Local and Distributed Rendezvous of Underactuated Rigid Bodies

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Abstract—This paper solves the rendezvous problem for a network of underactuated rigid bodies such as quadrotor helicopters. A control strategy is presented that makes the centres of mass of the vehicles converge to an arbitrarily small neighborhood of one another. The convergence is global, and each vehicle can compute its own control input using only an on-board camera and a three-axis rate gyroscope. No global positioning system is required, nor any information about the vehicles' attitudes.

Fig. 1. Vehicle class under consideration.

I. INTRODUCTION

Consider a network of flying robots, each propelled by a thrust vector and endowed with an actuation mechanism producing torques about three orthogonal body axes —see Figure 1. With six degrees-of-freedom and four actuators, each robot is underactuated with degree of underactuation two. A quadrotor helicopter is an example of such a robot. Suppose each robot mounts a camera and an inertial measurement unit (IMU) that includes a three-axis rate-gyroscope, so that the robot is able to measure, in the coordinates of its own frame, the relative displacements and velocities of nearby vehicles, and its own angular velocity. The rendezvous control problem is to get the robots to move to a common location using only the above on-board sensors. To this day, this problem is open. This paper presents the first solution.

Consider now $n \geq 2$ robots. The *rendezvous control problem* investigated in this paper is to find feedback laws making the relative distances and velocities become arbitrarily small for all $i, j \in \{1, \dots, n\}$, and for arbitrary initial conditions of all robots. Crucial in the problem statement is the requirement on sensing. If robot i can sense robot j, then robot i can sense the relative position and velocity of robot j in its own local frame. Robot i can also measure its own angular velocity in the coordinates of its body frame. Robot i can neither access its own inertial position and velocity, nor its own attitude. A feedback law satisfying the above sensing requirements is referred to as being *local and distributed*. Such feedbacks are translational and rotational invariant. In the case of agents in Euclidean space, the class of translational and rotational invariant feedbacks is classified in [1]. In this paper, the set of vehicles that robot i can sense is assumed to be constant. This assumption is questionable in practice, but is made to render the problem mathematically treatable. The rendezvous problem with distance-dependent neighbors remains a challenging
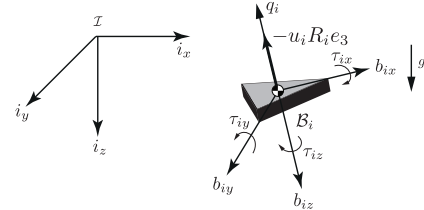
open problem for much simpler classes of robot models, such as double-integrators. The block diagram of the proposed controller is depicted in Figure 2. There are two nested loops. The outer loop treats each robot as a point-mass driven by a force input, and produces a double-integrator consensus controller which becomes a reference input for the inner loop. The inner loop assigns local and distributed feedbacks for the robots. More intuition is provided in Section V.

Besides having a simple expression making its real-time implementation feasible, the proposed controller meets the sensing requirements of the rendezvous control problem. In particular, it does not require any knowledge of the robots' absolute positions and velocities, or of their attitudes. It does not even require sensing of the relative attitudes. Finally, the controller does not require any communication among robots.

Our main result, Theorem 1, states that the proposed controller does indeed solve the rendezvous control problem, and in so doing it effectively reduces the problem to one of consensus for double-integrators. The latter problem has been researched extensively in the literature (e.g., [2], [3], [4]).

A. Related work

Typical coordination problems include attitude synchronization, rendezvous, flocking, and formation control. For networks of single or double-integrator systems, the rendezvous problem is referred to as *consensus* or *agreement*, and it has been investigated by many researchers, for instance [2], [3], [4], [5], [6], [7], [8], [9].

A passivity-based solution of the attitude synchronization problem for kinematic vehicle models is proposed in [10]. In [11], [12], [13], the same problem is investigated for dynamic vehicle models. The proposed controllers do not require measurements of the angular velocity, but they do require absolute attitude measurements. In [14], the authors use the energy shaping approach to design local and distributed controllers for attitude synchronization. The same approach
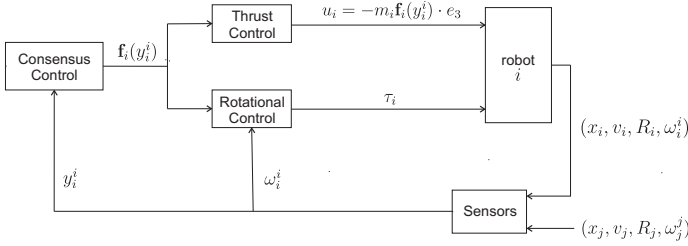
Fig. 2. Block diagram of the rendezvous control system for robot i. The outer loop assigns a desired thrust vector $\mathbf{f}_i(y_i^i)$. The inner loop thrust control uses $\mathbf{f}_i(y_i^i)$ to assign the vehicle input u_i while the rotational control uses $\mathbf{f}_i(y_i^i)$ to assign the torque input τ_i. The vector y_i^i contains the relative displacements and velocities of vehicles that robot i can sense, measured in the body frame of robot i.

is adopted in [15] to design two attitude synchronization controllers, both local and distributed. The first controller achieves almost-global synchronization for directed connected graphs. However, the controller design is based on distributed observers [16], and therefore requires auxiliary states to be communicated among neighboring vehicles. It also employs an angular velocity dissipation term that forces all vehicle angular velocities to zero in steady-state. The second controller in [15] does not restrict the final angular velocities, and does not require communication, but it requires an undirected sensing graph, and guarantees only local convergence.

The rendezvous problem for kinematic unicycles was solved in [17] using time-varying feedbacks. The solutions in [18], [19] use local and distributed, continuously differentiable, and time-independent feedbacks. The papers [17], [20], [21], [22] discuss the feasibility of achieving various formations using local and distributed feedback for kinematic unicycle models. Dynamic unicycle models are considered in [23], [24]. In [23], a two-mode formation control is presented in which the sensing graph has a spanning tree with a designated leader vehicle as the root. Each vehicle, however, has access to the acceleration of the leader through communication. The control strategy requires a switch between two control modes designed to deal with nonholonomic constraints in the system. The paper [24] presents a local and distributed control law making dynamic and kinematic unicycles converge to a common circle whose centre is stationary and dependent on the initial configuration of the vehicles. The spacing and ordering of unicycles on the circle is also controlled. The problem is solved using a three step hierarchical control based on a reduction theorem for the stabilization of sets.

The case of kinematic vehicles in three-space is investigated in [14], [25], [26], [27]. The authors of [14], [25] consider the problem of full attitude and position synchronization, but assume fully actuated vehicles. In [27], the authors propose distributed controllers to stabilize relative equilibria which, as shown in [28], [29], correspond to parallel, circular or helical formations. In [30], the authors consider directed graphs containing a globally reachable node and develop an adaptive feedback that is not local and distributed to achieve rendezvous for dynamic, underactuated vehicle models. Finally, in [31], [32] the authors consider formation control for dynamic, underactuated vehicle models. However, again, the feedbacks

are not local and distributed. In [32] the sensing graph is assumed to be undirected, and communication among vehicles is required, while in [31] the graph is balanced, and it is assumed that each vehicle has access to the thrust input of its neighbors, therefore requiring once again communication between vehicles. Both approaches in [31], [32] use a two-stage backstepping methodology in which the first stage treats each vehicle as a point-mass system to which a desired thrust is assigned. A desired thrust direction is then extracted and backstepping is used to design a rotational control such that vehicle rendezvous or formation control is achieved. Our previous work [33] investigates almost-global vehicle rendezvous making use of a two-stage hierarchical methodology similar to [31], [32]. In this approach, one can combine a consensus controller for a network of double-integrators and an attitude tracking controller satisfying certain assumptions to produce a rendezvous controller for underactuated vehicles. However, this approach requires that all vehicles can sense a common inertial vector in their own body frame, which requires additional on-board sensors. Moreover, the approach requires communication among vehicles. The solution presented in this paper overcomes all these limitations. To the best of our knowledge, a solution to the rendezvous control problem for underactuated flying vehicles stated earlier has not yet appeared in the literature.

B. Organization of the paper.

We begin, in Section II, by introducing some notation and presenting basic notions of homogeneity of functions and stability of sets. In Section III we review the vehicle model. In Section IV we formulate the rendezvous control problem. The main result, Theorem 1, is presented in Section V, and its proof in Section VI. In Section VII, we present simulation results showing that the proposed solution is robust against measurement errors, as well as force and torque disturbances. Finally, in Section VIII, we end the paper with some remarks. The proof of the main result relies on two technical lemmas that are proved in the appendix.

