Bang-Bang Hybrid Stabilization of Perturbed Double-Integrators

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Abstract

A bang-bang hybrid controller is presented that globally practically stabilizes the origin of a double-integrator affected by an unknown bounded uncertainty at the input side. The proposed controller has two key features: it guarantees a uniformly bounded number of switches over any compact time interval, and it is robust against bounded measurement errors. When disturbances are absent, through a proper choice of the control parameters it reduces to the time-optimal bang-bang controller for the double-integrator.

Key words: Bang-bang control; Hybrid systems; Switched systems

1 Introduction

In this paper we investigate the global practical stabilization of the perturbed double-integrator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x,t) + u, \end{aligned} \tag{1}$$

where $u \in U := \{-\bar{u}, 0, +\bar{u}\}$, with $\bar{u} > 0$, and $f(x,t)$ is a map in the class $\mathcal{F}$ of functions $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ that are locally Lipschitz with respect to x, measurable with respect to t, and bounded by a constant $\bar{f} > 0$, i.e., $\sup|f| \leq \bar{f}$. We denote $x = [x_1\ x_2]^\top \in \mathbb{R}^2$. We consider the following problem.

Stabilization by Constant Controls Problem (SCCP). Design a piecewise-constant feedback controller with values in U for system (1) such that the following properties hold:

(i) For all $f \in \mathcal{F}$, the point $x = 0$ is *globally practically stable* for the closed-loop system: For all $r > 0$ there exist controller parameters such that a compact set $\mathcal{Q} \subset B_r(0)$ with $0 \in \operatorname{int} \mathcal{Q}$ is globally asymptotically stable.

(ii) The number of controller switches is *uniformly bounded* over compact time intervals: For any $T > 0$, there exists $N \in \mathbb{N}$ such that for any $x_0 \in \mathbb{R}^2$ and for any $f \in \mathcal{F}$ the controller switches value at most N times over any time interval of length T.

The control of double-integrators plays an important role in control theory and applications. In particular, our formulation of SCCP was inspired by applications in the field of aerospace engineering. It is common to approximate the rotational dynamics of a rigid spacecraft in a neighborhood of its target configuration by a collection of decoupled double-integrators, [1], [6], [10]. Moreover, in [23] we have shown that the relative translational dynamics of two spacecraft flying in formation in a general multi-body gravitational field can be modeled as a collection of three perturbed double-integrators of the form (1). The requirement, in SCCP, that the controller be piecewise-constant is motivated by the fact that spacecraft motion control is usually performed by means of cold-gas jet thrusters, able to provide only on-off thrust forces. These actuators are commonly used to

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perform both attitude and position control on modern spacecraft ([1], [4], [6], [11], [23]). Finally, the requirement that the number of control switches be uniformly bounded over compact time intervals arises from the fact that, in practice, actuators can only switch value with bounded frequency. A solution to SCCP, therefore, is key in enabling a new generation of position and attitude controllers for spacecraft formations.

Despite its apparent simplicity, SCCP is a largely open problem. The majority of research on bang-bang stabilization of the origin of the perturbed system (1) relies on sliding mode control. In [19], it is shown that the time-optimal bang-bang controller for the double-integrator (e.g., [5]) preserves its finite-time stabilization property under a restrictive class of disturbances f, but in the presence of such disturbances it induces a sliding mode. In [20], a sliding mode controller is presented that achieves the same result for any measurable bounded perturbation while satisfying constraints on the state of the system. Sliding mode controllers, however, violate control specification (ii) of SCCP. Several methods have been proposed to alleviate the unbounded switching frequency typical of sliding mode controllers, for instance the state-dependent gain method in [12]. Such methods, however, often result in control laws that are not piecewise-constant. Alternatively, hysteresis bands around switching boundaries have been used to avoid unbounded switching frequency [12], [16], but they introduce a problematic coupling between the switching frequency and the asymptotic bound on the state.

To the best of our knowledge, the only attempts at solving SCCP are found in the context of second-order sliding mode control [2], [3], [24]. These control algorithms guarantee global finite-time attractivity of the origin (see, for example Proposition 4.1 in [24]) but in all cases the controller's switching frequency is unbounded at the origin. One could introduce a hysteresis mechanism at the origin to guarantee bounded switching frequency, making the origin globally *practically* attractive. The stability of the origin and the robustness of the proposed controllers against measurement error are not investigated in the above papers. Levant in [13],[14] presents dynamic feedbacks producing piecewise-constant controls resulting in global finite-time stability of the origin. The controllers in [14] have infinite switching frequency at the origin but, once again, one could introduce an hysteresis mechanism eliminating this problem, and turning these controllers into global practical stabilizers. With this modification, the controllers presented in [14] solve SCCP. Levant also shows that his controllers enjoy robustness against measurement error (see Theorem 3 in [14]). However, it is unclear whether the switching frequency remains bounded in the presence of measurement error.

Inspired by the fact, shown in [17], [18], [21], that hybrid feedback is advantageous over discontinuous feedback for its potential robustness against measurement error, in this paper we solve SCCP by means of a hybrid piecewise-constant feedback controller. Necessary and sufficient stability conditions are provided [1] in terms of the control magnitude $\bar{u}$. Further, we show that the proposed controller is robust against bounded measurement error, in the following sense. If the bound on the measurement error is sufficiently small, then the stability properties of the controller under exact state feedback and noisy state feedback are identical, and the switching frequency remains bounded.

In developing a solution to SCCP, we begin with the time-optimal bang-bang controller for the unperturbed double-integrator. We add another switching boundary, the set $\{x_2 = 0\}$, and define an automaton that selectively enables and disables switching boundaries in such a way that the resulting sequence of switching points contracts to the origin. The switching frequency remains bounded owing to this selective enabling of switching sets and to an hysteresis mechanism at the origin. If the hysteresis is removed, the origin becomes globally finite-time stable, but the switching frequency becomes infinite when solutions reach the origin, just as in [14]. When the perturbation is absent, i.e., $f \equiv 0$, with a suitable choice of the control parameters our hybrid controller reduces to the time-optimal bang-bang stabilizer for the double-integrator.

The paper is organized as follows. In Section 2 we present the solution of SCCP and state the main results, Theorem 1 and 3. In preparation for the proofs of these theorems, in Section 3 we review a basic result from [15] characterizing the boundary of attainable sets of planar nonlinear systems, and use it to characterize the attainable sets of the perturbed double-integrator (1) with constant controls. The proof of Theorem 1 is presented in Section 4. The proof of Theorem 3, characterizing the robustness of the proposed controller against measurement error, is presented in Section 5.

Notation: We denote $B_\epsilon(0) = \{x \in \mathbb{R}^2 : (x^\top x)^{1/2} < \epsilon\}$ and $\bar{B}_\epsilon(0) = \{x \in \mathbb{R}^2 : (x^\top x)^{1/2} \leq \epsilon\}$. These definitions imply that the set $B_0(0)$ is empty, while $\bar{B}_0(0) = \{0\}$. The boundary of a set A is defined as $\partial A = \bar{A} \setminus \operatorname{int} A$ where $\bar{A}$ is the closure of A and $\operatorname{int} A$ is its interior. We denote by A^c the set $A^c = \mathbb{R}^2 \setminus A$ and we denote by $-A$ the set $-A = \{x : -x \in A\}$. If A is a closed subset of $\mathbb{R}^2$, we define its enlargement $A_\sigma \subset \mathbb{R}^2$, $\sigma > 0$, as $A_\sigma = \{x \in \mathbb{R}^n : d(x, A) \leq \sigma\}$, where $d(x, A)$ denotes the euclidean point-to-set distance.

[1] A preliminary version of these results has been presented in [23] and [22].

2 Main Results

In this section we present a hybrid feedback control law that solves SCCP. We begin by assuming that the state $x(t)$ is available for feedback. Later, we assume that the state measurement is corrupted by a bounded error signal.

