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Brief paper

Distributed Nash equilibrium seeking: A gossip-based algorithm[☆]

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ABSTRACT

This paper presents an asynchronous gossip-based algorithm for finding a Nash equilibrium (NE) of a game in a distributed multi-player network. The algorithm is designed in such a way that players make decisions based on estimates of the other players' actions obtained from local neighbors. Using a set of standard assumptions on the cost functions and communication graph, the paper proves almost sure convergence to a NE for diminishing step sizes. For constant step sizes an error bound on expected distance from a NE is established. The effectiveness of the proposed algorithm is demonstrated via simulation for both diminishing and constant step sizes.

1. Introduction

Finding a NE in a distributed multi-player network game is a problem that has received increasing attention in recent years. Many important real-world applications in wired and wireless networks involve such a setup (Chen & Huang, 2012; Stankovic, Johansson, & Stipanovic, 2012; Yin, Shanbhag, & Mehta, 2011). Peer-to-peer (P2P) and mobile ad-hoc networks are two examples among many. In this problem each player pursues minimization of his cost function selfishly by taking an action in response to other players' actions. This requires full information on all other players' actions in the network. However, this is a stringent requirement in a distributed network. Players have to minimize their cost functions based on limited local information from the neighboring players.

Our goal is to design a locally distributed algorithm to find a NE in a networked continuous kernel game. In such a game, all the players share their information locally and update their actions in order to minimize their cost functions according to the limited information.

Literature review. Our work is related to the literature on Nash games (Alpcan & Başar, 2005; Yin et al., 2011). Distributed algorithms for computing NE have recently drawn significant attention due to a wide range of applications, to name only a few (Alpcan & Başar, 2005; Frihauf, Krstic, & Basar, 2012; Pan & Pavel, 2009; Pavel, 2007). In Kannan and Shanbhag (2010), an iterative regularization algorithm is studied for monotone game. A distributed algorithm for a class of generalized games is proposed in Zhu and Frazzoli (2012) which studies convergence to a NE for a *complete communication graph*. The paper (Gharesifard & Cortes, 2013) considers a distributed algorithm for NE seeking in a *two-network zero-sum game*. A new systematic methodology is presented in Li and Marden (2013) to find distributed algorithms for games with local-agent utility functions (proved to be state-based *potential games*). The algorithms are designed to be dependent on information from only a set of local neighboring agents. The authors in Bramoull, Kranton, and D'Amours (2014) generalize the problem of finding NE (in special games such as those involving strategic innovation, public goods, and social interactions) to the case in which players are considered to be linked if their payoffs are directly affected by the action of the others. A distributed learning algorithm is proposed in Chen and Huang (2012) for finding NE in a *spatial spectrum access game*, albeit for games with finite action spaces. In Gharehshiran, Krishnamurthy, and Yin (2013), regret-based reinforcement learning algorithms have been developed for equilibrium seeking over networks in finite action games. A fictitious play-based approach has been proposed in Swenson, Kar, and Xavier (2012), in which an average empirical distribution is tracked. In Wang et al. (2013) a distributed consensus protocol was proposed for finding a Nash equilibrium of a *congestion game*.

Gossip-based communication has been widely used in asynchronous algorithms due to simplicity and applicability particularly for distributed optimization (Lee & Nedic, 2016; Ram, Nedić, & Veeravalli, 2010). In Koshal, Nedic, and Shanbhag (2012), a gossip-based algorithm has been designed for finding a NE in *aggregative*

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games. Since the algorithm is designed for aggregative games, there is no need to estimate the other players' actions. However, in a general game the aggregate of the players' actions is not enough to update players' actions.

Contributions. Inspired by (Koshal et al., 2012), we propose an asynchronous gossip-based algorithm for a larger class of games. In the proposed algorithm each player maintains an estimate vector of the other players' actions. A communication protocol is designed for sharing estimates and actions between the local players so that they update their estimates and actions. In contrast to Koshal et al. (2012), in which the players take average of the scalar aggregate estimate including their own action, our algorithm excludes the players' actions from their estimates. This exclusion is appropriate in a general game, but it precludes exploiting doubly stochastic properties in the gossiping step. However, we overcome this drawback by using an extra intermediary variable.

In general, convergence properties of distributed algorithms depend on the selection of step sizes (Blatt, Hero, & Gauchman, 2007; Kvaternik & Pavel, 2011; Nedic & Ozdaglar, 2009). Diminishing step sizes typically lead as close as possible to the optimal point (Nedic, 2011), but can have slow convergence, while constant step sizes cause a fluctuation around the optimal point (Zhang, Zheng, & Chiang, 2008). The choice of step size usually demands a trade off between convergence speed and accuracy of convergence. A preliminary version of this work treating only diminishing step sizes has appeared in Salehisadaghiani and Pavel (2014). In this paper we consider both diminishing as well as constant step sizes. Using a set of standard assumptions on the cost functions and communication graph, for diminishing step sizes we prove almost sure (a.s.) convergence toward a NE of the game. For constant step sizes we establish an error bound on the expected distance from the NE.

The paper is organized as follows. In Section 2, the problem statement and assumptions are provided. An asynchronous gossip-based algorithm is proposed in Section 3. In Section 4, convergence of the algorithm with diminishing step sizes is discussed, while in Section 5 constant step sizes are considered. Simulation results are presented in Section 6 and conclusions in Section 7.

1.1. Notation

The $N \times N$ identity matrix and the $N \times 1$ vector of 1's are denoted by I_N and $\mathbf{1}_N$, respectively. We use e_i to denote a unit vector in $\mathbb{R}^N$ whose ith element is 1 and the others are 0. The limit superior of a sequence x_n is defined as $\limsup_{n\to\infty} x_n := \inf_{n\geq 0} \sup_{m\geq n} x_m$.