II. PRELIMINARIES AND NOTATION

We denote by $\mathbb{R}_+$ the set of positive real numbers. We use interchangeably the notation $v = [v_1 \; \cdots \; v_n]^\top$ or $(v_1, \ldots, v_n)$ for a column vector in $\mathbb{R}^n$. We denote by $\mathbf{1} \in \mathbb{R}^n$ the vector $(1, \ldots, 1)$. If v, w are vectors in $\mathbb{R}^3$, we denote by $v \cdot w := v^\top w$ their Euclidean inner product (also called the dot product), and by $\|v\| := (v \cdot v)^{1/2}$ the Euclidean norm of v. If $v = (v_x, v_y, v_z)$, we define

$$v^\times := \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}.$$

One has that $v^\times w = v \times w$. Let $\{e_1, e_2, e_3\}$ denote the natural basis of $\mathbb{R}^3$ and $\mathsf{SO}(3) := \{M \in \mathbb{R}^{3\times3} : M^{-1} = M^\top, \det(M) = 1\}$. If Γ is a closed subset of a Riemannian manifold $\mathcal{X}$, and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a distance metric on $\mathcal{X}$, we denote by $\|\chi\|_\Gamma := \inf_{\psi \in \Gamma} d(\chi, \psi)$ the point-to-set distance of $\chi \in \mathcal{X}$ to Γ. If $\varepsilon > 0$, we let $B_\varepsilon(\Gamma) := \{\chi \in \mathcal{X} :$

$\|\chi\|_\Gamma < \varepsilon\}$ and by $\mathcal{N}(\Gamma)$ we denote a neighborhood of Γ in $\mathcal{X}$. If $A, B \subset \mathcal{X}$ are two sets, denote by $A \backslash B$ the set-theoretic difference of A and B. If $I = \{i_1, \ldots, i_n\}$ is an index set, the ordered list of elements $(x_{i_1}, \ldots, x_{i_n})$ is denoted by $(x_j)_{j \in I}$.

Let U, V, W be finite-dimensional vector spaces. A function $f : U \to W$ is *homogeneous* if, for all $\rho > 0$ and for all $x \in U$, $f(\rho x) = \rho f(x)$. A function $f : U \times V \to W$, $f(x, y)$ is *homogeneous with respect to* x if for all $\rho > 0$ and for all $(x, y) \in U \times V$, $f(\rho x, y) = \rho f(x, y)$.

The following stability definitions are taken from [34]. Let $\Sigma : \dot{\chi} = f(\chi)$ be a smooth dynamical system with state space a Riemannian manifold $\mathcal{X}$. Let $\phi(t, \chi_0)$ denote its local phase flow. Let $\Gamma \subset \mathcal{X}$ be a closed set that is positively invariant for Σ, i.e., for all $\chi_0 \in \Gamma$, $\phi(t, \chi_0) \in \Gamma$ for all $t > 0$ for which $\phi(t, \chi_0)$ is defined.

Definition 1: The set Γ is *stable* for Σ if for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\phi(t, \chi_0) \in B_\varepsilon(\Gamma)$, for all $t > 0$ for which $\phi(t, \chi_0)$ is defined. The set Γ is *attractive* for Σ if there exists neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\lim_{t \to \infty} \|\phi(t, \chi_0)\|_\Gamma = 0$. The *domain of attraction of* Γ is the set $\{\chi_0 \in \mathcal{X} : \lim_{t \to \infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$. The set Γ is *globally attractive* for Σ if it is attractive with domain of attraction $\mathcal{X}$. The set Γ is *locally asymptotically stable (LAS)* for Σ if it is stable and attractive. The set Γ is *globally asymptotically stable* for Σ if it is stable and globally attractive. $\triangle$

Now consider a dynamical system $\Sigma(k) : \dot{\chi} = f(\chi, k)$, in which $k \in \mathbb{R}^p$ is a vector of constant parameters (typically, control gains) and f is a smooth vector field with state space a Riemannian manifold.

Definition 2: The set Γ is *globally practically stable* for $\Sigma(k)$ if for any $\varepsilon > 0$, there exists a gain $k^\star$ such that $B_\varepsilon(\Gamma)$ has a subset which is globally asymptotically stable for $\Sigma(k^\star)$. $\triangle$

The difference between global asymptotic stability and global practical stability of a set Γ is that with the former, solutions converge exactly to the set Γ, while with the latter, solutions converge to an arbitrarily small neighbourhood of Γ with appropriate choice of the control gain k. Global practical stability therefore relaxes the requirement of having asymptotic solutions, which has the advantage of simplifying control design.

III. MODELING

We now return to the i-th robot depicted in Figure 1, with the aim of deriving its equations of motion. We fix a right-handed orthonormal inertial frame $\mathcal{I}$, common to all robots, and attach at the centre of mass of robot i a right-handed orthonormal body frame $\mathcal{B}_i = \{b_{ix}, b_{iy}, b_{iz}\}$, as depicted in the figure. We denote by (x_i, v_i) the inertial position and velocity of robot i. We let g denote the gravity vector in frame $\mathcal{I}$.

We let R_i be the 3×3 matrix whose columns are the coordinate representations of b_{ix}, b_{iy}, b_{iz} (in this order) in frame $\mathcal{I}$, so that $R_i \in \mathsf{SO}(3)$. The unit vector $q_i := -R_i e_3$, depicted in Figure 1, is referred to as the *thrust direction vector* of robot i, and the matrix R_i is referred to as the *attitude* of

TABLE I
TABLE OF NOTATION

Quantity	Description
m_i, J_i	mass and inertia matrix of robot i
$x_i \in \mathbb{R}^3$	inertial position of robot i
$v_i \in \mathbb{R}^3$	linear velocity of robot i
$R_i \in \mathsf{SO}(3)$	attitude of robot i
$\omega_i \in \mathbb{R}^3$	angular velocity of robot i
$q_i = -R_i e_3$	thrust direction vector of robot i
$r^i = R_i^{-1} r$	coord. repr. of r in frame $\mathcal{B}_i$
$x_{ij} = x_j - x_i$	rel. displacement of robot j wrt robot i
$v_{ij} = v_j - v_i$	rel. velocity of robot j wrt robot i
$\mathbf{f}_i \in \mathbb{R}^3$	reference force of robot i
$\boldsymbol{\omega}_i \in \mathbb{R}^3$	reference angular velocity of robot i
$\mathcal{N}_i$	set of neighbors of robot i
$y_i = (x_{ij}, v_{ij})_{j \in \mathcal{N}_i}$	vector of rel. pos. and vel. available to robot i

the robot. We assume that a thrust force $u_i q_i$ is applied at the centre of mass of robot i. Notice that $u_i q_i$ has magnitude u_i, is directed opposite to b_{iz}, and has constant direction in body frame $\mathcal{B}_i$.

Robot i is assumed to have an actuation mechanism that induces control torques $\tau_{ix}, \tau_{iy}, \tau_{iz}$ about its body axes. We let $\tau_i := (\tau_{ix}, \tau_{iy}, \tau_{iz})$ be the torque vector, and ω_i denote the angular velocity of the robot with respect to frame $\mathcal{I}$ (the unique vector in $\mathbb{R}^3$ such that $\dot{R}_i(R_i)^{-1} = \omega_i^\times$).

In this paper we adopt the convention that if $r \in \mathbb{R}^3$ is an inertial vector, the coordinate representation of r in frame $\mathcal{B}_i$ is denoted by r^i, that is, $r^i := R_i^{-1} r$. In particular, the angular velocity of robot i in its own body frame is denoted by ω_i^i. Finally, we use boldface symbols to denote reference quantities. For instance, $\mathbf{f}_i$ is the reference force for vehicle i as in (5) and $\boldsymbol{\omega}_i$ is the reference angular velocity for vehicle i as in (9). The notation is summarized in Table I.

Picking $(x_i, v_i, R_i, \omega_i^i)$ as state for robot i, we obtain the equations of motion

$$\begin{aligned} \dot{x}_i &= v_i, \\ m_i \dot{v}_i &= -u_i R_i e_3 + m_i g, \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{R}_i &= R_i (\omega_i^i)^\times, \\ J_i \dot{\omega}_i^i &= \tau_i - \omega_i^i \times J_i \omega_i^i. \end{aligned} \tag{2}$$

In the above, m_i is the mass of robot i and $J_i = J_i^\top$ is its inertia matrix. We define the (inertial) relative positions and velocities as $x_{ij} := x_j - x_i$, $v_{ij} := v_j - v_i$. This model is standard and is widely used in the literature to model flying vehicles such as quadrotor helicopters. See, for instance, [35]. Sometimes researchers use alternative attitude representations, prominently quaternions [32] or Euler angles [36], [37]. The model (1)-(2) ignores aerodynamic effects such as drag and wind disturbances (such effects are included in [35]). It also ignores the dynamics of the actuators.