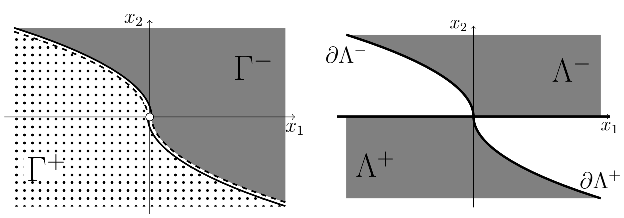
We define **initialization sets** Γ^+, Γ^- as

$$\begin{aligned}\Gamma^+ =& \{(x_1,x_2) : x_1 < 0,\ x_2 < \sqrt{-2\bar{u}x_1}\} \cup \\ & \{(x_1,x_2) : x_1 > 0,\ x_2 \le -\sqrt{2\bar{u}x_1}\}, \\ \Gamma^- =& -\Gamma^+.\end{aligned} \tag{2}$$

Define **switching sets** Λ^+, Λ^- as

$$\begin{aligned}\Lambda^+ =& \{(x_1,x_2) : x_1 \le 0,\ x_2 \le 0\} \cup \\ & \{(x_1,x_2) : x_1 > 0,\ x_2 \le -\sqrt{2\bar{u}x_1}\}, \\ \Lambda^- =& -\Lambda^+.\end{aligned} \tag{3}$$

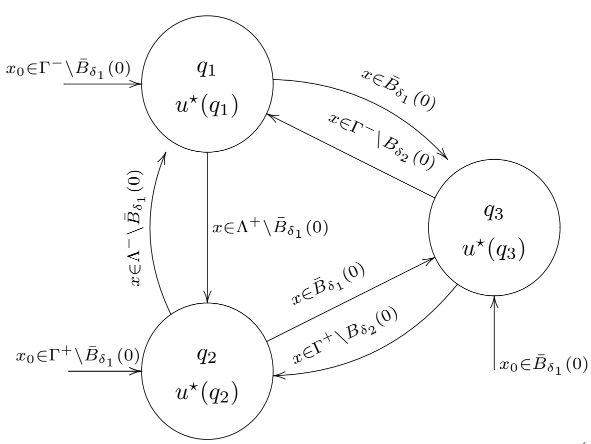
Defining the half-parabolas $S^+ = \{(x_1, -\sqrt{2\bar{u}x_1}) : x_1 \ge 0\}$ and $S^- = -S^+$, we have $\partial\Lambda^+ = S^+ \cup \{(x_1,0) : x_1 \le 0\}$ and $\partial\Lambda^- = S^- \cup \{(x_1,0) : x_1 \ge 0\}$. Next, consider



(a) (b)

Fig. 1. Sets Γ^- and Γ^+ (a) and sets Λ^- and Λ^+ (b).

the automaton $\mathcal{A}$ in (4), and denote by $Q := \{q_1, q_2, q_3\}$ the set of discrete states of $\mathcal{A}$.



(4)

Finally, the proposed control law $u^\star : Q \to \mathbb{R}$ is

$$\begin{aligned} u^\star(q_1) &= -\bar{u} \\ u^\star(q_2) &= \bar{u} \\ u^\star(q_3) &= 0. \end{aligned} \tag{5}$$

Thus the proposed controller is piecewise-constant with values in the set $\{-\bar{u}, 0, \bar{u}\}$, and has dynamics that are governed by the automaton $\mathcal{A}$ in (4). An initial condition x_0 of the double-integrator induces an initialization of the automaton according to the rules in (5). For example, if $x_0 \in \Gamma^- \setminus \bar{B}_{\delta_1}(0)$, then $\mathcal{A}$ is initialized at q_1. A state transition from state q_j to state q_k, with $k \neq j$, will be denoted as $q_j \to q_k$. Each edge of the automaton is associated with a transition condition that determines whether or not the transition occurs. For instance, a transition $q_1 \to q_3$ occurs at time t if and only if $x(t) \in \bar{B}_{\delta_1}(0)$.

The discrete states q_1 and q_2 in the automaton activate and deactivate the switching sets Λ^+ and Λ^-, so that a switch in the control value is allowed only when the trajectory enters the switching set which is currently active. This mutually exclusive activation of the switching sets eliminates sliding modes. Moreover, referring to Figure 1b, the gap between Λ^+ and Λ^- (white region) guarantees that when trajectories are away from the origin, the switching frequency is bounded. Near the origin, the boundedness of the switching frequency is guaranteed by a basic hysteresis mechanism implemented using two nested balls $B_{\delta_1}(0) \subset B_{\delta_2}(0)$ and the discrete state q_3.

To illustrate the selective activation mechanism described above, suppose that $x_0 \in \Gamma^- \setminus \bar{B}_{\delta_1}(0)$. Then the discrete state is initialized at q_1 and the control value is $u^\star(q_1) = -\bar{u}$. The only allowable state transition from q_1 occurs either when $x(t)$ enters $\bar{B}_{\delta_1}(0)$ ($q_1 \to q_3$), in which case the control value is switched to $u^\star(q_3) = 0$, or when $x(t)$ enters $\Lambda^+ \setminus \bar{B}_{\delta_1}(0)$, ($q_1 \to q_2$), in which case $u^\star$ is switched to $u^\star(q_2) = +\bar{u}$. Therefore, the switching set Λ^- is disabled when the discrete state is at q_1. Similarly, in q_2 the switching set Λ^+ is disabled and the control value can only switch when the state enters $\bar{B}_{\delta_1}(0)$ or when it enters set $\Lambda^- \setminus \bar{B}_{\delta_1}(0)$. In q_3, the controller is turned off, and it will be turned on only when the state exits $B_{\delta_2}(0)$. This is the hysteresis mechanism at the origin.

The parameters δ_1 and δ_2 in the automaton are chosen according to the following procedure. Let r be the radius of the ball in part (i) of SCCP. Then pick any number $\mu \in (0, \mu^\star)$, where $\mu^\star = \min\{1, ((\bar{u} - \bar{f})^2/\bar{f}^2 + 2\bar{u}(\bar{u} -$

$\bar{f})/(r\bar{f}))^{1/2}\}$). Pick $\delta_2 > 0$ such that

$$\begin{array}{ll} \delta_2 < \left(2\bar{f}\sqrt{\bar{u}^2+\mu^2r^2}-\bar{u}^2-\bar{f}^2\right)^{\frac{1}{2}}, & \text{if } h<0 \\ \delta_2 < |\frac{-\bar{f}}{\bar{u}-\bar{f}}|(-\bar{u}+\sqrt{\bar{u}^2+\mu^2r^2}), & \text{otherwise.} \end{array} \tag{6}$$

where $h = (\bar{u}-\bar{f})^2 + \bar{u}\bar{f} - \bar{f}\sqrt{\bar{u}^2+\mu^2r^2}$. Finally, pick $\delta_1 \in (0, \delta_2)$.

The next result shows controller (4)-(5), with δ_1, δ_2 chosen as above, solves SCCP.

Theorem 1 *Consider system* (1) *with perturbation $f \in \mathcal{F}$. Controller (4)-(5) solves* SCCP *if and only if $\bar{u} > \bar{f}(1+\sqrt{5})/2$. In particular, for any $r > 0$, if δ_1 and δ_2 are chosen so as to satisfy the inequalities in (6), then there exists a globally asymptotically stable compact subset of $B_r(0)$ containing the origin.*

Remark 2 *When the perturbation is absent, i.e., $f \equiv 0$, by setting $\delta_1 = 0$ and $\delta_2 > 0$ the proposed hybrid feedback reduces to the time-optimal bang-bang controller for the double-integrator. Moreover, for arbitrary $f \in \mathcal{F}$, it can be shown that setting $\delta_1 = 0$ and $\delta_2 > 0$ makes the origin globally finite-time stable, but the switching frequency becomes infinite when solutions reach the origin.*

Next, we consider the case when the measured state signal is

$$y(t) = x(t) + e(t), \tag{7}$$

where $e(t)$ is a bounded error signal satisfying $\sup \|e(t)\| \le \sigma$, for some $\sigma > 0$. Replacing $x(t)$ by $y(t)$ in the automaton $\mathcal{A}$ in (4), the question now is whether the stability properties of Theorem 1 persist in the presence of such measurement error. The answer is *yes*, and is contained in the following result.

Theorem 3 *Consider system (1) with controller (4)-(5) in the presence of bounded measurement error $e(t)$. If $\bar{u} > \bar{f}(1+\sqrt{5})/2$, the controller (4)-(5) solves* SCCP *in the following sense. For any $r > 0$, if δ_1 and δ_2 are chosen so as to satisfy the inequalities in* (6)*, then there exists $\sigma^\star > 0$ such that for all $\sigma \in [0, \sigma^\star)$, and for all $x_0 \in \mathbb{R}^2$, the following properties hold:*

(i) there exists a globally asymptotically stable compact subset of $B_r(0)$ containing the origin;
(ii) the number of controller switches is uniformly bounded over compact time intervals.

In essence, the sufficiency part of Theorem 1 remains unchanged in the presence of sufficiently small measurement error.