2. Problem statement

Consider a set of N players in a network specified by a *communication graph* $G_C(V, E_C)$ where $V = \{1, \dots, N\}$ denotes the set of players and $E_C \subset V \times V$ specifies the pairs of players that may communicate. The set of neighbors of player i in G_C, denoted by $N_C(i)$, is the set of vertices which are connected to vertex i by an edge, i.e., $N_C(i) := \{j \in V | (i, j) \in E_C\}$. For $i \in V$, $J_i : \Omega \to \mathbb{R}$ is the cost function of player i where $\Omega = \Omega_i \times \Omega_{-i} \subset \mathbb{R}^N$ is the action set of all players and $\Omega_i \subset \mathbb{R}$ is the action set of player i. The Nash game denoted by $\mathcal{G}(V, \Omega_i, J_i)$ is defined based on the set of players V, the action set Ω_i, $\forall i \in V$ and the cost function J_i, $\forall i \in V$. Let $x = (x_i, x_{-i}) \in \Omega$, with $x_i \in \Omega_i$, denote all players' actions. The cost function J_i depends on all (x_i, x_{-i}). The game is played such that for given $x_{-i} \in \Omega_{-i}$, each player i aims to minimize his own cost function selfishly to find an optimal action,

$$\begin{aligned} &\underset{y_i}{\text{minimize}} \quad J_i(y_i, x_{-i}) \\ &\text{subject to} \quad y_i \in \Omega_i. \end{aligned} \tag{1}$$

Note that the solution set of player i in (1), depends on the actions of the other players x_{-i}. We assume that the cost function J_i and the action set Ω are only available to player i, $i \in V$. Thus players are required to exchange some information to update their actions.

Assumption 1. The communication graph $G_C(V, E_C)$ is connected and undirected.

The connectivity assumption is critical in order to ensure that the information on each player is reached by all other players, infinitely often.

The NE of the game is defined as follows.

Definition 1. Consider an N-player game $\mathcal{G}(V, \Omega_i, J_i)$. A vector $x^* = (x_i^*, x_{-i}^*) \in \Omega$ is called a NE of this game if and only if,

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*) \quad \forall x_i \in \Omega_i,\ \forall i \in V. \tag{2}$$

We review next some basic results. A NE can be efficiently computed by solving the associated *Variational Inequality* (*VI*) problem.

Proposition 1 (*Proposition 1.4.2, Facchinei & Pang, 2003*)**.** *Let Ω_i be a closed convex subset of $\mathbb{R}$ for $i \in V$. Let also for $i \in V$, function $J_i(y_i, x_{-i})$ be convex and continuously differentiable in y_i for each fixed x_{-i}. Then a tuple $x^* = (x_i^*, x_{-i}^*)$ is a NE if and only if $x^* \in SOL(\Omega, F)$, where $SOL(\Omega, F)$ is the solution set of $VI(\Omega, F)$, $\Omega = \Omega_i \times \Omega_{-i}$ and $F(x)^T = [\nabla_{x_1}^T J_1(x), \dots, \nabla_{x_i}^T J_i(x), \dots, \nabla_{x_N}^T J_N(x)]$. ($F : \Omega \to \mathbb{R}^N$ is called a pseudo-gradient mapping.)*

Using Proposition 1, one can characterize a NE in terms of a *VI* problem as in the following lemma (Proposition 1.5.8, page 83 in Facchinei & Pang, 2003).

Lemma 1. *x^* is a NE of the game represented by (1) if and only if $x^* = T_\Omega[x^* - \alpha F(x^*)]$ for $\alpha > 0$, where $T_\Omega : \mathbb{R}^N \to \Omega$ is a Euclidean projection.*

In the following, we state a few assumptions including the existence and uniqueness conditions of a NE.

Assumption 2. The set Ω_i is non-empty, compact and convex subset of $\mathbb{R}$ for every $i \in V$. The cost function of player i, $J_i(x_i, x_{-i})$ is a continuously differentiable function in x_i for every $i \in V$. Also $J_i(x_i, x_{-i})$ is jointly continuous in x and convex in x_i for every x_{-i} and $i \in V$.

By Assumption 2, it follows that there exists $C > 0$ such that for all $i \in V$ and for all $x \in \Omega$,

$$\|\nabla_{x_i} J_i(x)\| \leq C. \tag{3}$$

Assumption 3. $F : \Omega \to \mathbb{R}^N$ is strictly monotone on Ω, i.e., $(F(x) - F(y))^T (x - y) > 0\ \forall x, y \in \Omega, x \neq y$.

Assumption 4. $\nabla_{x_i} J_i(x_i, u)$ is Lipschitz continuous in x_i (u), for every fixed $u \in \Omega_{-i}$ ($x_i \in \Omega_i$) and for every $i \in V$, that is, for some positive constant σ_i (L_i), $\|\nabla_{x_i} J_i(x_i, u) - \nabla_{x_i} J_i(y_i, u)\| \leq \sigma_i \|x_i - y_i\|$ $\forall x_i, y_i \in \Omega_i$ ($\|\nabla_{x_i} J_i(x_i, u) - \nabla_{x_i} J_i(x_i, z)\| \leq L_i \|u - z\| \forall u, z \in \Omega_{-i}$).

3. Asynchronous Gossip-based algorithm

We propose an asynchronous gossip-based algorithm to compute a NE of $\mathcal{G}(V, \Omega_i, J_i)$ over $G_C(V, E_C)$ using only partial information. As in Proposition 1, we obtain a NE by solving the associated *VI* problem using a projected gradient-based method.

In the algorithm players can build and maintain estimates of the other players' actions and locally communicate with the neighbors to exchange their estimates and update their actions.

1–***Initialization Step***

Each player i sets an initial *temporary* estimate vector $\tilde{x}^i(0) = [\tilde{x}_1^i(0), \ldots, \tilde{x}_N^i(0)]^T \in \Omega$, where $\tilde{x}^i(0) = [\tilde{x}_1^i(0), \ldots, \tilde{x}_j^i(0), \ldots, \tilde{x}_N^i(0)]^T \in \Omega$ with $\tilde{x}_j^i(0) \in \Omega_j$ is player i's initial temporary estimate of player j's action.