IV. RENDEZVOUS CONTROL PROBLEM

We begin by defining the *sensor digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of nodes labelled as $\{1, \ldots, n\}$, each representing a robot, and $\mathcal{E}$ is the set of edges. An edge from node i to node j indicates that robot i can sense robot j ($\mathcal{G}$ has no self-loops).

A node is *globally reachable* if there exists a path from any other node to it [1].

We denote by $\mathcal{N}_i \subset \mathcal{V}$ the set of vehicles that robot i can sense. In a realistic scenario, $\mathcal{N}_i$ is the set of robots within the field of view of robot i. For instance, if each robot mounted an omnidirectional camera, then one could define $\mathcal{N}_i$ to be the collection of robots that are within a given distance from robot i. With such a definition, the sensor digraph $\mathcal{G}$ would be state-dependent, making the stability analysis too hard at present[2].

In light of the above, in this paper we assume that $\mathcal{N}_i$ is constant for each $i \in \{1, \ldots, n\}$ (and hence $\mathcal{G}$ is constant as well). If $j \in \mathcal{N}_i$, then we say that robot j is a *neighbour* of robot i. If this is the case, then robot i can sense the relative displacement and velocity of robot j in its own body frame, i.e., the quantities x_{ij}^i, v_{ij}^i. Define the vector $y_i := (x_{ij}, v_{ij})_{j \in \mathcal{N}_i}$. The relative displacements and velocities available to robot i are contained in the vector $y_i^i := (x_{ij}^i, v_{ij}^i)_{j \in \mathcal{N}_i}$. We also assume that robot i can sense its own angular velocity in its own frame $\mathcal{B}_i$. To summarize, we have the definition below.

Definition 3: A *local and distributed feedback* (u_i, τ_i) for robot i is a locally Lipschitz function of y_i^i and ω_i^i. $\triangle$

The *local* property indicates that all quantities are represented in the body frame of robot i, while *distributed* indicates that only relative quantities with respect to neighboring robots are accessible. In applications, a local and distributed feedback for robot i can be computed with on-board cameras and rate gyroscopes.

We are now ready to define the Rendezvous Control Problem.

Rendezvous Control Problem: Consider a multi-agent system with sensor graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and agents with dynamics given by (1), (2). Let the *rendezvous manifold* be

$$\Gamma := \big\{ (x_i, v_i, R_i, \omega_i^i)_{i \in \mathcal{V}} \in \mathbb{R}^{3n} \times \mathbb{R}^{3n} \times \text{SO}(3)^n \times \mathbb{R}^{3n} \\ : x_{ij} = v_{ij} = 0, \ \forall \ i, j \in \mathcal{V} \big\}.$$
$$(3)$$

Find a local and distributed feedback policy $(u_i, \tau_i)_{i=1,\ldots,n}$ that globally practically stabilizes Γ, if one exists. $\triangle$

The goal of the rendezvous control problem is to achieve synchronization of the robot positions and velocities to any desired degree of accuracy from any initial configuration.

V. SOLUTION OF THE RENDEZVOUS CONTROL PROBLEM

Definition 4: Consider a collection of n double-integrators

$$\dot{x}_i = v_i \\ \dot{v}_i = f_i, \ i = 1 \ldots n,$$
$$(4)$$

where f_i is the control input of subsystem i. Suppose the double-integrators have the same sensor digraph $\mathcal{G}$ as the

[1] For a graph $\mathcal{G}$, existence of a globally reachable node is equivalent to having a directed spanning tree in the reverse graph.

[2] Relatively little research has been done on distributed coordination problems with state-dependent sensor graphs. In this context, in the simplest case when the robots are modelled as kinematic integrators, it has been shown in [38] that the circumcentre law of Ando et al. [39] preserves connectivity of the sensor graph and leads to rendezvous if the sensor graph is initially connected. Despite the simplicity of the robot model, the stability analysis in [38] is hard, and the control law is continuous but not Lipschitz continuous.

underactuated robots of Section III. A feedback $\mathbf{f}_i(y_i)$, $i = 1 \ldots n$, is a *double-integrator consensus controller* if $\mathbf{f}_i(y_i)$ has the form

$$\mathbf{f}_i(y_i) = \sum_{j \in \mathcal{N}_i} \big(a_{ij} x_{ij} + b_{ij} v_{ij} \big), \ i = 1, \ldots, n, \quad (5)$$

with $a_{ij}, \ b_{ij} \in \mathbb{R}$ and if, setting $f_i = \mathbf{f}_i(y_i)$ in (4), the set

$$\big\{ (x_i, v_i)_{i \in \{1 \ldots n\}} \in \mathbb{R}^{3n} \times \mathbb{R}^{3n} : x_{ij} = 0, v_{ij} = 0, \ \forall i, j \big\}$$

is globally asymptotically stable for (4). $\triangle$

Ren et al. in [2, Theorems 4.1, 4.2] and Yu et al. in [3, Theorem 1] have shown that a double-integrator consensus controller exists if and only if the sensor digraph $\mathcal{G}$ has a globally reachable node. Now the main result of this paper.

Theorem 1: If the sensor digraph $\mathcal{G}$ has a globally reachable node, then the rendezvous control problem is solvable for system (1)-(2), and a solution is given as follows. Let $\mathbf{f}_i(y_i)$, $i = 1, \ldots, n$, be a double-integrator consensus controller. The local and distributed feedback,

$$u_i = -m_i \mathbf{f}_i(y_i^i) \cdot e_3, \\ \tau_i = \omega_i^i \times J_i \omega_i^i - k_1 J_i \big((\omega_i^i \times \mathbf{f}_i(y_i^i)) \times e_3 \big) \\ - k_1^2 k_2 \big[\omega_i^i - k_1 (\mathbf{f}_i(y_i^i) \times e_3) \big], \ i = 1 \ldots n,$$
$$(6)$$

where $k_1, k_2 > 0$ are control parameters, makes the rendezvous manifold (3) globally practically stable. In particular, for any $\varepsilon > 0$, there exist $k_1^\star, k_2^\star > 0$ such that for all $k_1 > k_1^\star$, $k_2 > k_2^\star$, the set $B_\varepsilon(\Gamma)$ has a globally asymptotically stable subset.

The proof of Theorem 1 is presented in Section VI.

Explanation of proposed controller

Returning to the block diagram of Figure 2, we now explain in detail the operation of its two nested loops. We begin with the observation that a double-integrator consensus controller $\mathbf{f}_i(y_i)$, $i = 1 \ldots n$, for system (4) also makes the systems

$$\dot{x}_i = v_i \\ \dot{v}_i = f_i + g$$
$$(7)$$

rendezvous, since the addition of the gravity vector g does not affect the relative dynamics. Now compare system (7) to the translational dynamics of the flying robots,

$$\dot{x}_i = v_i \\ \dot{v}_i = -\frac{1}{m_i} u_i R_i e_3 + g.$$
$$(8)$$

If it were the case that $f_i = -(1/m_i)u_i R_i e_3$, systems (7) and (8) would be identical. Then, setting $-u_i R_i e_3 = m_i \mathbf{f}_i(y_i)$ in (8) would solve the rendezvous problem. Inspired by this observation, the outer loop of the block diagram in Figure 2 assumes that $-u_i R_i e_3$ is the control input of (8) and computes a desired double-integrator force $m_i \mathbf{f}_i(y_i)$ which becomes a reference signal for the inner loop.