We conclude this section with a remark concerning the switching frequency of the proposed hybrid controller. Although Theorems 1 and 3 state that the number of controller switches is uniformly bounded over compact time intervals, it may happen that the time interval between two subsequent switches is arbitrarily small. To take into account the characteristics of a real actuator, one would have to implement the controller (4)-(5) with a dwell-time. It turns out that the effects of dwell-time on the stability analysis are equivalent to those due to measurement error, so that the proposed controller is robust against sufficiently small dwell-time. More precisely, to take dwell-time into account, one may modify the statement of Theorem 3 by adding after the statement "there exists $\sigma^\star > 0$..." the statement "there exists a sufficiently small bound on the dwell-time." This fact, a direct consequence of results presented in Section 5, will not be proved here due to space limitations.

3 Boundaries of attainable sets

In preparation for the proofs of Theorems 1 and 3, we review a result in [15] characterizing the boundaries of attainable sets of planar single-input systems. Before presenting the formal definition of attainable set and its relevant properties, we briefly motivate their relevance as a tool to solve SCCP. Consider system (1) with the hybrid feedback (4)-(5). Fix an initial condition x_0 and, for the sake of argument, suppose the automaton state is fixed at either q_1 or q_2, so that $u = \pm\bar{u}$. The attainable set of (1) from x_0 is the set of states that (1) can reach from x_0 as the perturbation f ranges over the class $\mathcal{F}$. In order to prove that the closed-loop double-integrator enjoys certain stability properties independent of perturbations $f \in \mathcal{F}$, our strategy is to prove an analogous property for the relevant attainable sets.

For a fixed automaton state q_j, $j \in \{1, 2\}$, we may rewrite the closed-loop double-integrator as follows:

$$\dot{x} = \lambda(x,t)F_1^{q_j}(x) + (1-\lambda(x,t))F_2^{q_j}(x), \tag{8}$$

where $\lambda : \mathbb{R}^2 \times \mathbb{R} \to [0,1]$ is defined as $\lambda(x,t) = \left(\bar{f} - f(x,t)\right)/(2\bar{f})$, and

$$F_1^{q_j}(x) = \begin{bmatrix} x_2 \\ -\bar{f} + (-1)^j\bar{u} \end{bmatrix}, \quad F_2^{q_j}(x) = \begin{bmatrix} x_2 \\ \bar{f} + (-1)^j\bar{u} \end{bmatrix},$$

with $j \in \{1, 2\}$. Allowing the perturbation $f(x,t)$ to range over the class $\mathcal{F}$ corresponds to replacing $\lambda(x,t)$ in (8) by a generic measurable signal $\lambda : \mathbb{R} \to [0,1]$. In light of this observation, consider the planar system

$$\dot{x} = \lambda(t)F_1(x) + (1-\lambda(t))\,F_2(x) \tag{9}$$

where $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are C^1 vector fields and λ is an input signal with values in the interval $[0,1]$.

Definition 4 *The **attainable set** $\mathcal{A}(x_0,t)$ **from** x_0 **at time** t of system (9) is the set*

$$\mathcal{A}(x_0,t) = \{x(t) : x(t) \text{ is a solution of (9) through } x_0 \text{ for some measurable control signal } \lambda, \text{ with } \lambda : \mathbb{R} \to [0,1]\}.$$

*The **attainable set** $\mathcal{A}(x_0)$ **from** x_0 of system (9) is the set $\mathcal{A}(x_0) = \bigcup_{t\geq 0} \mathcal{A}(x_0,t)$.* △

Define sets $\mathcal{R}^-$ and $\mathcal{R}^+$ as

$$\begin{aligned} \mathcal{R}^- &= \{x \in \mathbb{R}^2 : \det[F_1(x)\ F_2(x)] < 0\}, \\ \mathcal{R}^+ &= \{x \in \mathbb{R}^2 : \det[F_1(x)\ F_2(x)] > 0\}. \end{aligned} \tag{10}$$

Definition 5 ([15]) *The **extremal vector fields** $F_L(x)$ and $F_R(x)$ are defined as*

$$\begin{aligned} F_L(x) &= \begin{cases} F_1(x), & x \in \mathcal{R}^+ \\ F_2(x), & x \in \mathcal{R}^- \end{cases} \\ F_R(x) &= \begin{cases} F_2(x), & x \in \mathcal{R}^+ \\ F_1(x), & x \in \mathcal{R}^-. \end{cases} \end{aligned} \tag{11}$$

*The solutions at time t with initial condition x_0 of the extremal vector fields $F_L(x)$ and $F_R(x)$ are called **extremal solutions** and are denoted by $\phi_L(t,x_0)$ and $\phi_R(t,x_0)$, respectively. The images of extremal solutions on the plane are called **extremal arcs**. In particular, the **L-arc (resp. R-arc) through** x_0, denoted by $\gamma_L(x_0)$ (resp. $\gamma_R(x_0)$), is the image of the map $t \mapsto \phi_L(t,x_0)$ (resp. $t \mapsto \phi_R(t,x_0)$) for t ranging over some interval over which the map is defined.* △

Extremal arcs of (9) are the phase curves of (9) with minimum and maximum slope. The next lemma states that extremal arcs form the boundary of attainable sets. Before stating the lemma we recall that system (9) is said to be **small time locally controllable (STLC) from** x_0 if, for all $T > 0$, x_0 lies in the interior of $\mathcal{A}(x_0,[0,T])$.

Lemma 6 ([15]) *Let $x_0 \in \mathbb{R}^2$ be such that system (9) is not STLC from x_0. Suppose that for some $T > 0$ a solution $x(t)$ of (9) with initial conditions x_0 has the property that $x(t) \in \partial\mathcal{A}(x_0,t)$ for all $t \in [0,T]$ and that system (9) is not STLC from $x(t)$, for all $t \in [0,T]$. Then $x(t)$ is a concatenation of extremal solutions.*

Now we return to system (8) with fixed automaton state q_j, $j \in \{1,2\}$. If $\bar{u} > \bar{f}$, $\dot{x}_2$ is bounded away from zero, which implies that system (8) with input λ is not STLC from x_0. We may therefore apply Lemma 6 to system (8). In this context, the sets $\mathcal{R}^+$, $\mathcal{R}^-$ are given by $\mathcal{R}^+ = \{(x_1,x_2) : x_2 > 0\}$, $\mathcal{R}^- = \{(x_1,x_2) : x_2 < 0\}$. For each fixed q_j, $j \in \{1,2\}$, the extremal vector fields of (8) are given by

$$\begin{aligned} F_L^{q_j}(x) &= \begin{bmatrix} x_2 \\ -\operatorname{sign}(x_2)\bar{f} + (-1)^j\bar{u} \end{bmatrix} \\ F_R^{q_j}(x) &= \begin{bmatrix} x_2 \\ \operatorname{sign}(x_2)\bar{f} + (-1)^j\bar{u} \end{bmatrix}. \end{aligned} \tag{12}$$

The associated extremal solutions $\phi_L^{q_j}(s,x_0)$ and $\phi_R^{q_j}(s,x_0)$ through x_0 for $s \geq 0$, can be computed analytically. They are concatenations of arcs of parabolas $X_s^{q_j}(x_0)$ and $Y_s^{q_j}(x_0)$ defined as

$$\begin{aligned} X_s^{q_j}(x_0) &= \begin{bmatrix} \left(-\bar{f} + (-1)^j\bar{u}\right)\frac{s^2}{2} + x_{20}s + x_{10} \\ \left(-\bar{f} + (-1)^j\bar{u}\right)s + x_{20} \end{bmatrix} \\ Y_s^{q_j}(x_0) &= \begin{bmatrix} \left(\bar{f} + (-1)^j\bar{u}\right)\frac{s^2}{2} + x_{20}s + x_{10} \\ \left(\bar{f} + (-1)^j\bar{u}\right)s + x_{20} \end{bmatrix}, \end{aligned}$$

where the concatenation occurs when the solution hits $\{x_2 = 0\}$. More precisely, for all $x_0 \in \mathcal{R}^-$, we have

$$\begin{aligned} \phi_L^{q_j}(s,x_0) &= \begin{cases} Y_s^{q_j}(x_0), & \text{if } Y_s^{q_j}(x_0) \in \mathcal{R}^- \\ X_{s-s_Y^j(x_0)}^{q_j} \circ Y_{s_Y^j(x_0)}^{q_j}(x_0), & \text{if } Y_s^{q_j}(x_0) \in \mathcal{R}^+ \end{cases} \\ \phi_R^{q_j}(s,x_0) &= \begin{cases} X_s^{q_j}(x_0), & \text{if } X_s^{q_j}(x_0) \in \mathcal{R}^- \\ Y_{s-s_X^j(x_0)}^{q_j} \circ X_{s_X^j(x_0)}^{q_j}(x_0), & \text{if } X_s^{q_j}(x_0) \in \mathcal{R}^+, \end{cases} \end{aligned} \tag{13}$$