2–***Gossiping Step***

At the gossiping step, player i_k wakes up at the kth time interval $T(k)$ and finds a neighbor j_k with probability $p_{i_k j_k}$. Then they exchange their temporary estimate vectors and construct their estimate vectors $\hat{x}^i(k) = [\hat{x}_1^i(k), \ldots, \hat{x}_N^i(k)]^T \in \Omega$, $i \in \{i_k, j_k\}$ as follows:

$$\begin{cases} \hat{x}_{i_k}^{i_k}(k) = \tilde{x}_{i_k}^{i_k}(k) \\ \hat{x}_{-i_k}^{i_k}(k) = \dfrac{\tilde{x}_{-i_k}^{i_k}(k) + \tilde{x}_{-i_k}^{j_k}(k)}{2}, \end{cases} \tag{4}$$

and similarly for player j_k. Note that $\tilde{x}_i^i(k) = x_i(k)$ for all $i \in V$ in every iteration $T(k)$. For all other players $i \notin \{i_k, j_k\}$, the temporary estimate is maintained, i.e.,

$$\hat{x}^i(k) = \tilde{x}^i(k), \quad \forall i \notin \{i_k, j_k\}. \tag{5}$$

3–***Local Step***

Player i uses $\hat{x}^i(k)$ as his estimate of all other players' actions (due to imperfect information) and updates his action as follows: if $i \in \{i_k, j_k\}$,

$$x_i(k+1) = T_{\Omega_i}[x_i(k) - \alpha_{k,i} \nabla_{x_i} J_i(x_i(k), \hat{x}_{-i}^i(k))], \tag{6}$$

otherwise, $x_i(k+1) = x_i(k)$. In (6), $T_{\Omega_i} : \mathbb{R} \to \Omega_i$ is a Euclidean projection and $\alpha_{k,i}$ is a diminishing step size such that,

$$\sum_{k=1}^{\infty} \alpha_{k,i}^2 < \infty, \qquad \sum_{k=1}^{\infty} \alpha_{k,i} = \infty \quad \forall i \in V. \tag{7}$$

Note that $\alpha_{k,i}$ is inversely related to the number of updates that player i has made until time k which is denoted by $\nu_k(i)$ (i.e., $\alpha_{k,i} = \frac{1}{\nu_k(i)}$).

Let p_i denote the probability with which player i updates his action, then $p_i = \frac{1}{N} + \frac{1}{N} \sum_{j \in N_C(i)} p_{ij}$ $\forall i \in V$, where $p_{ij} > 0$ is the probability that players i and j contact each other.

Lemma 2 (Nedic, 2011). *Let Assumption 1 hold and $\alpha_{k,i} = \frac{1}{\nu_k(i)}$ for all k, i and let $q \in (0, 1/2)$. Let also $p_{\min} = \min_{i \in V} p_i$. Then $\exists \tilde{k} = \tilde{k}(q, N)$ such that for all $k \geq \tilde{k}$ and $i \in V$ a.s., $|\alpha_{k,i} - \frac{1}{kp_i}| \leq \frac{2}{k^{3/2-q} p_{\min}^2}$.*

After updating his action, player i updates his temporary estimate vector as,

$$\tilde{x}^i(k+1) = \hat{x}^i(k) + \Big(x_i(k+1) - x_i(k)\Big) e_i, \quad \forall i \in V. \tag{8}$$

In (8), for the case when $i \notin \{i_k, j_k\}$, we have $\tilde{x}^i(k+1) = \hat{x}^i(k)$, that is, the estimate vector of player i remains unchanged at the next iteration.

4. Convergence for diminishing step sizes

In this section we prove convergence of the algorithm for diminishing step size as in (7). Consider a memory in which the history of the decision making is recorded. Let $\mathcal{M}_k$ denote the *sigma-field* generated by the history up to time $k-1$ with $\mathcal{M}_0 = \mathcal{M}_1 = \{\tilde{x}^i(0), i \in V\}$ and $\mathcal{M}_k = \mathcal{M}_0 \cup \{(i_l, j_l); 1 \leq l \leq k-1\}$, $\forall k \geq 2$. In the following we use a well-known result on supermartingale convergence, (Lemma 11, Chapter 2.2, Polyak, 1987).

Lemma 3. *Let V_k, u_k, β_k and ζ_k be non-negative random variables adapted to σ-algebra $\mathcal{M}_k$. If $\sum_{k=0}^{\infty} u_k < \infty$, $\sum_{k=0}^{\infty} \beta_k < \infty$, and $\mathbb{E}[V_{k+1}|\mathcal{M}_k] \leq (1+u_k)V_k - \zeta_k + \beta_k$ for all $k \geq 0$, then V_k converges a.s. and $\sum_{k=0}^{\infty} \zeta_k < \infty$.*

Convergence is shown in two parts. First, we prove convergence of $\tilde{x}^i$ to an average consensus, shown to be the average of temporary estimate vectors. Then we prove a.s. convergence of $x(k)$ toward a NE. For convenience, we rewrite the algorithm via an intermediary variable $\bar{x}(k) = (\bar{x}^i(k), \bar{x}^{-i}(k)) \in \Omega^N$ with $\bar{x}^i(k) \in \Omega$,

$$\bar{x}(k) = (W(k) \otimes I_N)\tilde{x}(k). \tag{9}$$

In (9), $\tilde{x}(k) = (\tilde{x}^i(k), \tilde{x}^{-i}(k)) \in \Omega^N$ is the overall temporary estimate at $T(k)$ and $W(k) := [w_{ij}(k)]_{i,j \in V}$ is a doubly stochastic weight matrix ($W^T \mathbf{1}_N = W\mathbf{1}_N = \mathbf{1}_N$) defined as $W(k) = I_N - (e_{i_k} - e_{j_k})(e_{i_k} - e_{j_k})^T/2$. One can write (9) component-wise as follows:

$$\begin{cases} \bar{x}^{i_k}(k) = \bar{x}^{j_k}(k) = \dfrac{\tilde{x}^{i_k}(k) + \tilde{x}^{j_k}(k)}{2} \\ \bar{x}^i(k) = \tilde{x}^i(k), \quad i \notin \{i_k, j_k\}. \end{cases} \tag{10}$$

Using the intermediary variable $\bar{x}$ we rewrite the algorithm as the following:

(1) Each player i chooses an initial temporary estimate vector $\tilde{x}^i(0) = [\tilde{x}_1^i(0), \ldots, \tilde{x}_N^i(0)]^T$.

(2) The gossiping rule is as follows:

$$\bar{x}(k) = (W(k) \otimes I_N)\tilde{x}(k). \tag{11}$$

(3) Each player i executes the following updating:

$$x_i(k+1) = T_{\Omega_i}[x_i(k) - \alpha_{k,i} \nabla_{x_i} J_i(x_i(k), \bar{x}_{-i}^i(k))], \tag{12}$$

if $i \in \{i_k, j_k\}$, otherwise, $x_i(k+1) = x_i(k)$. Moreover,

$$\tilde{x}^i(k+1) = \bar{x}^i(k) + \Big(x_i(k+1) - \bar{x}_i^i(k)\Big) e_i, \forall i \in V. \tag{13}$$

It can be seen that steps 2 and 3 have been slightly changed since $\bar{x}_i^i(k) \neq \hat{x}_i^i(k)$ for $i \in \{i_k, j_k\}$.