We now explore in more detail the operation of the inner loop. First we observe that since $\mathbf{f}_i(y_i)$ is a linear function, we have $R_i \mathbf{f}_i(y_i^i) = \mathbf{f}_i(R_i y_i^i) = \mathbf{f}_i(y_i)$. Moreover, using the fact that dot products are invariant under rotations, we have

$$u_i = -m_i \mathbf{f}_i(y_i^i) \cdot e_3 = m_i (R_i \mathbf{f}_i(y_i^i)) \cdot (-R_i e_3) = m_i \mathbf{f}_i(y_i) \cdot q_i,$$

where q_i is the thrust direction vector. Thus, the thrust magnitude is the projection of the desired thrust $m_i\mathbf{f}_i(y_i)$ onto the thrust direction vector—see Figure 3. Now let $\boldsymbol{\omega}_i^i(y_i, R_i) := R_i^\top k_1 \left(\mathbf{f}_i(y_i) \times R_i e_3\right) = k_1 \left(\mathbf{f}_i(y_i^i) \times e_3\right)$. Then we have

$$\begin{aligned}
\tau_i =& \omega_i^i \times J_i\omega_i^i - k_1 J_i \left((\omega_i^i \times \mathbf{f}_i(y_i^i)) \times e_3\right) \\
& - k_1^2 k_2 \left(\omega_i^i - \boldsymbol{\omega}_i^i(y_i, R_i)\right).
\end{aligned} \tag{9}$$

For simplicity of notation, we drop the arguments of $\boldsymbol{\omega}_i^i(y_i, R_i)$. We will show in the proof of Theorem 1 that the torque inputs τ_i make ω_i^i converge to an arbitrarily small neighborhood of $\boldsymbol{\omega}_i^i$, $i = 1, \ldots, n$. Thus, $\boldsymbol{\omega}_i^i$ can be seen as a reference angular velocity for the inner loop. Using the fact that, for all $a, b \in \mathbb{R}^3$ and all $R \in \mathsf{SO}(3)$, $R(a \times b) = (Ra) \times (Rb)$, we have

$$\begin{aligned}
\boldsymbol{\omega}_i(y_i, R_i) &= R_i\boldsymbol{\omega}_i^i = R_i k_1 \left(\mathbf{f}_i(y_i^i) \times e_3\right) \\
&= k_1 \left((R_i\mathbf{f}_i(y_i^i)) \times (R_i e_3)\right) \\
&= k_1(\mathbf{f}_i(y_i) \times -q_i) = k_1(q_i \times \mathbf{f}_i(y_i)).
\end{aligned}$$

Thus $\boldsymbol{\omega}_i$ is perpendicular to the plane formed by the thrust direction vector q_i and the desired thrust force $m_i\mathbf{f}_i(y_i)$—see Figure 3. Since the angular velocity vector identifies an instantaneous axis of rotation, it follows that if $\omega_i = \boldsymbol{\omega}_i$, then robot i rotates about $\boldsymbol{\omega}_i$ according to the right-hand rule. Referring to Figure 3, we see that such a rotation closes the gap between $u_i q_i$ and $m_i\mathbf{f}_i(y_i)$, and the speed of rotation is proportional to $\sin\varphi$, where φ is the angle between $u_i q_i$ and $m_i\mathbf{f}_i(y_i)$ marked in the figure. When the gap is closed, we have $u_i = \|m_i\mathbf{f}_i(y_i)\|$, $q_i = m_i\mathbf{f}_i(y_i)/\|m_i\mathbf{f}_i(y_i)\|$, and thus $u_i q_i = m_i\mathbf{f}_i(y_i)$. In conclusion, the inner loop assigns (u_i, τ_i) to make ω_i approximately converge to $\boldsymbol{\omega}_i$ in the sense of global practical stability, so that $u_i q_i = -u_i R_i e_3$ approximately converges to $m_i\mathbf{f}_i(y_i)$, which is computed by the outer loop.

While the intuition behind the proposed controller is simple, the proof that the interplay between the two nested loop results in global practical stability of the rendezvous manifold is rather delicate, and it crucially relies on the homogeneity of the functions $\mathbf{f}_i(y_i)$, $i = 1, \ldots, n$.

Remark 1: Theorem 1 proves global practical stability of the rendezvous manifold Γ. The reason that the stability is practical and not asymptotic is roughly as follows. In order to achieve rendezvous of the rigid bodies, $u_i q_i$ is driven approximately to $m_i\mathbf{f}_i(y_i)$. What's important is not so much the difference in magnitude of these vectors but rather the difference in angle between them. In Figure 3, one can see that $\boldsymbol{\omega}_i$ acts to reduce this angle with a rate proportional to the magnitude of $\boldsymbol{\omega}_i$. Since $\boldsymbol{\omega}_i$ is a linear function of $\mathbf{f}_i(y_i)$, as the robots approach consensus $\boldsymbol{\omega}_i$ converges to zero at the same rate as $\mathbf{f}_i(y_i)$. This leads to increasing inaccuracy in closing the gap between the vectors $u_i q_i$ and $m_i\mathbf{f}_i(y_i)$ insomuch that in a very small neighborhood of rendezvous, $\boldsymbol{\omega}_i$ is so small that it fails to make the translational dynamics act as double integrators. More detailed reasoning is provided in Remark 2.

Features of the proposed controller

(i) The proposed controller has a number of advantages over our previous work in [33]. Unlike [33], the inner control loop does not require any derivatives of the reference thrust
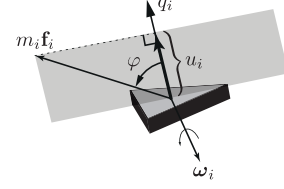

Fig. 3. Illustration of the control input u_i and reference angular velocity $\boldsymbol{\omega}_i$ in (6).

force $\mathbf{f}_i(y_i)$. In [33], the large expressions resulting from such derivatives pose difficulty in real-time computation of the control law. More importantly, the computation of such derivatives requires communication between neighboring robots, a problem that has been overcome in the present approach. The approach in [33] requires that robots have access to a common inertial vector. This requirement is absent in this paper.

(ii) The feedback of Theorem 1 is static. It does not depend on dynamic compensators that require communication between neighboring robots.

(iii) The feedback of Theorem 1 is local and distributed in the sense of Definition 3. Interestingly, it does not require sensing of relative attitudes, which can be computed using on-board cameras, but are harder to compute than relative displacements.

(iv) On the rendezvous manifold Γ there is no prespecified thrust direction q_i for robot i and the robot thrust directions do not need to align at rendezvous. This is desirable if one wants to employ the proposed controller in a hierarchical control setting to enforce additional control specifications.

VI. PROOF OF THEOREM 1

The feedback in (6) is local and distributed because it is a smooth function of y_i^i and ω_i^i only. By Theorems 4.1 and 4.2 in [2] (or Theorem 1 in [3]), if $\mathcal{G}$ has a globally reachable node then there exists a double-integrator consensus controller, and the feedback (6) is well-defined. We need to show that it renders the rendezvous manifold Γ in (3) globally practically stable. We begin by expressing the translational portion of the dynamics in coordinates relative to robot 1, i.e., in terms of the variables $(x_{1j}, v_{1j})_{j=2,\ldots,n}$,

$$\begin{aligned}
\dot{x}_{1j} &= v_{1j}, \\
\dot{v}_{1j} &= -\frac{1}{m_j} R_j e_3 u_j + \frac{1}{m_1} R_1 e_3 u_1, \quad j = 2, \ldots, n,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\dot{R}_i &= R_i(\omega_i^i)^\times, \\
J_i\dot{\omega}_i^i &= \tau_i - \omega_i^i \times J_i\omega_i^i, \quad i = 1, \ldots, n.
\end{aligned} \tag{11}$$

Since all relative states (x_{ij}, v_{ij}) can be expressed in terms of the variables above through the identity $(x_{ij}, v_{ij}) = (x_{1j} - x_{1i}, v_{1j} - v_{1i})$, perfect rendezvous occurs if and only if the vector $(x_{1j}, v_{1j})_{j=2,\ldots,n}$ is zero. Denoting,

$$\begin{aligned}
X &:= (x_{1j}, v_{1j})_{j=2,\ldots,n} \in \mathsf{X} := \mathbb{R}^{3(n-1)} \times \mathbb{R}^{3(n-1)}, \\
R &:= (R_1, \ldots, R_n) \in \mathsf{R} := \mathsf{SO}(3)^n, \\
\omega &:= (\omega_1^1, \ldots, \omega_n^n) \in \Omega := \mathbb{R}^{3n},
\end{aligned}$$

the new collective state is $(X, R, \omega) \in \mathsf{X} \times \mathsf{R} \times \Omega$. The meaning of the new state is this: X contains all translational states

(positions and velocities) relative to robot 1, R contains all the attitudes, and ω contains all body frame angular velocities.