while for all $x_0 \in \mathcal{R}^+$, we have

$$\begin{aligned} \phi_L^{q_j}(s,x_0) &= \begin{cases} X_s^{q_j}(x_0), & \text{if } X_s^{q_j}(x_0) \in \mathcal{R}^+ \\ Y_{s-s_X^j(x_0)}^{q_j} \circ X_{s_X^j(x_0)}^{q_j}(x_0), & \text{if } X_s^{q_j}(x_0) \in \mathcal{R}^- \end{cases} \\ \phi_R^{q_j}(s,x_0) &= \begin{cases} Y_s^{q_j}(x_0), & \text{if } Y_s^{q_j}(x_0) \in \mathcal{R}^+ \\ X_{s-s_Y^j(x_0)}^{q_j} \circ Y_{s_Y^j(x_0)}^{q_j}(x_0), & \text{if } Y_s^{q_j}(x_0) \in \mathcal{R}^- \end{cases} \end{aligned} \tag{14}$$

where $s_X^j(x_0) = -x_{02}/((-1)^j\bar{u} - \bar{f})$, $s_Y^j(x_0) = -x_{02}((-1)^j\bar{u} + \bar{f})$.

The existence of extremal solutions for each $x_0 \in \mathbb{R}^2$ and each fixed q_j, $j \in \{1,2\}$, is guaranteed by the theory of Filippov in [8] (see Lemma 4.1 in [15]). We denote by $\gamma_L^{q_j}(x_0)$ and $\gamma_R^{q_j}(x_0)$ the extremal arcs generated by $\phi_L^{q_j}(s,x_0)$ and $\phi_R^{q_j}(s,x_0)$, respectively. Further, we denote by $\mathcal{A}^{q_j}(x_0)$ the attainable set from x_0 of system (8) for fixed q_j, $j \in \{1,2\}$.

In conclusion, assuming that the automaton state is either at q_1 or q_2, by Lemma 6 we have that $\partial\mathcal{A}^{q_j}(x_0)$ is the union of extremal arcs $\gamma_L^{q_j}$ and $\gamma_R^{q_j}$. When the automaton state is at q_3, the controller is turned off (i.e., $u = 0$) and there is no need to characterize attainable sets.

4 Proof of Theorem 1

The proof of Theorem 1 unfolds in four steps.

(1) We present necessary and sufficient conditions on the control value $\bar{u}$ so that any solution of the double-integrator (1) with hybrid feedback (5)-(4) gives rise to a well-defined sequence of switching points $\{x^i\}$, with $i \in I \subset \mathbb{N}$. This result, stated in Lemma 8, allows us to reduce the problem of proving convergence to the origin of state trajectories to the much simpler study of convergence of a sequence of switching points.
(2) We prove in Lemma 10 that for any disturbance $f \in \mathcal{F}$, the sequence of switching points $\{x^i\}_{i\in I}$ induced by controller (4)-(5) contracts to the origin if and only if $\bar{u} > \bar{f}(1+\sqrt{5})/2$.
(3) We prove in Lemma 11 that for any $r > 0$, if δ_1 and δ_2 are chosen according to condition (6), then there exists a compact positively invariant set [2] $\mathcal{Q} \subset B_r(0)$.
(4) Finally, we prove Theorem 1 by showing that for any $x_0 \in \mathbb{R}^2$ the solution enters set $\mathcal{Q}$ in finite time. Moreover the set $\mathcal{Q}$ is stable. It is also shown that if δ_1 and δ_2 are chosen according to condition (6), then the switching frequency of the controller remains uniformly bounded.

Definition 7 *Let $x(t)$ be a solution of system (1) with hybrid feedback (5)-(4). A time instant t_i is called a* **switching time of $x(t)$**, *if $x(t_i) \in \left(S^+ \cup S^- \cup \bar{B}_{\delta_1}(0)\right)$ and $x(t_i)$ induces a state transition $q_j \to q_k$, with $j, k \in \{1,2,3\}$, $j \neq k$. The value of the state at a switching time, $x^i = x(t_i)$ is called a* **switching point of $x(t)$**. △

Lemma 8 *Let $0 \le \delta_1 < \delta_2$. If, and only if, $\bar{u} > \bar{f}$, then for any $f \in \mathcal{F}$ and any initial condition in $(\bar{B}_{\delta_1}(0))^c$, the solution $x(t)$ of (1) with hybrid feedback (5)-(4) induces a switching sequence $\{x^i\}$, $i \in I \subset \mathbb{N}$ nonempty, with the following property:*

$$(x^1, \dots, x^i \in (\bar{B}_{\delta_1}(0))^c) \implies i+1 \in I. \qquad (15)$$

In other words, as long as the solution $x(t)$ does not enter $\bar{B}_{\delta_1}(0)$, there will be new switching points. Therefore, $x(t) \to \infty$ if and only if $I = \mathbb{N}$ and $x^i \to \infty$, and $x(t)$ enters $\bar{B}_{\delta_1}(0)$ if and only if $\{x_i\}$ enters $\bar{B}_{\delta_1}(0)$.

PROOF. See also [22]. The proof here is omitted due to space limitations. □

2 In this paper, a set $K \subset \mathbb{R}^2$ is said to be positively invariant for system (1) with controller (4)-(5) if for any $(x_0, t_0) \in K \times \mathbb{R}$ and for any $f \in \mathcal{F}$ bounded by $\bar{f}$, the closed-loop solution $x(t)$ remains in K for all $t \ge t_0$. This notion is sometimes referred to as strong invariance [7].

A byproduct of Lemma 8 is that, when $\bar{u} > \bar{f}$, only three types of switching points are possible. They are classified in the next definition.

Definition 9 *Let $x^i \in (S^+ \cup S^-) \setminus \bar{B}_{\delta_1}(0)$ be a switching point of a solution $x(t)$ of (1) with hybrid feedback (5)-(4) and $\bar{u} > \bar{f}$, and consider the next switching point x^{i+1}, whose existence is guaranteed by Lemma 8. x^{i+1} is a* **1-switch from** x_i *if one of the points $\{x^i, x^{i+1}\}$ belongs to S^+, and the other one belongs to S^-; x^{i+1} is a* **2-switch from** x_i *if $\{x^i, x^{i+1}\}$ belong to the same arc of parabola, S^+ or S^-; x^{i+1} is a* **0-switch from** x_i *if $x^{i+1} \in \bar{B}_{\delta_1}(0)$.*

In Lemma 8 we have shown that hybrid feedback (5)-(4) induces a sequence of switching points $\{x^i\}_{i\in I}$. We show in the following that this sequence is contracting (i.e., there exists $\alpha \in (0,1)$ such that $\|x^{i+1}\| \le \alpha \|x^i\|$ for all $i \in I$) for sufficiently large control value $\bar{u}$.

Lemma 10 *Consider system (1) with hybrid feedback (5)-(4), and pick δ_1, δ_2 such that $0 \le \delta_1 < \delta_2$. The following are equivalent:*

(i) There exists $\alpha \in (0,1)$ such that for any $f \in \mathcal{F}$ and any initial condition, the sequence $\{x^i\}_{i\in I}$ of switching points of the solution $x(t)$ of (1) with hybrid feedback (5)-(4) is contracting as long as $x^i \notin \bar{B}_{\delta_1}(0)$: $x^i, x^{i+1} \in (\bar{B}_{\delta_1(0)})^c \implies \|x^{i+1}\| \le \alpha \|x^i\|$;
(ii) $\bar{u} > \bar{f}\left(1+\sqrt{5}\right)/2$.