4.1. Convergence of temporary estimates

We show next that under Assumptions 1–2, $\tilde{x}^i(k)$ converges a.s. toward the average of all temporary estimates,

$$Z(k) = \frac{1}{N}(\mathbf{1}_N^T \otimes I_N)\tilde{x}(k). \tag{14}$$

Theorem 1. *Let $\tilde{x}(k)$ be the overall temporary estimate vector and $Z(k)$ be its average as in (14). Let also $\alpha_{k,\max} = \max_{i \in V} \alpha_{k,i}$. Then under Assumptions 1–2,*

(i) $\sum_{k=0}^{\infty} \alpha_{k,\max} \|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\| < \infty$,

(ii) $\sum_{k=0}^{\infty} \|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|^2 < \infty$.

Proof. See Appendix A.

Using $x(k) = [\tilde{x}_1^1(k), \ldots, \tilde{x}_N^N(k)]^T$ yields the following.

Corollary 1. *For the players' actions $x(k)$, the following hold using Assumptions 1–2,*

(i) $\sum_{k=0}^{\infty} \alpha_{k,\max} \|x(k) - Z(k)\| < \infty$,

(ii) $\sum_{k=0}^{\infty} \|x(k) - Z(k)\|^2 < \infty$.

The next result shows convergence of $\bar{x}^i(k)$. This will be used to prove the convergence of the algorithm to the NE.

Lemma 4. *Let $\tilde{x}(k)$ and $Z(k)$ be as in Theorem 1. Then for $\bar{x}(k)$ (11) the following holds under Assumptions 1–2,*

$$\sum_{k=0}^{\infty}\mathbb{E}\Big[\|\bar{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|^2\Big|\mathcal{M}_k\Big]<\infty. \tag{15}$$

Proof. The proof follows from Theorem 1 part (ii), the doubly stochastic property of $W(k)$ and $\|W(k)\otimes I_N\|=1$.

4.2. Convergence of players actions to NE

Next we prove convergence of actions to the NE.

Theorem 2. *Let $x(k)$ and x^* be the players' actions and the NE of $\mathcal{G}$. Using Assumptions 1–4, the sequence $\{x(k)\}$ generated by the algorithm converges a.s. to x^*.*

Proof. We will show that $\|x_i(k)-x_i^*\|\rightarrow 0$ as $k\rightarrow\infty$. By (12), Lemma 1 and the projection's non-expansive property, for $i\in\{i_k,j_k\}$ we obtain,

$$\|x_i(k+1)-x_i^*\|^2 \leq \Big\|x_i(k)-x_i^*-\alpha_{k,i}\Big(\nabla_{x_i}J_i(x_i(k),\bar{x}_{-i}^i(k))-\nabla_{x_i}J_i(x_i^*,x_{-i}^*)\Big)\Big\|^2.$$

One can expand the RHS of the foregoing inequality, and add and subtract $\nabla_{x_i}J_i(x_i(k),Z_{-i}(k))$ and $\nabla_{x_i}J_i(x_i(k),x_{-i}(k))$ from the inner product term. Then use (3) and $\pm 2a^Tb\leq\|a\|^2+\|b\|^2$. Recall that, this holds only for $i\in\{i_k,j_k\}$. When $i\notin\{i_k,j_k\}$, $x_i(k+1)=x_i(k)$. One can combine these two cases together with a given p_i (player i's update probability). For $i\in V$,

$$\begin{aligned}
\mathbb{E}\Big[\|x_i(k+1)-x_i^*\|^2\Big|\mathcal{M}_k\Big] &\leq (1+2p_i\alpha_{k,i}^2)\|x_i(k)-x_i^*\|^2\\
&+4C^2p_i\alpha_{k,i}^2\\
&+p_i\mathbb{E}\Big[\Big\|\nabla_{x_i}J_i(x_i(k),\bar{x}_{-i}^i(k))-\nabla_{x_i}J_i(x_i(k),Z_{-i}(k))\Big\|^2\Big|\mathcal{M}_k\Big]\\
&+p_i\Big\|\nabla_{x_i}J_i(x_i(k),Z_{-i}(k))-\nabla_{x_i}J_i(x_i(k),x_{-i}(k))\Big\|^2\\
&-2p_i\alpha_{k,i}\Big(\nabla_{x_i}J_i(x_i(k),x_{-i}(k))-\nabla_{x_i}J_i(x_i^*,x_{-i}^*)\Big)^T(x_i(k)-x_i^*).
\end{aligned}\tag{16}$$

Let $p_{\max}=\max_{i\in V}p_i$ and $\alpha_{k,\min}=\min_{i\in V}\alpha_{k,i}$. We add and subtract $\frac{1}{kp_i}$ from $\alpha_{k,i}$ in the last term of the RHS of (16) and use Lemma 2 for $q\in(0,1/2)$. Then summing over $i\in V$ and using Assumption 4 yields,

$$\begin{aligned}
\mathbb{E}\Big[\|x(k+1)-x^*\|^2\Big|\mathcal{M}_k\Big] &\leq 4NC^2p_{\max}\alpha_{k,\max}^2\\
&+\Big(1+2p_{\max}\alpha_{k,\max}^2+\frac{2p_{\max}}{k^{3/2-q}p_{\min}^2}\Big)\|x(k)-x^*\|^2\\
&+p_{\max}L^2\sum_{i\in V}\mathbb{E}\Big[\|\bar{x}_{-i}^i(k)-Z_{-i}(k)\|^2\Big|\mathcal{M}_k\Big]\\
&+p_{\max}L^2\cdot\sum_{i\in V}\|Z_{-i}(k)-x_{-i}(k)\|^2\\
&+\frac{2p_{\max}}{k^{3/2-q}p_{\min}^2}\|F(x(k))-F(x^*)\|^2\\
&-\frac{2}{k}(F(x(k))-F(x^*))^T(x(k)-x^*),
\end{aligned}\tag{17}$$