Due to the identity $(x_{ij}, v_{ij}) = (x_{1j} - x_{1i}, v_{1j} - v_{1i})$, the vector $y_i = (x_{ij}, v_{ij})_{j \in \mathcal{N}_i}$ is a linear function of X which we will denote $y_i = h_i(X)$. Similarly, the vector $y_i^i = (x_{ij}^i, v_{ij}^i)_{j \in \mathcal{N}_i}$ is a function of X and R, linear with respect to X. We will denote this function $y_i^i = h_i^i(X, R)$.

Using the definitions above, we may now express $\mathbf{f}_i(y_i^i)$ and $\boldsymbol{\omega}_i^i(y_i, R_i) = k_1(\mathbf{f}_i(y_i^i) \times e_3)$ (the latter function was discussed in Section V) in terms of states. Accordingly, we define $\mathbf{g}_i : \mathsf{X} \to \mathbb{R}^3$, $\mathbf{g}_i^i : \mathsf{X} \times \mathsf{R} \to \mathbb{R}^3$ and $\boldsymbol{\omega} : \mathsf{X} \times \mathsf{R} \to \Omega$ as follows:

$$\mathbf{g}_i(X) := (\mathbf{f}_i \circ h_i)(X),$$
$$\mathbf{g}_i^i(X, R) := R_i^{-1} \mathbf{g}_i(X) = (\mathbf{f}_i \circ h_i^i)(X, R), \quad (12)$$
$$\boldsymbol{\omega}(X, R) := \left(\boldsymbol{\omega}_i^i(h_i(X), R_i)\right)_{i=1,\dots,n}.$$

We remark that $\mathbf{g}_i$ is linear and $\mathbf{g}_i^i$ is linear with respect to its first argument. The second identity in the definition of $\mathbf{g}_i^i$ is due to the linearity of $\mathbf{f}_i$.

Finally, we can rewrite the rendezvous manifold in new coordinates as,

$$\Gamma^\star = \{(X, R, \omega) \in \mathsf{X} \times \mathsf{R} \times \Omega : X = 0\}. \quad (13)$$

We will prove that $\Gamma^\star$ is globally practically stable, which will imply that Γ is globally practically stable as well.

A. Lyapunov function

Consider the n double-integrators (4) with control $\mathbf{f}_i(y_i)$ in (5), expressed in X coordinates:

$$\dot{x}_{1j} = v_{1j}$$
$$\dot{v}_{1j} = \mathbf{f}_j(y_j) - \mathbf{f}_1(y_1) = \mathbf{g}_j(X) - \mathbf{g}_1(X), \quad j = 2, \dots, n. \quad (14)$$

By Definition 4, the origin of this linear time-invariant system is globally asymptotically stable. Thus, there exists a quadratic Lyapunov function $V : \mathsf{X} \to \mathbb{R}$, $V(X) = X^\top P X$, where P is a symmetric positive definite matrix, such that the derivative of V along the vector field in (14) is negative definite.

Let $\mathbb{J} \in \mathbb{R}^{3n \times 3n}$ be the block-diagonal matrix with the i-th block equal to J_i, and consider the function $W : \mathsf{X} \times \mathsf{R} \times \Omega \to \mathbb{R}$ defined as

$$W(X, R, \omega) = \alpha W_{\mathsf{tran}}(X) + W_{\mathsf{rot}}(X, R, \omega), \quad (15)$$

where $\alpha > 0$ is a parameter to be assigned later and

$$W_{\mathsf{tran}}(X) = \sqrt{V(X)} + \frac{1}{2}V(X),$$

$$W_{\mathsf{rot}}(X, R, \omega) = \sum_{i=1}^{n} \mathbf{g}_i^i(X, R) \cdot e_3$$
$$+ \frac{1}{2}(\omega - \boldsymbol{\omega}(X, R))^\top \mathbb{J}(\omega - \boldsymbol{\omega}(X, R)).$$

Lemma 1: Consider the continuous function W defined in (15). Then

$$\alpha^\star := \sup_{(X, R) \in (\mathsf{X} \setminus \{0\}) \times \mathsf{R}} \sum_{i=1}^{n} |\mathbf{g}_i^i(X/\sqrt{V(X)}, R) \cdot e_3| < \infty,$$

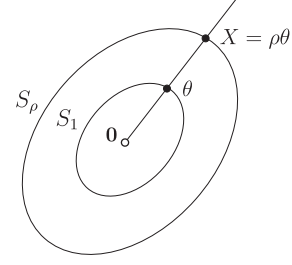and for all $\alpha > \alpha^\star$, the following properties hold:

Fig. 4. Illustration of the sets S_1 and S_ρ.

(i) $W \geq 0$ and $W^{-1}(0) \subset \Gamma^\star$.
(ii) For all $c > 0$, the sublevel set $W_c := \{(X, R, \omega) : W(X, R, \omega) \leq c\}$ is compact.
(iii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that $W_\delta \subset B_\varepsilon(\Gamma^\star)$.

The proof is in the appendix.

From now on we assume $\alpha > \alpha^\star$. In light of the lemma, if we show that W is nonincreasing outside a certain compact region of the state space, then all trajectories of (10)-(11) with feedback (6) are bounded, ruling out finite escape times. Moreover, in light of part (iii) of the lemma, to prove that $\Gamma^\star$ is practically stable it suffices to prove that for every $\delta > 0$, there exists a gain vector (k_1, k_2) such that W_δ is globally asymptotically stable. For this, we need to show that $W \geq \delta \implies \dot{W} < 0$.

B. Coordinate transformation

We now construct a coordinate transformation on the translational states X that leverages the homogeneity property of $\mathbf{f}_i$. Return to the Lyapunov function $V(X) = X^\top P X$ associated with the double-integrator consensus controller. Since V is a positive definite quadratic form, its level sets are compact and convex. For $\rho > 0$, let S_ρ denote the set $S_\rho := \{X \in \mathsf{X} : V(X) = X^\top P X = \rho^2\}$. The sets S_1 and S_ρ, for some $\rho > 1$ are depicted in Figure 4. By convexity of S_1, any point $X \in \mathsf{X} \setminus \{0\}$, can be uniquely represented as $X = \rho\theta$, $\rho \in \mathbb{R}_+$, $\theta \in S_1$, where $\rho = \sqrt{X^\top P X}$ and $\theta = X/\rho$. In the above decomposition, one can think of ρ as a scaling factor determining the size of the neighborhood of zero where X belongs, while θ is a shape variable determining the relative positions and velocities of the robots modulo scaling. We use this construction to transform the coordinates of the relative translational states in X as follows. Define the map

$$F : \mathsf{X} \setminus \{0\} \times \mathsf{R} \times \Omega \to \mathbb{R}_+ \times S_1 \times \mathsf{R} \times \Omega,$$
$$F(X, R, \omega) = (\rho, \theta, R, \omega), \ \rho := \sqrt{V(X)}, \theta := X/\sqrt{V(X)}.$$

Clearly F is a smooth bijection. Moreover its inverse $F^{-1}(\rho, \theta, R, \omega) = (\rho\theta, R, \omega)$ is smooth as well, so F is a diffeomorphism[3]. The new state is $(\rho, \theta, R, \omega) \in \mathbb{R}_+ \times S_1 \times \mathsf{R} \times \Omega$. Rendezvous in these coordinates would correspond to having $\rho = 0$, which is outside of the image of F. This is not a problem though, since we want to show practical stability of

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[3]F is a diffeomorphism of smooth manifolds. The set S_1 is diffeomorphic to the unit sphere of dimension $6(n-1) - 1$. All other sets involved in the Cartesian products are smooth manifolds

the rendezvous manifold, for which it suffices to show that ρ can be made arbitrarily small.

Having defined a coordinate transformation, our next objective is to represent the Lyapunov function candidate W in new coordinates. The new representation is $\hat{W} = W \circ F^{-1}$, which amounts to simply replacing X by $\rho\theta$. Doing so we obtain

$$\hat{W}(\rho,\theta,R,\omega) = \alpha\hat{W}_{\text{tran}}(\rho) + \hat{W}_{\text{rot}}(\rho,\theta,R,\omega), \qquad (16)$$

where $\hat{W}_{\text{tran}}(\rho) = \rho + \frac{\rho^2}{2}$,

$$\hat{W}_{\text{rot}}(\rho,\theta,R,\omega) = \rho \sum_{i=1}^{n} \mathbf{g}_i^i(\theta,R) \cdot e_3$$
$$+ \frac{1}{2}\big(\omega - \boldsymbol{\omega}(\rho\theta,R)\big)^\top \mathbb{J}\big(\omega - \boldsymbol{\omega}(\rho\theta,R)\big).$$

In writing the above, we used the identity $\rho = \sqrt{V(X)}$ and the fact that the function $\mathbf{g}_i^i(X,R)$ is linear with respect to X, implying that $\mathbf{g}_i^i(\rho\theta,R) = \rho\mathbf{g}_i^i(\theta,R)$. In what follows, we let $\hat{W}_\delta := \{(\rho,\theta,R,\omega) \in \mathbb{R}_+ \times S_1 \times \mathsf{R} \times \Omega : \hat{W}(\rho,\theta,R,\omega) < \delta\}$. Thus, $\hat{W}_\delta = F(W_\delta)$.