PROOF. (ii) ⇒ (i). Assume that $x^i \in S^+$, so that the automaton $\mathcal{A}$ is at q_2 (the argument for the case $x^i \in S^-$ is analogous). If $x^{i+1} \in \bar{B}_{\delta_1}(0)$, then part (i) trivially holds. Suppose that $x^{i+1} \notin \bar{B}_{\delta_1}(0)$. Either $x^{i+1} \in S^-$ (i.e., x^{i+1} is a 1-switch from x^i) or $x^{i+1} \in S^+$ (i.e., x^{i+1} is a 2-switch from x^i). Suppose first that $x^{i+1} \in S^-$, from which it follows that $x^{i+1} \in \mathcal{A}^{q_2}(x^i) \cap S^-$. Let $p = \gamma_R^{q_2}(x^i) \cap S^-$. Then x^{i+1} lies on the arc

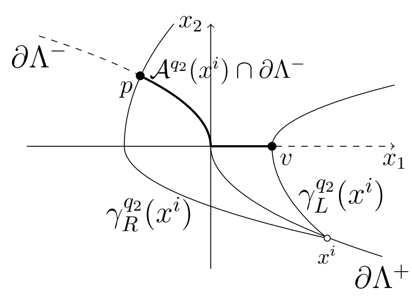


Fig. 2. Attainable switching set from x^i.

of parabola S^- delimited by 0 and p, implying that $\|x^{i+1}\| \le \|p\|$. Using the expression for $\phi_R^{q_2}(s, x^i)$ in (13) one can show that p exists and its first component p_1 is related to the first component x_1^i of x^i as $p_1 = -\alpha_1^2 x_1^i$, where $\alpha_1 = \left((\bar{f}^2 + \bar{f}\bar{u})/(2\bar{u}^2 - \bar{u}\bar{f} - \bar{f}^2)\right)^{1/2}$. Since $\bar{u} >$

$\bar{f}\left(1+\sqrt{5}\right)/2$, it holds that $\alpha_1 \in (0,1)$, and therefore $\|x^{i+1}\| \leq \|p\| \leq \alpha_1 \|x^i\|$, with $\alpha_1 \in (0,1)$.

Now suppose that $x^{i+1} \in S^+$ is a two-switch from x^i. The switching point x^{i+1} is reached from x^i through the following sequence of events. (A) The solution from x^i with $u^\star(q_2) = \bar{u}$ hits the positive x_1 axis at a point z, a state transition $q_2 \to q_1$ occurs, and the control value becomes $u^\star(q_1) = -\bar{u}$. (B) The solution from z intersects S^+ in x^{i+1}. Consider the point $v = \gamma_L^{q_2}(x^i) \cap \{(x_1, 0) : x_1 \geq 0\}$ depicted in Figure 2. The point z defined above must lie on the segment of the x_1 axis delimited by 0 and v. Therefore, $\|z\| \leq \|v\|$. Using the expression for $\phi_L^{q_2}(x^i)$ from (13) it can be shown that the first component v_1 of v satisfies $v_1 = \alpha_2^2 x_1^i$, with $\alpha_2 = \left(1 - \bar{u}/(\bar{u}+\bar{f})\right)^{1/2} \in (0,1)$. Therefore, $\|z\| \leq \|v\| \leq \alpha_2^2 x_1^i$. Now we turn our attention to event (B) above. The point x^{i+1} lies in the segment $S^+ \cap \mathcal{A}^{q_1}(z)$. The extremal solutions from z are arcs of parabolas given by $\phi_L^{q_1}(s,z) = Y_s^{q_1}(z)$ and $\phi_R^{q_1}(s,z) = X_s^{q_1}(z)$, defined in Section 3. In particular, the first component of both functions is decreasing with s. This implies that the first component x_1^{i+1} of x^{i+1} satisfies $x_1^{i+1} < z_1 \leq \alpha_2^2 x_1^i$. Thus $\|x^{i+1}\| \leq \alpha_2 \|x^i\|$. By setting $\alpha = \max\{\alpha_1, \alpha_2\}$, and noting that $\alpha \in (0,1)$, the proof that (ii) $\Rightarrow$ (i) is complete.

(i) $\Rightarrow$ (ii). Let $\{x^i\}$ be a contracting switching sequence and suppose, by way of contradiction, that $\bar{u} \leq \bar{f}\left(1+\sqrt{5}\right)/2$. Let $x^i, x^{i+1} \in (\bar{B}_{\delta_1}(0))^c$. Assume $x^i \in S^+$ and let $f \in \mathcal{F}$ be defined as $f(x,t) = \bar{f}\,\mathrm{sign}(x_2(t))$. Then $x(t) = \phi_R^{q_2}(t - t_i, x^i)$ for all $t \in [t_i, t_{i+1}]$. Therefore $x^{i+1} = p \in S^-$, as defined in the proof of sufficiency. Recall that $p_1 = -\alpha_1^2 x_1^i$, with $\alpha_1 = \left((\bar{f}^2 + \bar{f}\bar{u})/(2\bar{u}^2 - \bar{u}\bar{f} - \bar{f}^2)\right)^{1/2}$. Since $\bar{u} \leq \bar{f}(1+\sqrt{5})/2$ we have $\alpha_1 \geq 1$ which contradicts the hypothesis that the switching sequence is contracting. $\square$

Next we show that for any $r > 0$, there exists a compact positively invariant subset of $B_r(0)$. This will be used to prove practical stability.

Lemma 11 *Consider system* (1) *with the hybrid feedback* (5)-(4)*. If $\bar{u} > \bar{f}(1+\sqrt{5})/2$ then for any $p \in S^+$ there exists a compact set $\mathcal{Q}_p$ and parameters $0 < \delta_1 < \delta_2$ in automaton (4) such that $\mathcal{Q}_p$ is positively invariant. Moreover, for any $r > 0$, pick $\delta_1, \delta_2 > 0$ according to conditions* (6)*. Then the point p can be chosen so that $\bar{B}_{\delta_2}(0) \subset$ int $\mathcal{Q}_p \subset \mathcal{Q}_p \subset B_r(0)$.*

PROOF. Let $p \in S^+$ be arbitrary, Let $\mathcal{P}_p$ be the compact region depicted in Figure 3a, delimited by the extremal arcs $\gamma_R^{q_2}(p)$, $\gamma_L^{q_2}(p)$, and by $\partial\Lambda^-$. Let $\mathcal{Q}_p = \mathcal{P}_{\bar{x}} \cup -\mathcal{P}_{\bar{x}}$. Clearly, $0 \in \mathrm{int}\,\mathcal{Q}_p$. We claim that, if $\delta_1 = \delta_2 = 0$, any solution of (1) with hybrid feedback (5)-(4) originating in $\mathcal{P}_p$ can only exit $\mathcal{P}_p$ through $\partial\Lambda^-$ and, similarly, that any solution originating in $-\mathcal{P}_p$ can only exit it through $\partial\Lambda^+$. Referring to Figure 3a, the boundary of $\mathcal{P}_p$ is formed by $\partial\Lambda^-$ and two extremal arcs, $\gamma_R^{q_2}(p)$ and $\gamma_R^{q_2}(p)$. By the definition of extremal arcs, all solutions of (1) with hybrid feedback (5)-(4) cross (or are tangent to) $\gamma_R^{q_2}(p)$ from left to right, and $\gamma_L^{q_2}(p)$ from right to left. Therefore solutions in $\mathcal{P}_p$ can only exit it through $\partial\Lambda^-$. The analogous statement for $-\mathcal{P}_p$ can be proved in the same way, and the claim is proved.

In light of the claim above, since $\mathcal{P}_p \cap \partial\Lambda^- \subset -\mathcal{P}_p$ and $-\mathcal{P}_p \cap \partial\Lambda^+ \subset \mathcal{P}_p$, the set $\mathcal{Q}_p$ is positively invariant when $\delta_1 = \delta_2 = 0$. Moreover, if one chooses $0 < \delta_1 < \delta_2$ such that $\bar{B}_{\delta_2}(0) \subset \mathrm{int}\,\mathcal{Q}_p$, $\mathcal{Q}_p$ remains positively invariant.

It can be shown that there exists $c > 0$ such that for any $p \in S^+$, $\mathcal{Q}_p \subset B_{c\|p\|}(0)$. Then, if we pick p such that $c\|p\| < r$, we have $\mathcal{Q}_p \subset B_r(0)$. We have thus established that for any $r > 0$, there exist $p \in S^+$ and $\delta_2 > 0$ such that $\mathcal{Q}_p$ is positively invariant, $\bar{B}_{\delta_2}(0) \subset \mathrm{int}\,\mathcal{Q}_p$, and $\mathcal{Q}_p \subset B_r(0)$. As a matter of fact, one such δ_2 is given in (6), and is motivated by the following considerations. The positive scalar $\mu > 0$ in (6) guarantees that, setting $p = \partial B_{\mu r}(0) \cap S^+$, we have $\mathcal{Q}_p \subset B_r(0)$. The scalar δ_2 in (6) guarantees that $\bar{B}_{\delta_2}(0) \subset \mathrm{Int}\mathcal{Q}_p$. $\square$

We are ready to prove the main result of this paper.

Proof of Theorem 1

($\Rightarrow$) Similar to the proof of necessity of Lemma 10.