where $L=\max_{i\in V}L_i$. By Assumption 4 it follows for F,

$$\|F(x)-F(y)\|\leq\rho\|x-y\|\quad\forall x,y\in\Omega,\tag{18}$$

where $\rho=\sqrt{2\sum_{i\in V}(L_i^2+\sigma_i^2)}$. Using (18) for the 5th term in the RHS of (17), we obtain,

$$\begin{aligned}
\mathbb{E}\Big[\|x(k+1)-x^*\|^2\Big|\mathcal{M}_k\Big] &\leq\Big(1+2p_{\max}\alpha_{k,\max}^2+\frac{2p_{\max}}{k^{3/2-q}p_{\min}^2}+\frac{2p_{\max}\rho^2}{k^{3/2-q}p_{\min}^2}\Big)\|x(k)-x^*\|^2\\
&+4NC^2p_{\max}\alpha_{k,\max}^2+p_{\max}L^2\sum_{i\in V}\mathbb{E}\Big[\|\bar{x}_{-i}^i(k)-Z_{-i}(k)\|^2\Big|\mathcal{M}_k\Big]\\
&+p_{\max}L^2\cdot\sum_{i\in V}\|Z_{-i}(k)-x_{-i}(k)\|^2\\
&-\frac{2}{k}(F(x(k))-F(x^*))^T(x(k)-x^*).
\end{aligned}$$

We then apply Lemma 3 for

$$V_k:=\|x(k)-x^*\|^2,$$

$$u_k:=2p_{\max}\alpha_{k,\max}^2+\frac{2p_{\max}}{k^{3/2-q}p_{\min}^2}+\frac{2p_{\max}\rho^2}{k^{3/2-q}p_{\min}^2},$$

$$\begin{aligned}
\beta_k &:= p_{\max}L^2\Big(\sum_{i\in V}\mathbb{E}\Big[\Big\|\bar{x}_{-i}^i(k)-Z_{-i}(k)\Big\|^2\Big|\mathcal{M}_k\Big]\\
&\quad+\sum_{i\in V}\Big\|Z_{-i}(k)-x_{-i}(k)\Big\|^2\Big)+4NC^2p_{\max}\alpha_{k,\max}^2,
\end{aligned}$$

$$\zeta_k:=\frac{2}{k}\Big(F(x(k))-F(x^*)\Big)^T(x(k)-x^*).$$

By (7), $\sum_{k=0}^{\infty}u_k<\infty$. Also by Lemma 4 and Corollary 1, $\sum_{k=0}^{\infty}\beta_k<\infty$. Then by Lemma 3,

(1) $\|x(k)-x^*\|^2$ converges a.s.,

(2) $\sum_{k=0}^{\infty}\frac{2}{k}\Big(F(x(k))-F(x^*)\Big)^T(x(k)-x^*)<\infty$.

To complete the proof it remains to be shown that $\|x(k)-x^*\|\xrightarrow{a.s.}0$. This follows from Assumption 2 (the compactness of Ω) and Assumption 3. ■

5. Error bound for constant step sizes

In Sections 3 and 4, diminishing step sizes were employed. In order to prevent scenarios in which the step size becomes too small, we consider constant step sizes. Constant step sizes typically cause the iterates oscillate in the neighborhood of the NE. The magnitude of oscillation is proportional to the step size.

In the following, we use a constant step size $\alpha_{k,i}=\alpha_i\in[\alpha_{\min},\alpha_{\max}]$ for all $k\geq 0$ and $i\in V$. We find a bound on the expected distance between $\tilde{x}^i(k)$ and $Z(k)$. We use this bound to obtain an error bound on the expected distance between $x(k)$ and x^*. For constant step sizes, instead of Assumption 3, we consider the following assumption.

Assumption 5. $F:\Omega\rightarrow\mathbb{R}^N$ is strongly monotone on Ω with a constant $\mu>0$, i.e., $(F(x)-F(y))^T(x-y)\geq\mu\|x-y\|^2$, $\forall x,y\in\Omega$.

Theorem 3. *Let $\tilde{x}(k)$ be the overall temporary estimate vector and $Z(k)$ be as in (14). Then under Assumptions 1–2,*

$$\limsup_{k\rightarrow\infty}\mathbb{E}\Big[\|\tilde{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|^2\Big]\leq\frac{8+(16\sqrt{2}+16)\sqrt{\gamma}}{(1-\gamma)(1-\sqrt{\gamma})}\alpha_{\max}^2C^2,\tag{19}$$

where C is as in (3), $\gamma=\mathbb{E}\Big[\|Q(k)\|^2\Big|\mathcal{M}_k\Big]$ and $Q(k)=[(W(k)-\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^TW(k))\otimes I_N]$.

Proof. See Appendix B.

Corollary 2. *Let C and γ be as in Theorem 3. For the players' actions $x(k)$, and $Z(k)$, the following inequality holds under Assumptions 1–2,*

$$\limsup_{k\to\infty} \mathbb{E}\Big[\|x(k)-Z(k)\|^2\Big] \le \frac{8+(16\sqrt{2}+16)\sqrt{\gamma}}{(1-\gamma)(1-\sqrt{\gamma})}\alpha_{\max}^2 C^2.$$

Lemma 5. *Let $\tilde{x}(k)$, $Z(k)$, C and γ be as in Theorem 3. Then for $\bar{x}(k)$, as in (11), the following holds under Assumptions 1–2,*

$$\limsup_{k\to\infty} \mathbb{E}\Big[\|\bar{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|^2\Big] \le \frac{8+(16\sqrt{2}+16)\sqrt{\gamma}}{(1-\gamma)(1-\sqrt{\gamma})}\alpha_{\max}^2 C^2.$$

The proof is based on (19) and the properties of $W(k)$.

Now, we are ready to state the main theorem of this section, which provides an error bound on the expected distance between $x(k)$ and x^*.