C. Stability analysis

Let $\delta > 0$ be arbitrary. We have $\hat{W} \le \alpha(\rho + \rho^2/2) + \rho\sup_{(\theta,R)}|\mathbf{g}_i^i(\theta,R)\cdot e_3| + (1/2)(\omega-\boldsymbol{\omega})^\top\mathbb{J}(\omega-\boldsymbol{\omega})$. Using the definition of $\alpha^\star$ in Lemma 1 and the fact that $\alpha > \alpha^\star$, we get

$$\hat{W} \le \alpha(2\rho + \rho^2/2) + (1/2)(\omega-\boldsymbol{\omega})^\top\mathbb{J}(\omega-\boldsymbol{\omega}).$$

It readily follows that there exists $\varrho \in (0, \min\{1,\delta\})$ such that

$$\Lambda_\varrho := \{(\rho,\theta,R,\omega) : \rho \in (0,\varrho), \|\omega-\boldsymbol{\omega}(\rho\theta,R)\|^2 < \varrho\} \subset \hat{W}_\delta.$$

We will show that there exist $\alpha > 0$ and a gain vector (k_1,k_2) such that $\dot{\hat{W}} < 0$ outside the set Λ_ϱ. This will imply that $\hat{W} \ge \delta \implies \dot{\hat{W}} < 0$, proving that $\hat{W}_\delta$ is globally asymptotically stable.

Lemma 2: Consider the closed-loop system (10)-(11) with feedback (6). If $k_1 > 1$, $k_2 > 0$, then there exist scalars $M_1,\ldots,M_4 > 0$ such that the derivatives of ρ and $\hat{W}_{\text{rot}}$ along the closed-loop system in (ρ,θ,R,ω) coordinates satisfy the following inequalities:

$$\dot{\rho} \le \rho\left[-M_2 + M_1\sum_{i=1}^{n}\|\mathbf{g}_i^i(\theta,R)\times e_3\|\right],$$
$$\dot{\hat{W}}_{\text{rot}} \le \rho^2\sum_{i=1}^{n}\Big[-k_1\|\mathbf{g}_i^i(\theta,R)\times e_3\|^2 \qquad (17)$$
$$+\frac{M_4}{k_2}\Big] + \rho M_3 - \frac{k_1^2 k_2}{2}\|\omega-\boldsymbol{\omega}(\rho\theta,R)\|^2.$$

The proof is in the appendix.

From now on we let $k_1 > 1$. Using the inequalities in Lemma 2, we get

$$\dot{\hat{W}} \le (\rho+\rho^2)\left[-\alpha M_2 + \alpha M_1\sum_{i=1}^{n}\|\mathbf{g}_i^i(\theta,R)\times e_3\|\right]$$
$$+\rho^2\sum_{i=1}^{n}\left[-k_1\|\mathbf{g}_i^i(\theta,R)\times e_3\|^2 + \frac{M_4}{k_2}\right]$$
$$+\rho M_3 - \frac{k_1^2 k_2}{2}\|\omega-\boldsymbol{\omega}(\rho\theta,R)\|^2.$$

Denote $\boldsymbol{\beta}_i(\theta,R) := \|\mathbf{g}_i^i(\theta,R)\times e_3\|$, and $\boldsymbol{\beta}(\theta,R) := (\boldsymbol{\beta}_1(\theta,R),\ldots,\boldsymbol{\beta}_n(\theta,R))$. For notational convenience, we omit the arguments of the functions $\boldsymbol{\beta}$ and $\boldsymbol{\omega}$. With these definitions, the inequality above may be rewritten as

$$\dot{\hat{W}} \le (\rho+\rho^2)\left(-\alpha M_2 + \alpha M_1\mathbf{1}^\top\boldsymbol{\beta}\right) + \rho^2\left(-k_1\|\boldsymbol{\beta}\|^2 + \frac{M_4 n}{k_2}\right)$$
$$+\rho M_3 - \frac{k_1^2 k_2}{2}\|\omega-\boldsymbol{\omega}\|^2.$$

For every $k_2 > nM_4/M_3$, we have

$$\dot{\hat{W}} \le (\rho+\rho^2)\left(-\alpha M_2 + M_3 + \alpha M_1\mathbf{1}^\top\boldsymbol{\beta}\right) - \rho^2 k_1\|\boldsymbol{\beta}\|^2$$
$$-\frac{k_1^2 k_2}{2}\|\omega-\boldsymbol{\omega}\|^2.$$

If we further pick $\alpha > \max\{\alpha^\star, 3M_3/M_2\}$, we have

$$\dot{\hat{W}} \le (\rho+\rho^2)\left(-2M_3 + \alpha M_1\mathbf{1}^\top\boldsymbol{\beta}\right) - \rho^2 k_1\|\boldsymbol{\beta}\|^2$$
$$-\frac{k_1^2 k_2}{2}\|\omega-\boldsymbol{\omega}\|^2.$$

Splitting the term $-\rho^2 k_1\|\boldsymbol{\beta}\|^2$ into two parts and collecting terms for ρ and ρ^2, we obtain

$$\dot{\hat{W}} \le \rho^2\left(-2M_3 + \alpha M_1\mathbf{1}^\top\boldsymbol{\beta} - \frac{k_1}{2}\|\boldsymbol{\beta}\|^2\right)$$
$$+\rho\left(-2M_3 + \alpha M_1\mathbf{1}^\top\boldsymbol{\beta} - \rho\frac{k_1}{2}\|\boldsymbol{\beta}\|^2\right)$$
$$-\frac{k_1^2 k_2}{2}\|\omega-\boldsymbol{\omega}\|^2.$$

Consider now the expression

$$M_3 - \alpha M_1\mathbf{1}^\top\boldsymbol{\beta} + \frac{k_1\varrho}{2}\|\boldsymbol{\beta}\|^2 = \begin{bmatrix}\mathbf{1}^\top & \boldsymbol{\beta}^\top\end{bmatrix}\begin{bmatrix}\frac{M_3}{n}I & -\alpha\frac{M_1}{2}I \\ -\alpha\frac{M_1}{2}I & \frac{k_1\varrho}{2}I\end{bmatrix}\begin{bmatrix}\mathbf{1} \\ \boldsymbol{\beta}\end{bmatrix}.$$

If $k_1 > 2n(\alpha M_1/2)^2/(\varrho M_3)$, the above quadratic form is positive definite, implying that

$$M_3 - \alpha M_1\mathbf{1}^\top\boldsymbol{\beta} + \frac{k_1\varrho}{2}\|\boldsymbol{\beta}\|^2 \ge 0. \qquad (18)$$

Since $\varrho < 1$, we also have $M_3 - \alpha M_1\mathbf{1}^\top\boldsymbol{\beta} + (k_1/2)\|\boldsymbol{\beta}\|^2 \ge 0$. Using the latter inequality, we get a further upper bound for $\dot{\hat{W}}$,

$$\dot{\hat{W}} \le -\rho^2 M_3 + \rho\left(-2M_3 + \alpha M_1\mathbf{1}^\top\boldsymbol{\beta} - \rho\frac{k_1}{2}\|\boldsymbol{\beta}\|^2\right)$$
$$-\frac{k_1^2 k_2}{2}\|\omega-\boldsymbol{\omega}\|^2. \qquad (19)$$

Using (19), we now prove that outside Λ_ϱ, $\dot{\hat{W}} < 0$. In other words, when either $\rho \ge \varrho$ or $\|\omega-\boldsymbol{\omega}\|^2 \ge \varrho$, $\dot{\hat{W}} < 0$.

Remark 2: If the derivative $\dot{\hat{W}}$ were negative definite, then the rendezvous manifold $\Gamma^\star$ would be globally asymptotically stable. However, this is not guaranteed in (19). The reason is as follows. Suppose ρ is very small and $\|\omega-\boldsymbol{\omega}\| = 0$. Then all terms multiplied by ρ^2 become negligible and what remains in (19) is, $\dot{\hat{W}} \le \rho\left(-2M_3 + \alpha M_1\mathbf{1}^\top\boldsymbol{\beta}\right)$. As we have no control over the value of the constants M_1 and M_3 in the equation above, $\dot{\hat{W}}$ can be greater than zero if the second term dominates the first.