($\Leftarrow$) We prove first global practical stability. For any $r > 0$, by Lemma 11 there exists $p \in S^+$ and $0 < \delta_1 < \delta_2$ such that the compact set $\mathcal{Q}_p$ is positively invariant and $\bar{B}_{\delta_2}(0) \subset \mathrm{int}\,\mathcal{Q}_p \subset \mathcal{Q}_p \subset B_r(0)$. We claim that $\mathcal{Q}_p$ is stable. To this end, we need to show that for any neighborhood V of $\mathcal{Q}_p$, there exists a neighborhood U of $\mathcal{Q}_p$ such that all solutions of the closed-loop systems originating in U remain in V for all positive time. For each p in S^+, the boundary of the set $\mathcal{Q}_p$ of Lemma 11 is formed by arcs of trajectories of a differential equation that depend continuously on initial conditions. Therefore, for any V as above, one can find $q \in S^+$ with $\|q\| > \|p\|$ such that $\mathcal{Q}_p \subset \mathrm{int}\,\mathcal{Q}_q \subset \mathcal{Q}_q \subset V$. Setting $U = \mathrm{int}\,\mathcal{Q}_q$ one obtains the desired stability property.

We now show that set $\mathcal{Q}_p$ is globally attractive. By Lemma 8, for any initial condition in $(\bar{B}_{\delta_1}(0))^c$ and any $f \in \mathcal{F}$, the solution $x(t)$ gives rise to a well-defined switching sequence $\{x^i\}_{i \in I}$. By Lemma 10, this sequence is contracting as long as $x^i \notin \bar{B}_{\delta_1}(0)$. Since $\bar{B}_{\delta_1}(0) \subset \bar{B}_{\delta_2}(0) \subset \mathcal{Q}_p$, $x^i \in \mathcal{Q}_p$ for sufficiently large i. By Lemma 11, $x(t) \in \mathcal{Q}_p \subset B_r(0)$ for all $t \geq t_i$. This proves that $\mathcal{Q}_p$ is globally asymptotically stable.

We are left to show that property (ii) of SCCP holds: for any $T > 0$, there exists $N \in \mathbb{N}$ such that for all $x_0 \in \mathbb{R}^2$ and for any $f \in \mathcal{F}$, the controller switches value at most N times over any time interval of length T. In other words, the automaton $\mathcal{A}$ in (4) performs at most N discrete state transitions $q_j \to q_k$ over an interval of length T. To begin, consider state transitions not involving the state q_3. Suppose that at time t_1 a state transition $q_2 \to q_1$ occurs, and that a subsequent state transition $q_1 \to q_2$ occurs at time $t_2 > t_1$. Thus, $x(t_1) \in \Lambda^- \backslash \bar{B}_{\delta_1}(0)$ and $x(t_2) \in \Lambda^+ \backslash \bar{B}_{\delta_1}(0)$. In order to reach $\Lambda^+ \backslash \bar{B}_{\delta_1}(0)$ from $\Lambda^- \backslash \bar{B}_{\delta_1}(0)$, $x_2(t)$ must cover a minimum distance which is bounded away from zero. Since $|\dot{x}_2| \leq \bar{u} + \bar{f}$, it follows that $t_2 - t_1$ is lower bounded by a constant $T_1 > 0$, independent of $x(t_1)$. Therefore the minimum time between the two consecutive state transitions above is $T_1 > 0$. By symmetry, the same holds for state transitions $q_1 \to q_2$ followed by $q_2 \to q_1$. Similarly, the time between two consecutive state transitions of the type $q_j \to q_3$ followed by $q_3 \to q_k$, with $j, k \in \{1, 2\}$, is bounded from below by a positive constant, $T_2 > 0$. Indeed, the time between two such transitions is lower bounded by the minimum time it takes a closed-loop trajectory initialized in $\bar{B}_{\delta_1}(0)$ to exit the ball $\bar{B}_{\delta_2}(0)$. We are left with the analysis of transitions of the form $q_j \to q_k$ followed by $q_k \to q_3$ or $q_3 \to q_j$ followed by $q_j \to q_k$, with $j \neq k$, $j, k \in \{1, 2\}$. In this case, there is no lower bound on the time between such transitions. For instance, at the time of a transition $q_1 \to q_2$, the state $x(t)$ may be arbitrarily close to the set $\Lambda^+ \cap \bar{B}_{\delta_1}(0)$, and may enter $\bar{B}_{\delta_1}(0)$ after arbitrarily small time, triggering a transition $q_2 \to q_3$. However, the next transition must have the form $q_3 \to q_j$, $j \in \{1, 2\}$ which, as we have proved above, cannot occur before time T_2. A similar reasoning can be repeated for all other sequences of transitions described above. If we let $T^\star = \min\{T_1, T_2\}$, over a time interval of length $T \in (0, T^\star)$ there can be at most $N = 2$ state transitions (for instance, the ones discussed earlier, $q_1 \to q_2$ followed by $q_2 \to q_3$). For a time interval of length $T \in [0, 2T^\star)$, one may have at most $N = 4$ state transitions (the prototypical worst case is the sequence $q_1 \to q_2$, $q_2 \to q_3$, $q_3 \to q_1$, $q_1 \to q_2$; the time it takes for the first pair and second pair of transitions to occur may be arbitrarily small, but there must be at least $T^\star$ units of time between the first pair and the second pair of transitions). Thus, over a time interval $T > 0$ there can be at most $N = 2\lfloor (T/T^\star) \rfloor + 2$ state transitions, where $\lfloor \cdot \rfloor$ denotes the floor function. □

5 Proof of Theorem 3

The proof of Theorem 3 unfolds in three steps:

(1) In Lemma 13 we show that if the measurement error is small enough, the number of controller switches is uniformly bounded over compact time intervals.
(2) In Lemma 14 we show that if σ is small enough, then there exists a compact positively invariant set $\mathcal{Q}^\sigma \subset B_r(0)$ containing the origin, that is also stable.
(3) In Lemma 15 we show that if σ is small enough, the set $\mathcal{Q}^\sigma$ is globally attractive, in particular we show that controller (5)-(4) induces a switching sequence $\{x^i\}_{i \in I}$ such that $x^N \in \mathcal{Q}^\sigma$ for some $N > 0$.

We begin our analysis with the following observation. The identity (7) implies that $x(t) \in \bar{B}_\sigma(y(t))$. In the presence of measurement error, state transitions in the automaton $\mathcal{A}$ may occur each time the ball $B_\sigma(x(t))$ intersects a switching boundary. For instance, suppose that $y(t)$ enters $\Lambda^- \backslash \bar{B}_{\delta_1}(0)$, triggering a transition to q_1. The location of $x(t)$ is uncertain. We only know that, at the time of the transition to q_1, $x(t)$ lies on a neighborhood of radius σ of the set $\Lambda^- \backslash \bar{B}_{\delta_1}(0)$. In order to analyze the effects of measurement error, it is therefore necessary to consider the enlargements of the various switching boundaries (see Section 1 for the notion of enlargement of a set). Accordingly, let S^+_σ, S^-_σ, $(\partial\Lambda^+)_\sigma$, and $(\partial\Lambda^-)_\sigma$ denote the enlargements of sets S^+, S^-, $\partial\Lambda^+$, and $\partial\Lambda^-$, respectively. Finally, let $S_\sigma = S^+_\sigma \cup S^-_\sigma$.

Definition 12 *Let $x(t)$ be a solution of system* (1) *with hybrid feedback* (5)-(4) *in the presence of measurement error. A time instant t_i is called a* **switching time of $x(t)$** *if $x(t_i) \in \left(S_\sigma \cup \bar{B}_{\delta_1+\sigma}(0)\right)$ and at time $t = t_i$ a state transition $q_j \to q_k$, with $j, k \in \{1, 2, 3\}$, $k \neq j$ occurs. The value of the state at a switching time, $x^i = x(t_i)$ is called a* **switching point of $x(t)$**. △

Lemma 13 *Consider system* (1) *with controller* (4)-(5) *in the presence of measurement error $e(t)$ satisfying $\sup \|e(t)\| \leq \sigma$. For any $r > 0$, pick $\delta_1, \delta_2 > 0$ according to conditions* (6)*. If $\bar{u} > \bar{f}(1 + \sqrt{5})/2$, then there exists $\sigma > 0$ such that property (ii) of* SCCP *holds.*

PROOF. The presence of measurement error can induce two kinds of undesirable high-frequency switching. First, $y(t)$ could repeatedly enter Λ^+ and Λ^-, inducing high-frequency switching between q_1 and q_2. This can only happen when $x(t) \in \Lambda^+_\sigma \cap \Lambda^-_\sigma$. On the other hand, $x(t) \in \Lambda^+_\sigma \cap \Lambda^-_\sigma$ only if $y(t) \in \Lambda^+_{2\sigma} \cap \Lambda^-_{2\sigma}$. Pick σ small enough that

$$\Lambda^+_{2\sigma} \cap \Lambda^-_{2\sigma} \subset \bar{B}_{\delta_1}(0). \tag{16}$$

Then, when $x(t) \in \Lambda^+_\sigma \cap \Lambda^-_\sigma$ we are guaranteed that $u(t) = 0$, and therefore the controller does not switch value.

