmented with such compensator yields the trivial system in output coordinates:

$$y_i^{(k_i)}=w_i,\quad k_1+\ldots+k_m=n+r.$$

As shown in [2], a practical internal model satisfying A3 is given by (24) augmented with m integrators at the input side[1] (one for each input channel). Viewing (23) as a system with input u, consider the following dynamic extension

$$\begin{aligned}&\dot{x}=f_1(x)+f_2(x)m(x,d(\xi,x),\Delta)\\ &\dot{\xi}=c(\xi,x,w),\quad \xi\in\mathbb{R}^q\\ &y=h(x),\end{aligned} \tag{25}$$

where $q=r+m$ and $(c(\cdot,\cdot,\cdot),d(\cdot,\cdot))$ denote the vector field and output function of the augmented compensator above (clearly here, referring to A3, $(a(\cdot,\cdot,\cdot),b(\cdot,\cdot))=(c(\cdot,\cdot,\cdot),d(\cdot,\cdot)))$. From the dynamic feedback linearizability property and the fact that $m(x,u,0)=u$, when $\Delta=0$ we have a well-defined vector relative degree $\{k_1+1,\ldots,k_m+1\}$. Additionally, when $\Delta=0$, A3 ensures that $(x,\xi)=\mathcal{H}_X^{-1}(y_{X_2})$, and thus in particular ξ can be expressed as a function of y_{X_2}. In Lemma 2

[1] The integrators are not needed when d' is independent of w, i.e., when $d'=d'(\xi,x)$.

$$
\dot{\hat{X}}_i^P = \begin{cases} \left[\dfrac{\partial \mathcal{H}_X}{\partial \hat{X}_i^P}\right]^{-1} \left\{ (L_{\hat{F}}\, \mathcal{H}_X)^i - \Gamma^i \dfrac{N^i(\hat{y}_{X_i}) L_G\, g^i}{N^i(\hat{y}_{X_i})^\top \Gamma^i N^i(\hat{y}_{X_i})} \right\} & \text{if } L_G\, g^i \geq 0 \\ & \text{and } \hat{y}_{X_i} \in \partial \mathcal{C}^i \\ \hat{F}(\hat{X}_i, y^i) = F(\hat{X}_i, 0) + \left[\dfrac{\partial \mathcal{H}_X(\hat{X}_i)}{\partial \hat{X}_i}\right]^{-1} (\mathcal{E}^i)^{-1} L^i \left(y^i - H(\hat{X}_i)\right) & \text{otherwise} \end{cases} \tag{29}
$$

$\hat{y}_{X_i} = \mathcal{H}_X(\hat{X}_i^P)$	$(L_{\hat{F}}\, \mathcal{H}_X)^i = \frac{\partial \mathcal{H}_X}{\partial \hat{X}_i^P} \hat{F}(\hat{X}_i^P, y^i)$ $G(\hat{X}_i^P, v^i) = (L_{\hat{F}}\, \mathcal{H}_X)^i$	$L_G\, g^i = \frac{\partial g^i}{\partial \hat{y}_{X_i}} G(\hat{X}_i^P, v^i)$
$N^i(\hat{y}_{X_i}) = \left(\frac{\partial g^i(\hat{y}_{X_i})}{\partial \hat{y}_{X_i}}\right)^\top$	$\mathcal{E}^i = block\text{-}diag[\mathcal{E}_1^i, \ldots, \mathcal{E}_p^i]$ $\mathcal{E}_j^i = diag[\rho_i, \ldots, \rho_i^{\bar{k}_j}]$	$L^i = block\text{-}diag[L_1^i, \ldots, L_p^i]$ L_j^i Hurwitz $(\bar{k}_j \times 1)$
$\Gamma^i = (S^i \bar{\mathcal{E}}^i)^{-1} (S^i \bar{\mathcal{E}}^i)^{-\top}$	$\mathcal{E}^i = block\text{-}diag[\mathcal{E}_1^i, \ldots, \mathcal{E}_p^i]$ $\bar{\mathcal{E}}_j^i = 1/\rho_i^{\bar{k}_j}\ \mathcal{E}_j^i$	$S^i = (P^i)^{1/2}$ P^i satisfies:

$$
A^{i\top} P^i + P^i A^i = -I_{(n+q)\times(n+q)}
$$

$$
A^i = \begin{bmatrix} 0_{(n+q-1)\times 1} & I_{(n+q-1)\times(n+q-1)} \\ \multicolumn{2}{c}{0_{1\times(n+q)}} \end{bmatrix} - L^i [1, 0_{1\times n+q-1}]
$$

TABLE I

DEFINITIONS OF VARIOUS PARAMETERS IN OBSERVER (29).

we show that if the two properties above are preserved when Δ and its derivatives are not zero, then besides providing a practical internal model satisfying A3, the pair $(c(\cdot,\cdot,\cdot), d(\cdot,\cdot))$ is a valid input dynamic extension fulfilling the requirements in A4.

Lemma 2 *Assume that (23) is dynamic feedback linearizable. If there exist a smooth function φ and a positive integer n_Δ such that*

$$
\xi = \varphi(y_{X_2}, \Delta, \ldots, \Delta^{(n_\Delta)}) \tag{26}
$$

and

$$
y_i^{(k_i+1)} = w_i + g_i(x, \xi, \Delta, \ldots, \Delta^{(k_i)}), \quad i = 1, \ldots, m, \tag{27}
$$

with smooth g_i vanishing when $(\Delta, \ldots, \Delta^{(k_i)}) = (0, \ldots, 0)$, then the pair $(c(\cdot,\cdot,\cdot), d(\cdot,\cdot))$ satisfies A4.

IV. SOLUTION TO THE PRACTICAL OUTPUT TRACKING PROBLEM

In this section we solve Problem 1 using the separation principle in [8]. Consider the dynamic output feedback controller

$$
\begin{aligned} \dot{\xi} &= c(\xi, \hat{x}^P, \bar{w}(\hat{X}_1^P, \hat{X}_2^P, \xi)) \\ u &= d(\xi, \hat{x}^P), \end{aligned} \tag{28}
$$

where $X_1^P = \text{col}((\hat{x}^r)^P, (\hat{\zeta}^r)^P)$, $X_2^P = \text{col}(\hat{x}^P, \hat{\zeta}^P)$ are given in (29), for $i = 1, 2$, and various parameters are defined in Table I. The estimator (29) incorporates a high-gain component to guarantee convergence, and a dynamic projection to avoid peaking and confine the estimator state to within the observable region $\mathcal{X}$ (see [8] for more details).

Theorem 1 *Suppose that A1-A6 hold. Then, for any smooth bounded $\Delta(t)$ with bounded derivatives, (28), (29) solve Problem 1 on a compact set $\mathcal{A}$ whose size depends on c^* and the sets $\mathcal{C}_1$ and $\mathcal{C}_2$. If A4 holds for arbitrarily large c^* and a radially unbounded V, and A3 holds globally (i.e., $\mathcal{X} = \mathbb{R}^{n+q}$) with $\mathcal{H}_X(\mathbb{R}^{n+q})$ a convex set, then the solution of Problem 1 is semiglobal in that $\mathcal{A}$ can be chosen to be an arbitrarily large compact set.*

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