TABLE II
SIMULATION INITIAL CONDITIONS

Vehicle i	$x_i(0)$ (m)	$v_i(0)$ (m/s)	$R_i(0)$
1	$(0, -10, 10)$	$(0, 0, 0)$	side 1
2	$(0, 10, 10)$	$(0, 0, 0)$	side 2
3	$(0, 0, 0)$	$(0, 0, 0)$	down
4	$(-10, 0, -10)$	$(0, 0, 0)$	up
5	$(10, 0, -10)$	$(0, 0, 0)$	up

TABLE III
CONTROL EFFORT

	Figure 6	Figure 7
$\max_i \sup_t |u_i(t)|$ (N)	20.4	17.21
$\max_i \sup_t \|\tau_i(t)\|$ (N·m)	15.27	16.47
$\max_i \mathrm{rms}(|u_i(t)|)$ (N)	1.72	4.31
$\max_i \mathrm{rms}(\|\tau_i(t)\|)$ (N·m)	1.43	2.24

Suppose first that $\rho \geq \varrho$. Then from (19) we have

$$\dot{\hat{W}} \leq -\rho^2 M_3 + \rho\left(-2M_3 + \alpha M_1 \mathbf{1}^\top \boldsymbol{\beta} - \frac{k_1 \varrho}{2}\|\boldsymbol{\beta}\|^2\right) - \frac{k_1^2 k_2}{2}\|\omega - \boldsymbol{\omega}\|^2.$$

By inequality (18) we conclude that

$$\dot{\hat{W}} \leq -\rho^2 M_3 - \rho M_3 - \frac{k_1^2 k_2}{2}\|\omega - \boldsymbol{\omega}\|^2 < 0.$$

Next, suppose that $\|\omega - \boldsymbol{\omega}\|^2 \geq \varrho$. Then from (19),

$$\dot{\hat{W}} \leq -\rho^2 M_3 + \rho\alpha M_1 \mathbf{1}^\top \boldsymbol{\beta} - \frac{k_1^2 k_2}{2}\varrho$$
$$\leq -\rho^2 M_3 + \rho\alpha M_1 M_5 - \frac{k_1^2 k_2}{2}\varrho,$$

where $M_5 := \max_{(\theta,R)\in S_1\times\mathsf{R}}\{\mathbf{1}^\top\boldsymbol{\beta}(\theta,R)\}$. The maximum exists because $\boldsymbol{\beta}$ is continuous and $S_1 \times \mathsf{R}$ is a compact set. If $k_2 > (\alpha M_1 M_5/k_1)^2/\varrho$ then $\dot{\hat{W}} < 0$.

We have therefore proved that, if $\alpha > \max\{\alpha^\star, 3M_3/M_2\}$, $k_1 > \max\{1, 2n(\alpha M_1/2)^2/(\varrho M_3)\}$, and $k_2 > \max\{nM_4/M_3, (\alpha M_1 M_5/k_1)^2/\varrho\}$, then $\hat{W} > \delta$ implies that $\dot{\hat{W}} < 0$. Therefore, for any initial condition, the solution of (10)-(11) with feedback (6) is bounded and the set $\hat{W}_\delta$ is globally asymptotically stable. $\square$

VII. SIMULATION RESULTS

We consider a group of five robots with the sensor digraph in Figure 5. The robot masses and inertia matrices are: $m_1 = 3$ Kg, $m_2 = 3$ Kg, $m_3 = 3.4$ Kg, $m_4 = 3.2$ Kg, $m_5 = 3.2$ Kg and $J_1 := \mathrm{diag}(0.13, 0.13, 0.04)$ Kg·m^2, as in [32], $J_2 = J_1$, $J_3 = 1.4J_1$, $J_4 = 1.2J_1$, $J_5 = 1.2J_1$. We use the double-integrator consensus controller of Ren and Atkins [2], $\mathbf{f}_i(y_i) = \sum_{j=1}^n a_{ij}(x_{ij} + \gamma v_{ij})$ where $a_{ij} \geq 0$, $\gamma > 0$. It is shown in [2] that for sufficiently large γ the above controller does indeed achieve consensus. We pick $a_{ij} = 0.3$ for all $j \in \mathcal{N}_i$ and $\gamma = 30$. The control gains k_1 and k_2 in (6) are chosen to be $k_1 = 2$ and $k_2 = 0.45$. The initial conditions of the robots are shown in Table II. The initial attitudes $R_i(0)$
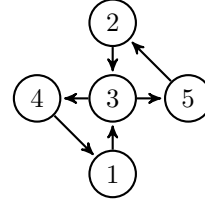

Fig. 5. Sensor digraph used in the simulation results.

of the robots are: up(right), side(ways) 1, side(ways) 2 and (upside)down respectively given by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Figure 6 shows the simulation without the presence of disturbances while Figure 7 shows the simulation when disturbances are present. The disturbances are: an additive random noise with maximum magnitude of 0.25 N on the applied force; an additive random noise with maximum magnitude of 0.25 N·m on the applied torque; an additive measurement error for the angular velocity, with maximum magnitude of 0.25 rad/s; an additive random noise on the quantity $\mathbf{f}_i(y_i^i)$ accounting for errors in measurements of relative displacements and velocities of the vehicles. The direction of this vector has been rotated within 0.25 rad and the magnitude is scaled between 0.75 to 1.25 times the actual magnitude. The disturbances are updated 10 times per second. In both cases of Figure 6 and Figure 7, the vehicles' positions and velocities converge to a neighborhood of one another.

In Figure 6 the vehicles remain within 0.25m of one another while in Figure 7 the vehicles remain within 1m of one another at steady state. These neighborhoods can be made even smaller by further increasing the control gains k_1 and k_2. However, this would result in having higher control inputs. Metrics related to the thrust and torque inputs are presented in Table III. The first two rows show peak control norms and the last two show the root mean square (rms) of the control norms. In these simulations we considered zero gravity, i.e., $g = 0$. This was done to improve visibility of the simulation results. In the presence of gravity, the vehicles would still converge to the same neighborhood of one another, however at steady state they would accelerate in the direction of gravity since gravity is not compensated through the control inputs in (6).

VIII. CONCLUSIONS

We have presented the first local and distributed feedback solving the rendezvous control problem for a class of underactuated robots modelling vertical take-off and landing (VTOL) vehicles such as quadrotor helicopters. The main result, Theorem 1, relies on the assumption that the sensor digraph is constant. As we have discussed in the paper, this assumption is questionable in practice, but a stability analysis in the presence of a state-dependent sensor digraph is beyond the scope of this paper. We believe that solutions in the literature for consensus of double-integrators with time-dependent sensor digraphs could be extended to rigid bodies
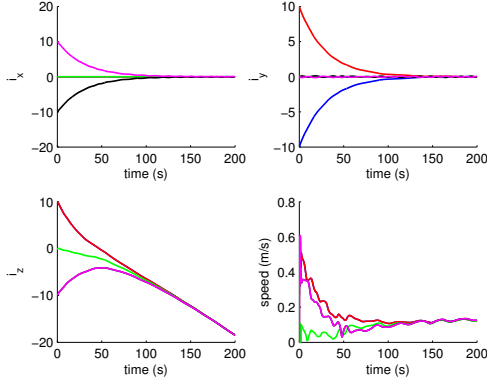
Fig. 6. Rendezvous control simulation without the presence of disturbances. At the top-left, top-right and bottom-left: positions of the five robots expressed in the inertial frame $\mathcal{I}$. At the bottom-right: linear speeds $\|v_i\|$, $i = 1, \ldots, 5$.
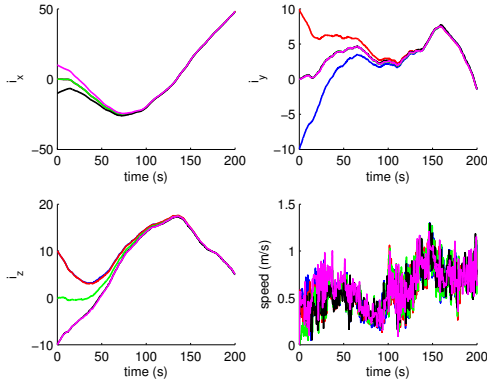

Fig. 7. Rendezvous control simulation with the presence of disturbances. At the top-left, top-right and bottom-left: positions of the five robots expressed in the inertial frame $\mathcal{I}$. At the bottom-right: linear speeds $\|v_i\|$, $i = 1, \ldots, 5$.

using the framework in this paper. However the Lyapunov function used in the analysis would need to be modified extensively. Since this makes the problem even more difficult than it already is, we leave it as a possible future research direction. In this paper we limited ourselves to the control specification of rendezvous. The proposed control law, in particular, does not guarantee hovering of the vehicles. While the robots converge to each other, nothing can be said about the motion of the ensemble. This cannot be otherwise, for it would be impossible to solve the rendezvous problem with hovering without additional sensors. One would need some measurement of the gravity vector, for example provided by a three-axis accelerometer. The point of view of these authors is that the proposed solution of the rendezvous problem will serve as a layer in a hierarchy of higher-level control specifications such as hovering, formation stabilization, and path following.