The second kind of high-frequency switching is induced when the ball $\bar{B}_\sigma(x(t))$ intersects both $\bar{B}_{\delta_1}(0)$ (possibly inducing a $q_j \to q_3$ transition) and $(B_{\delta_2}(0))^c$ (possibly inducing a $q_3 \to q_k$ transition). This cannot occur if the following condition is satisfied

$$\delta_2 - \delta_1 > 2\sigma. \tag{17}$$

If $\sigma > 0$ is small enough that conditions (16) and (17) hold, then the analysis of the number of switches over compact time intervals reduces to that in the proof of Theorem 1. This concludes the proof. □

Lemma 14 *Consider system* (1) *with controller* (4)-(5) *in the presence of measurement error* $e(t)$ *satisfying* $\sup \|e(t)\| \le \sigma$. *Let* $\bar{u} > \bar{f}(1+\sqrt{5})/2$ *and fix* $r > 0$. *Let* $\delta_1, \delta_2 > 0$ *be chosen according to conditions* (6). *Then there exists* $\sigma > 0$, *a point* $p \in S^+$, *and a compact positively invariant set* $\mathcal{Q}_p^\sigma$ *which is stable and such that* $\bar{B}_{\delta_2+\sigma}(0) \subset \mathcal{Q}_p^\sigma \subset B_r(0)$.

PROOF. Let $r > 0$ be arbitrary, and choose δ_1, δ_2 according to conditions (6). By Lemma 11, there exists $p \in S^+$ and a set $\mathcal{Q}_p \subset B_r(0)$ which is positively invariant in the absence of measurement error. We will now construct a larger set $\mathcal{Q}_p^\sigma$ which is positively invariant in the presence of measurement error. Let $L_p^\sigma = \gamma_R^{q_1}(p) \cap S_\sigma^+$. L_p^σ is the segment of extremal arc $\gamma_R^{q_1}(p)$ through p, contained in S_σ^+, as shown in Figure 3b. Let $\mathcal{P}_p^\sigma$ be the compact region defined as $\mathcal{P}_p^\sigma = \mathcal{A}^{q_2}(L_p^\sigma) \cap \overline{(\Lambda^- \setminus (\partial\Lambda^-)_\sigma)^c}$ ($\mathcal{P}_p^\sigma$ is the shaded region in Figure 3b). Let $\mathcal{Q}_p^\sigma = \mathcal{P}_p^\sigma \cup -\mathcal{P}_p^\sigma$. Note that $\mathcal{P}_p^0$ and $\mathcal{Q}_p^0$ coincide with the sets $\mathcal{P}_p$ and $\mathcal{Q}_p$ defined in the proof of Lemma 11. We claim that there

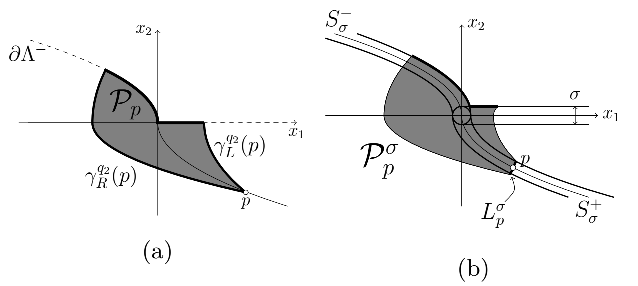


Fig. 3. Pictorial representation of sets $\mathcal{P}_p$, (a), and $\mathcal{P}_p^\sigma$, (b).

exists sufficiently small $\sigma > 0$ such that the following properties hold:

(a) $\mathcal{Q}_p^\sigma \subset B_r(0)$.
(b) $\mathcal{P}_p^\sigma \cap (\partial\Lambda^-)_\sigma \subset \operatorname{int} \mathcal{Q}_p^\sigma$ and $-\mathcal{P}_p^\sigma \cap (\partial\Lambda^+)_\sigma \subset \operatorname{int} \mathcal{Q}_p^\sigma$.
(c) $\bar{B}_{\delta_2+\sigma}(0) \subset \operatorname{int} \mathcal{Q}_p^\sigma$.

Indeed, the above set inclusions hold when $\sigma = 0$. Since the boundaries of $\mathcal{P}_p^\sigma$ and $\mathcal{Q}_p^\sigma$ are formed by arcs of trajectories of differential equations that depends continuously on initial conditions, the same inclusions continue to hold for sufficiently small σ.

Consider a discrete state transition $q_k \to q_2$ at time $\bar{t}$, with $k \in \{1, 3\}$ and with $\bar{x} = x(\bar{t}) \in \mathcal{P}_p^\sigma$. Let $\tau > \bar{t}$ be the time of the next state transition. We claim that $x(\tau) \in \mathcal{P}_p^\sigma$. First, by property (c), if the solution exits $\mathcal{P}_p^\sigma$ before time τ, then the solution cannot be in $\bar{B}_{\delta_2+\sigma}(0)$, and hence the state transition must be $q_2 \to q_1$. Moreover, in the time interval $(\bar{t}, \tau)$, the solution cannot exit $\mathcal{P}_p^\sigma$ through L_p^σ or through the two extremal arcs in Figure 3b. It can only exit $\mathcal{P}_p^\sigma$ through the portion of the boundary of $(\partial\Lambda^-)_\sigma$ which is contained in $\mathcal{P}_p^\sigma$ (the thick line in Figure 3b). At the same time, a state transition $q_2 \to q_1$ must occur before the state can exit $(\partial\Lambda^-)_\sigma$. Therefore, before any solutions can exit $\mathcal{P}_p^\sigma$ there must be a state transition. We have thus shown, as claimed, that $\bar{x} \in \mathcal{P}_p^\sigma \implies x(\tau) \in \mathcal{P}_p^\sigma$. Similarly, $\bar{x} \in (-\mathcal{P}_p^\sigma) \implies x(\tau) \in (-\mathcal{P}_p^\sigma)$. These two implications and property (b) give the implication $\bar{x} \in \mathcal{Q}_p^\sigma \implies x(\tau) \in \mathcal{Q}_p^\sigma$. At time τ, the discrete state switches to q_1 and the reasoning above reveals that the next state transition must still occur in $\mathcal{Q}_p^\sigma$. Since the switching times are a subset of the automaton transition times, the above gives the following implication: $x^i \in \mathcal{Q}_p^\sigma \implies x^{i+1} \in \mathcal{Q}_p^\sigma$. Since solutions cannot exit $\mathcal{Q}_p^\sigma$ between state transitions, we conclude that $\mathcal{Q}_p^\sigma$ is positively invariant. By properties (a) and (c), $\bar{B}_{\delta_2+\sigma}(0) \subset \mathcal{Q}_p^\sigma \subset B_r(0)$, as required.

We are left with proving that $\mathcal{Q}_p^\sigma$ is stable. The argument is the same as in the proof of Theorem 1. Namely, for every neighborhood V of $\mathcal{Q}_p^\sigma$ there exists $q \in S^+$ such that $\mathcal{Q}_q^\sigma$ is positively invariant and $\mathcal{Q}_p^\sigma \subset \operatorname{int} \mathcal{Q}_q^\sigma \subset \mathcal{Q}_q^\sigma \subset V$. Therefore all solutions of the closed-loop system originating in $\operatorname{int} \mathcal{Q}_q^\sigma$ remain in V for all positive time. □

Lemma 15 *Consider system* (1) *with controller* (4)-(5) *under the hypotheses of Lemma 14. For any* $r > 0$, *let* $\sigma > 0$ *and* $\mathcal{Q}_p^\sigma$ *be as in Lemma 14. Then, by possibly making* σ *smaller, for any initial condition the resulting solution* $x(t)$ *induces a switching sequence* $\{x^i\}_{i \in I}$, *such that* $x^N \in \mathcal{Q}_p^\sigma$ *for some* $N > 0$.

The proof of Lemma 15 makes use of the following fact. The proof is a matter of rote computation and is omitted due to space limitations.