Theorem 4. *Let $x(k)$ and x^* be the players' actions and the NE of $\mathcal{G}$, respectively. Let also Assumptions 1, 2, 4 and 5 hold. Moreover, let α_i satisfy the following condition:*

$$0 < (1+\rho^2+2\mu)p_{\min}\alpha_{\min} - (1+\rho^2+2\alpha_{\max})p_{\max}\alpha_{\max} < 1, \quad (20)$$

where ρ is the Lipschitz constant of F and μ is the positive constant for the strong monotonicity property of F. The sequence $\{x(k)\}$ generated by the algorithm with constant step-size α_i satisfies the following:

$$\limsup_{k\to\infty} \mathbb{E}\Big[\|x(k)-x^*\|^2\Big] \le \frac{\eta C^2 p_{\max}\alpha_{\max}^2}{(1+\rho^2+2\mu)p_{\min}\alpha_{\min} - (1+\rho^2+2\alpha_{\max})p_{\max}\alpha_{\max}}, \quad (21)$$

where $\eta = 4N + 2L^2\frac{8+(16\sqrt{2}+16)\sqrt{\gamma}}{(1-\gamma)(1-\sqrt{\gamma})}$, $L = \max_{i\in V} L_i$.

Proof. The proof follows by finding an upper bound for $\mathbb{E}[\|x_i(k+1)-x_i^*\|^2|\mathcal{M}_k]$ and then using Lemma 6 (Appendix B). Employing the same technique as in (16) with Assumption 4, the following inequality is obtained,

$$\begin{aligned}\mathbb{E}\Big[\|x_i(k+1)-x_i^*\|^2\Big|\mathcal{M}_k\Big] &\le (1+2p_i\alpha_{\max}^2)\|x_i(k)-x_i^*\|^2 \\ &+4C^2p_i\alpha_{\max}^2 + L^2p_i\mathbb{E}\Big[\Big\|\bar{x}_{-i}^i(k)-Z_{-i}(k)\Big\|^2\Big|\mathcal{M}_k\Big] \\ &+L^2p_i\Big\|Z_{-i}(k)-x_{-i}(k)\Big\|^2 \\ &-2p_i\alpha_i\Big(\nabla_{x_i}J_i(x_i(k),x_{-i}(k))-\nabla_{x_i}J_i(x_i^*,x_{-i}^*)\Big)^T(x_i(k)-x_i^*). \quad (22)\end{aligned}$$

Adding and subtracting $\alpha_{\min}p_{\min}$ from $\alpha_i p_i$ in the last term of (22) yields,

$$\begin{aligned}\mathbb{E}\Big[\|x(k+1)-x^*\|^2\Big|\mathcal{M}_k\Big] &\le (1+p_{\max}\alpha_{\max}-p_{\min}\alpha_{\min}+2p_{\max}\alpha_{\max}^2)\|x(k)-x^*\|^2 \\ &+4NC^2p_{\max}\alpha_{\max}^2 + L^2p_{\max}\sum_{i\in V}\mathbb{E}\Big[\Big\|\bar{x}_{-i}^i(k)-Z_{-i}(k)\Big\|^2\Big|\mathcal{M}_k\Big] \\ &+L^2p_{\max}\sum_{i\in V}\Big\|Z_{-i}(k)-x_{-i}(k)\Big\|^2 \\ &+(p_{\max}\alpha_{\max}-p_{\min}\alpha_{\min})\|F(x(k))-F(x^*)\|^2 \\ &-2p_{\min}\alpha_{\min}\Big(F(x(k))-F(x^*)\Big)^T(x(k)-x^*).\end{aligned}$$

Using (18) in the 5th term and Assumption 5 in the last term of the RHS and taking the expected value yields,

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}\Big[\|x(k+1)-x^*\|^2\Big|\mathcal{M}_k\Big]\Big] &= \mathbb{E}\Big[\|x(k+1)-x^*\|^2\Big] \\ &\le \Big(1+(1+\rho^2+2\alpha_{\max})p_{\max}\alpha_{\max}-(1+\rho^2+2\mu)p_{\min}\alpha_{\min}\Big) \\ &\times\mathbb{E}\Big[\|x(k)-x^*\|^2\Big]+4NC^2p_{\max}\alpha_{\max}^2 \\ &+L^2p_{\max}\sum_{i\in V}\mathbb{E}\Big[\Big\|\bar{x}_{-i}^i(k)-Z_{-i}(k)\Big\|^2\Big] \\ &+L^2p_{\max}\sum_{i\in V}\mathbb{E}\Big[\Big\|Z_{-i}(k)-x_{-i}(k)\Big\|^2\Big].\end{aligned}$$

Using (20), applying Lemma 6, Corollary 2 and Lemma 5, it can be shown that (21) holds. ■

The error bound (21) depends on the step sizes, the number of players, the parameters associated with the cost functions, as well as the communication graph and the players' update probability. It can be seen that the bound grows as α_i's become larger, γ gets closer to 1 and N increases. As a special case, we consider $\alpha_i = \alpha$, $p_i = p \forall i \in V$. Then (20) is simplified into $0 < \alpha(\mu-\alpha) < 1/2p$ and (21) becomes $\limsup_{k\to\infty}\mathbb{E}[\|x(k)-x^*\|^2] \le \frac{\eta C^2\alpha}{2(\mu-\alpha)}$, where η is as in (21).

6. Simulation results

We consider an *optical signal-to-noise ratio* (OSNR) model on a *wavelength-division multiplexing* (WDM) optical link (Pavel, 2012). In this optical model, a set $V = \{1,\dots,10\}$ of $N = 10$ channels are transmitted over an optically amplified link, (Fig. 1(a)). We denote by x_i (y_i) and n_i^0 (n_i) the signal power and the noise power of channel $i \in V$ at the transmitter Tx (the receiver Rx), respectively. Every x_i is typically limited, i.e., $\Omega_i = [0, u_{\max}]$. We consider an ONSR Nash game where each channel is a player that aims to maximize its individual channel OSNR by adjusting the transmission power. Each player communicates with the neighbors through the communication graph G_C that is represented in Fig. 1(b). We consider a total cost function of channel i as in Pan and Pavel (2007),

$$J_i(x_i,x_{-i}) = P_i(x_i,x_{-i}) - U_i(x_i,x_{-i}), \quad (23)$$

where $P_i : \Omega \to \mathbb{R}$ is a pricing function and $U_i : \Omega \to \mathbb{R}$ denotes a utility function, monotonic in OSNR. The pricing function P_i is given as

$$P_i(x_i,x_{-i}) = a_i x_i + \frac{1}{P^0 - \sum_{j\in V} x_j}, \quad \forall i \in V,$$

where $a_i > 0$ is a pricing parameter and P^0 is the total power target of the link. The utility $U_i(x_i,x_{-i})$ is given as

$$U_i(x_i,x_{-i}) = b_i \ln\Big(1 + c_i\frac{x_i}{n_i^0 + \sum_{j\in V, j\ne i}\Gamma_{ij}x_j}\Big), \quad \forall i \in V,$$

where b_i is a positive parameter and $\Gamma = [\Gamma_{ij}]$ is the link system matrix.

We investigate the effectiveness of our algorithm with both diminishing and constant step sizes over G_C. We consider $u_{\max} = 2$ mW, $P^0 = 2.5$ mW, Γ and the other channel parameters be as in Pan and Pavel (2007), $a = 0.001 \times \mathbf{1}_{10}$, $c = \mathbf{1}_{10}$ and $b = [1,3,2,1,3,3,2,2,1,1]$. Fig. 2(a) shows convergence of our algorithm with diminishing step sizes over G_C. The dashed lines represent the NE obtained by the gradient algorithm (GA) in Pan and Pavel (2007), which runs over a centralized (complete) communication graph. The normalized error compared to the

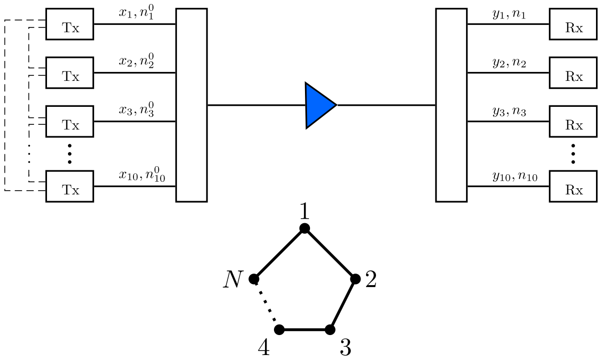

Fig. 1. (a) Optical WDM fiber link. (b) G_C.

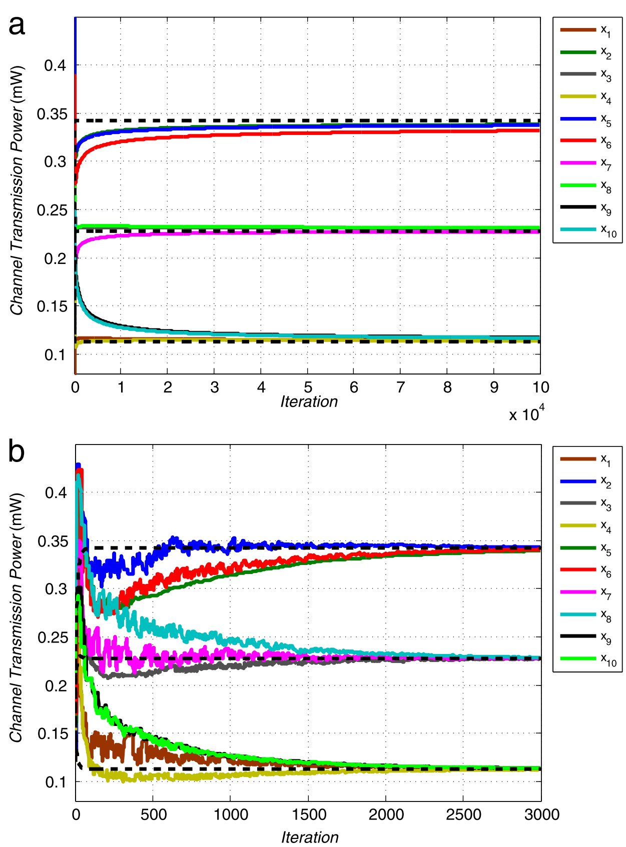

Fig. 2. Convergence of channel powers to NE over G_C: (a) diminishing (b) constant step sizes.

NE obtained in Pan and Pavel (2007) ($\frac{\|x-x^*\|}{\|x^*\|} \times 100\%$) is 1.51%. Now, consider randomly generated constant α_i with uniform distribution over [0.01, 0.02]. Fig. 2(b) shows the transmission power of the channels; the normalized error is 0.49%. Comparing the diminishing and constant step size algorithms, yields that the convergence is faster for a constant step size.

7. Conclusions

An asynchronous gossip algorithm is proposed to find a NE over a distributed multi-player network. At each iteration, players maintain estimates of the other players' actions and share them with the local neighbors to update their estimates and actions. Using standard assumptions on the cost functions and communication graph we proved that, for diminishing step sizes, the algorithm converges a.s. to a NE of the game. For constant step sizes an error bound is derived. Future work will consider the case in which cost functions are not affected by some players' actions. This could improve the algorithm by avoiding unnecessary information exchange.

Appendix A

Proof of Theorem 1, Part (i). We repeatedly use Lemma 3 to show that a term is absolutely summable. Using (13) and (11) yields,

$$\tilde{x}(k+1) = (W(k) \otimes I_N)\tilde{x}(k) + \mu(k+1), \tag{24}$$

with $\mu(k+1) = [(x_i(k+1) - \tilde{x}_i^i(k))e_i]_{i\in V}^T$. By (14), (24) and the double stochasticity of $W(k)$, we obtain,

$$\begin{aligned}&\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\\ &\quad = Q(k)\Big(\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\Big) + R\mu(k+1),\end{aligned} \tag{25}$$

where $Q(k) = [(W(k) - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T W(k)) \otimes I_N]$ and $R = [(I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T) \otimes I_N]$. Then,

$$\begin{aligned}&\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\|\Big|\mathcal{M}_k\Big]\\ &\quad \le \underbrace{\mathbb{E}\Big[\|Q(k)(\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k))\|\Big|\mathcal{M}_k\Big]}_{\text{Term 1}}\\ &\qquad + \underbrace{\mathbb{E}\Big[\|R\mu(k+1)\|\Big|\mathcal{M}_k\Big]}_{\text{Term 2}}.\end{aligned} \tag{26}$$

Let $\gamma = \mathbb{E}\Big[\|Q(k)\|^2\Big|\mathcal{M}_k\Big]$. By Lemma 2 in Nedic (2011), $\gamma < 1$. Then for Term 1 we obtain,

$$\text{Term 1} \le \sqrt{\gamma}\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|. \tag{27}$$