Fact 16 *Consider system* (1) *with controller (4)-(5) in the absence of measurement error. Let* $\bar{u} > \bar{f}(1+\sqrt{5})/2$. *Suppose the switching sets* Λ^+, Λ^- *in* (3) *are replaced by*

$$\begin{aligned}\Lambda^+ =& \{(x_1,x_2) : x_1 \le 0,\ x_2 \le 0\} \cup \\ & \{(x_1,x_2) : x_1 \ge 0,\ x_2 \le -\sqrt{2u_+ x_1}\}, \\ \Lambda^- =& \{(x_1,x_2) : x_1 \ge 0,\ x_2 \ge 0\} \cup \\ & \{(x_1,x_2) : x_1 \le 0,\ x_2 \ge \sqrt{-2u_- x_1}\},\end{aligned}$$

where u_+, u_- *are two positive parameters. Redefine* $S^+ = \{(x_1,x_2) : x_1 \ge 0, x_2 = -\sqrt{2u_+ x_1}\}$. *If* $x(t)$ *is any solution inducing a switching sequence* $\{x^k\}$, *the following holds.*

(i) If x^{i+1} *is a 1-switch from* $x^i \in S^+$, *then* $|x_1^{i+1}| \le$

$\alpha_1(u_+, u_-)|x_1^i|$, *where*

$$\alpha_1(u_+, u_-) = \frac{\bar{f} + \bar{u}}{-\bar{f} + \bar{u}} \left(\frac{\bar{f} - \bar{u} + u_-}{\bar{f} + \bar{u} + u_+} \right).$$

(ii) If x^{i+1} is a 2-switch from $x^i \in S^+$, then the arc of trajectory between x^i and x^{i+1} intersects the positive x_1 axis at a point $(p_1, 0)$ such that $|p_1| \leq \alpha_2(u_+)|x_1^i|$, where $\alpha_2(u_+) = \left(\bar{f} + \bar{u} - u_+\right) / \left(\bar{u} + \bar{f}\right)$.

Since the functions $\alpha_1(u_+, u_-)$ and $\alpha_2(u_+)$ are continuous, and since $\alpha_1(\bar{u}, \bar{u}) < 1$, $\alpha_2(\bar{u}) < 1$, it follows that there exists $\Delta > 0$ such that, letting $V = [\bar{u} - \Delta, \bar{u} + \Delta]$, we have $\bar{\alpha}_1 := \max\{\alpha_1(u_+, u_-) : (u_+, u_-) \in V \times V\} < 1$, and $\bar{\alpha}_2 := \max\{\alpha_2(u_+) : u_+ \in V\} < 1$. The interpretation of this result is that the contraction property of switching sequences is preserved under small perturbations of the concavity of parabolas defining the switching boundaries. This is the key idea behind robustness against measurement noise (and against dwell-time, as discussed at the end of Section 2).

Proof of Lemma 15. Suppose $x^i \notin \mathcal{Q}_p^\sigma$. Then $x^i \in S_\sigma$. Without loss of generality, we assume throughout the proof that $x^i \in S_\sigma^+$. Suppose first x^{i+1} is a 1-switch from x^i. Then $x^i \in S_\sigma$. Let Θ be defined as follows (see the shaded set in Figure 4): $\Theta = \{(x, -\text{sign}(x)\sqrt{2u|x|}) : x \in \mathbb{R},\ u \in V\}$. By part (i) of Fact 16, if $x^i, x^{i+1} \in \Theta$ then $|x_1^{i+1}| \leq \bar{\alpha}_1|x_1^i|$, with $\bar{\alpha}_1 \in (0,1)$. There exists $\rho > 0$ so that $S_\sigma \cap \{(x_1, x_2) : |x_1| \geq \rho\} \subset \Theta$ holds (see Figure 4). Then the uniform contraction property $|x_1^{i+1}| \leq \bar{\alpha}_1|x_1^i|$ holds as long as $|x_1^i|, |x_1^{i+1}| \geq \rho$. Moreover, $\rho \to 0$ as $\sigma \to 0$.

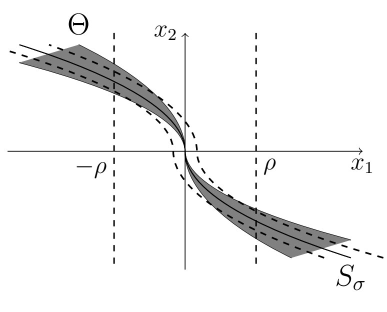


Fig. 4. Illustration of the sets used in the proof of Lemma 15.

Suppose now that x^{i+1} is a 2-switch from x^i. As in the proof of Lemma 10, we have events (A) and (B) depicted in Figure 5. (A) The solution from x^i remains to the left of the extremal arc $\gamma_L^{q_2}(x^i)$ until the state transition $q_2 \to q_1$ occurs. In the worst-case scenario, due to measurement error this transition occurs at the point p in the figure. Let $w = (w_1, 0)$ be the point of intersection of $\gamma_L^{q_2}(x^i)$ and the positive x_1 axis, as shown in Figure 5. (B) After the state transition, the solution remains to the left of $\gamma_R^{q_1}(p)$. Let $z = (z_1, 0)$ be the point of intersection of $\gamma_R^{q_1}(p)$ and the positive x_1 axis, as shown in the figure. Then, $\kappa := z_1 - w_1$ is constant independent of x^i, and $\kappa \to 0$ as $\sigma \to 0$. Suppose that $|x_1^i| \geq \rho$. Then $x^i \in \Theta$,

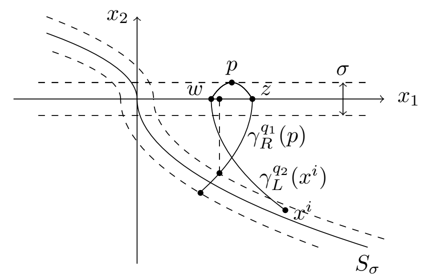


Fig. 5. Worst-case scenario for a 2-switch in the presence of measurement error.

and by part (ii) of Fact 16 we have $w_1 \leq \bar{\alpha}_2|x_1^i|$, with $\bar{\alpha}_2 \in (0,1)$. Moreover, $|x_1^{i+1}| \leq z_1$. Since $z_1 = w_1 + \kappa$ and $|w_1| \leq \bar{\alpha}_2|x_1^i|$, we conclude that $|x_1^{i+1}| \leq \bar{\alpha}_2|x_1^i| + \kappa$.

Now we put everything together. Let $\bar{\alpha} = \max\{\bar{\alpha}_1, \bar{\alpha}_2\}$. If $x^i \notin \mathcal{Q}_p^\sigma$ and $|x_1^i|, |x_1^{i+1}| \geq \rho$, then $|x_1^{i+1}| \leq \bar{\alpha}|x_1^i| + \kappa$. Either this sequence of upper bounds converges to $\kappa/(1-\bar{\alpha})$, or there exists $M > 0$ such that $|x_1^M| < \rho$. Since ρ and κ tend to zero as $\sigma \to 0$, there exists $\sigma > 0$ such that the sets $\{(x_1, x_2) \in S_\sigma : |x_1| < \rho\}$ and $\{(x_1, x_2) \in S_\sigma : |x_1| < \kappa/(1-\bar{\alpha})\}$ are contained in int $\mathcal{Q}_p^\sigma$. Thus, the sequence $\{x^i\}$ enters $\mathcal{Q}_p^\sigma$. □

Proof of Theorem 3

By Lemma 13, property (ii) of SCCP holds. By Lemma 14, for any $r > 0$ for the chosen values of $\delta_1, \delta_2 > 0$, there exists $\sigma > 0$ and a compact set $\mathcal{Q}_p^\sigma \subset B_r(0)$ which is stable and positively invariant. By Lemma 15, by possibly making σ smaller, all solutions of the closed-loop system enter $\mathcal{Q}_p^\sigma$ in finite time, and by positive invariance they remain there. Therefore, $\mathcal{Q}_p^\sigma$ is globally attractive, and hence globally asymptotically stable. □

6 Conclusions

We presented a hybrid bang-bang controller that globally practically stabilizes the origin of a double-integrator affected by unknown bounded an uncertainty at the input side. The controller was proved to be robust against bounded measurement errors, and has a guaranteed uniform bound on the number of switches over compact time intervals. Our controller is a hybrid enhancement of the classical time-optimal stabilizer for the double-integrator. Instead of parabolas, we could have used different switching boundaries obtaining the same results. An avenue for future research is to adapt the technique presented in this paper to derive a *class* of hybrid bang-bang controllers with the stability properties stated in Theorems 1, 3.

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