To find an upper bound for Term 2, we use $\|R\| = 1$ and $\mathbb{E}[\|x\|]^2 \le \mathbb{E}[\|x\|^2]$. From (24) and $x_i(k+1) = x_i(k) = \tilde{x}_i^i(k)$, $\forall i \notin \{i_k, j_k\}$, $\|\mu(k+1)\|^2 = \sum_{i\in\{i_k,j_k\}} \|x_i(k+1) - \tilde{x}_i^i(k)\|^2$. Using (12), (3), the projection's non-expansive property and $(a+b)^2 \le 2a^2 + 2b^2$ yields,

$$\text{Term 2} \le \sqrt{2}\sum_{i\in\{i_k,j_k\}} \|x_i(k) - \tilde{x}_i^i(k)\| + 2\alpha_{k,\max}C. \tag{28}$$

Next we show that the 1st term in the RHS is absolutely summable. Using (10), it follows that,

$$\|x_i(k) - \tilde{x}_i^i(k)\| = \frac{1}{2}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|, \tag{29}$$

for $i, j \in \{i_k, j_k\}$, $i \neq j$. Then we upper bound $\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|$ rather than $\|x_i(k) - \tilde{x}_i^i(k)\|$. Using (12), (13), (10) and projection's non-expansive property yields for $i, j \in \{i_k, j_k\}$, $i \neq j$,

$$\begin{aligned}&\|\tilde{x}_i^i(k+1) - \tilde{x}_i^j(k+1)\|\\ &\quad \le \frac{1}{2}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\| + \alpha_{k,i}\Big\|\nabla_{x_i}J_i(x_i(k), \tilde{x}_{-i}^i(k))\Big\|.\end{aligned} \tag{30}$$

We take expected value of (30) and multiply the LHS and the RHS of the resultant by $\alpha_{k+1,i}$ and $\alpha_{k,i}$, respectively ($\alpha_{k+1,i} < \alpha_{k,i}$). Then using (3), we obtain,

$$\begin{aligned}&\alpha_{k+1,i}\mathbb{E}\Big[\|\tilde{x}_i^i(k+1) - \tilde{x}_i^j(k+1)\|\Big|\mathcal{M}_k\Big]\\ &\quad \le \alpha_{k,i}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\| - \frac{\alpha_{k,i}}{2}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\| + \alpha_{k,i}^2 C,\end{aligned} \tag{31}$$

where we split $\frac{\alpha_{k,i}}{2}$ into $\alpha_{k,i}$ and $-\frac{\alpha_{k,i}}{2}$. Applying Lemma 3 and using (7) and (3) yields,

$$\sum_{k=0}^{\infty} \alpha_{k,i}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\| < \infty, \tag{32}$$

which, by (29), implies that $\sum_{k=0}^{\infty} \alpha_{k,i}\|x_i(k) - \tilde{x}_i^i(k)\| < \infty$. From (26)–(29),

$$\begin{aligned}
&\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\| \Big| \mathcal{M}_k\Big] \\
&\le \sqrt{\gamma}\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\| \\
&\quad + \frac{\sqrt{2}}{2}\sum_{i\in\{i_k,j_k\}}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\| + 2\alpha_{k,\max}C.
\end{aligned} \tag{33}$$

We multiply the LHS and RHS of (33) by $\alpha_{k+1,\max}$ and $\alpha_{k,\max}$, use (7) and Lemma 3 to complete Part i.

Proof of Part (ii). Taking conditional expected value in (25), using $\mathbb{E}[\|x\|]^2 \le \mathbb{E}[\|x\|^2]$, (27)–(29), yields,

$$\begin{aligned}
&\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\|^2 \Big| \mathcal{M}_k\Big] \\
&\le \gamma\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|^2 \\
&\quad + \frac{1}{2}\sum_{i\in\{i_k,j_k\}}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|^2 + 4\alpha_{k,\max}^2 C^2 \\
&\quad + 2\sqrt{\gamma}\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\| \\
&\quad \times \Big(\frac{\sqrt{2}}{2}\sum_{i\in\{i_k,j_k\}}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\| + 2\alpha_{k,\max}C\Big).
\end{aligned} \tag{34}$$

We bound the terms in (34) as in the following steps:

- Step 1: Prove that $\sum_{k=0}^{\infty}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|^2 < \infty$ by using (30) and (3) and then Lemma 3, (32) and (7).
- Step 2: Prove that $\sum_{k=0}^{\infty}\Big[\Big(\sum_{i\in\{i_k,j_k\}}\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|\Big)\cdot\Big(\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|\Big)\Big] < \infty$ by using Part i, Step 1, and applying Lemma 3 and (7).

Part ii is completed by applying Lemma 3, (7) to (34) and using Step 1 and Step 2. ■

Appendix B

We use the following lemma on scalar sequences.

Lemma 6 (Nedic, 2011). *Let v_k and u_k be scalar sequences such that $v_{k+1} \le cv_k + u_k$ $\forall k \ge 0$ and $c \in (0, 1)$. Then $\limsup_{k\to\infty} v_k \le (1/(1-c))\limsup_{k\to\infty} u_k$.*

Proof of Theorem 3. Taking expected value in (33) and using $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$ yields,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\| \Big| \mathcal{M}_k\Big]\Big] \\
&= \mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\|\Big] \\
&\le \sqrt{\gamma}\mathbb{E}\Big[\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|\Big] \\
&\quad + \frac{\sqrt{2}}{2}\sum_{i\in\{i_k,j_k\}}\mathbb{E}\Big[\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|\Big] + 2\alpha_{\max}C.
\end{aligned} \tag{35}$$

We find an upper bound for the 2nd term by taking expected value of (30) and using Lemma 6,

$$\limsup_{k\to\infty}\mathbb{E}\Big[\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|\Big] \le 2\alpha_{\max}C. \tag{36}$$

Applying Lemma 6 to (35) and using (36) and $\gamma < 1$, we obtain $\limsup_{k\to\infty}\mathbb{E}\Big[\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|\Big] \le \frac{2\sqrt{2}+2}{1-\sqrt{\gamma}}\alpha_{\max}C$. Using (34), Lemma 6 and (36) yields $\limsup_{k\to\infty}\mathbb{E}\Big[\|\tilde{x}_i^i(k) - \tilde{x}_i^j(k)\|^2\Big] \le 4\alpha_{\max}^2 C^2$. After a few manipulations (19) is obtained. ■

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