APPENDIX

A. Proof of Lemma 1

Recall the definition of $W(X, R, \omega)$, and assume that $X \neq 0$,

$$
\begin{aligned}
W =& \alpha \left(\sqrt{V(X)} + \frac{1}{2} V(X) \right) + \sum_{i=1}^{n} \mathbf{g}_i^i(X, R) \cdot e_3 \\
& + \frac{1}{2} (\omega - \boldsymbol{\omega}(X, R))^\top \mathbb{J} (\omega - \boldsymbol{\omega}(X, R)) \\
=& \sqrt{V(X)} \left(\alpha + \frac{\sum_{i=1}^{n} \mathbf{g}_i^i(X, R) \cdot e_3}{\sqrt{V(X)}} \right) + \frac{\alpha}{2} V(X) \\
& + \frac{1}{2} (\omega - \boldsymbol{\omega}(X, R))^\top \mathbb{J} (\omega - \boldsymbol{\omega}(X, R)).
\end{aligned}
$$

Since $\mathbf{g}_i^i(X, R)$ is linear with respect to X, we have

$$
\begin{aligned}
W =& \sqrt{V(X)} \left(\alpha + \sum_{i=1}^{n} \mathbf{g}_i^i(\mu(X), R) \cdot e_3 \right) + \frac{\alpha}{2} V(X) \\
& + \frac{1}{2} (\omega - \boldsymbol{\omega}(X, R))^\top \mathbb{J} (\omega - \boldsymbol{\omega}(X, R)),
\end{aligned}
$$

where $\mu(X) := X / \sqrt{V(X)}$ is continuous on $\mathsf{X} \backslash \{0\}$ and bounded as follows

$$
\|\mu(X)\| = \frac{\|X\|}{\sqrt{V(X)}} = \frac{\|X\|}{\sqrt{X^\top P X}} \leq \frac{1}{\sqrt{\lambda_{\min}(P)}}.
$$

Since $\mathbf{g}_i^i$ is continuous, $\mu(X)$ is bounded, and $R \in \mathsf{R}$, a compact set, it follows that the function $\sum_{i=1}^{n} |\mathbf{g}_i^i(\mu(X), R) \cdot e_3|$ has a bounded supremum. Accordingly, let

$$
\alpha^\star = \sup_{(X,R) \in (\mathsf{X} \backslash \{0\}) \times \mathsf{R}} \sum_{i=1}^{n} \left| \mathbf{g}_i^i(\mu(X), R) \cdot e_3 \right|.
$$

For all $\alpha > \alpha^\star$, we have $W(X, R, \omega) > \underline{W}(X, R, \omega)$,

$$
\begin{aligned}
\underline{W}(X, R, \omega) :=& \frac{\alpha}{2} V(X) \\
& + \frac{1}{2} (\omega - \boldsymbol{\omega}(X, R))^\top \mathbb{J} (\omega - \boldsymbol{\omega}(X, R)) \geq 0.
\end{aligned}
$$

We derived the bound above for $X \neq 0$, but since $\mathbf{g}_i^i(0, R) = 0$ (by linearity of $\mathbf{g}_i^i$ with respect to X), the bound also holds for $X = 0$. The above inequality implies that $W \geq 0$ and $W^{-1}(0) \subset \underline{W}^{-1}(0)$. But $\underline{W} = 0$ if and only if $V(X) = 0$ (i.e., $X = 0$) and $\omega = \boldsymbol{\omega}$. Thus $W^{-1}(0) \subset \Gamma^\star$, proving part (i) of the lemma.

For part (ii), note that for all $c > 0$, $W_c \subset \{\underline{W} \leq c\}$. Since $\underline{W}$ is a positive definite quadratic form in the variables $(X, \omega - \boldsymbol{\omega})$, its sublevel sets are compact in $(X, \omega - \boldsymbol{\omega})$ coordinates. Thus if $(X, R, \omega) \in W_c$, X and $\omega - \boldsymbol{\omega}(X, R)$ are bounded. Since $\boldsymbol{\omega}$ is continuous and $R \in \mathsf{R}$, a compact set, $\boldsymbol{\omega}$ is bounded, implying that ω is also bounded. Therefore the set W_c is bounded. Continuity of W implies that W_c is compact. This concludes the proof of part (ii) of the lemma.

For part (iii), let $\varepsilon > 0$ be arbitrary. Since $\underline{W}$ is a positive definite quadratic form in the variables $(X, \omega - \boldsymbol{\omega})$, there exists $\delta > 0$ such that $\underline{W}(X, R, \omega) \leq \delta$ implies $\|(X, \omega - \boldsymbol{\omega}(X, R))\| \leq \varepsilon$, which implies $\|X\| \leq \varepsilon$. Now consider any point $(X, R, \omega) \in \{\underline{W} \leq \delta\}$. Note that $(X, R, \omega) \in \mathsf{X} \times \mathsf{R} \times \Omega$ lies on the product of metric spaces X, R and Ω. Respectively, the metrics are d_X, d_R and d_Ω (d_X

and d_Ω are Euclidean metrics). As such, choosing to use the 2-product metric,

$$\|(X, R, \omega)\|_{\Gamma^\star}$$
$$= \inf_{(X_0, R_0, \omega_0) \in \Gamma^\star} \left(d_{\mathsf{X}}(X, X_0)^2 + d_{\mathsf{R}}(R, R_0)^2 + d_\Omega(\omega, \omega_0)^2 \right)^{\frac{1}{2}}.$$

The point $(0, R, \omega)$ is contained in the set $\Gamma^\star$ and therefore,

$$\|(X, R, \omega)\|_{\Gamma^\star} \leq d_{\mathsf{X}}(X, 0) \leq \|X\| \leq \varepsilon.$$

This implies that $(X, R, \omega) \in B_\varepsilon(\Gamma^\star)$. Thus, $W_\delta \subset \{\underline{W} \leq \delta\} \subset B_\varepsilon(\Gamma^\star)$, as required. This concludes the proof of Lemma 1. $\qquad\square$

B. Proof of Lemma 2

We will use a standard result from differential geometry relating the Lie derivatives of smooth functions along F-related vector fields [40, Proposition 8.16]. In our context, recalling that $\rho = \sqrt{V}|_{X=\rho\theta}$ and $\hat{W} = W|_{X=\rho\theta}$, the result has the following implication:

$$\dot{\rho} = \frac{d}{dt}\sqrt{V}\Big|_{X=\rho\theta} \quad \text{and} \quad \dot{\hat{W}}_{\mathsf{rot}} = \frac{d}{dt}W_{\mathsf{rot}}\Big|_{X=\rho\theta}. \qquad (20)$$

Rewrite the dynamics of X in (10) as

$$\dot{x}_{1j} = v_{1j}$$
$$\dot{v}_{1j} = [\mathbf{g}_j(X) - \mathbf{g}_1(X)]$$
$$\quad + R_j \left[(\mathbf{g}_j^j(X, R) \cdot e_3)e_3 - \mathbf{g}_j^j(X, R) \right]$$
$$\quad + R_1 \left[(\mathbf{g}_1^1(X, R) \cdot e_3)e_3 - \mathbf{g}_1^1(X, R) \right].$$

To get the identities above, we added and subtracted in (10) the ideal force feedbacks $\mathbf{f}_j(y_j) = \mathbf{g}_j(X)$ and $\mathbf{f}_1(y_1) = \mathbf{g}_1(X)$, and we replaced u_j and u_1 in (10) by the assigned feedbacks in (6). Finally, we used the identity $R_i\mathbf{g}_i^i = \mathbf{g}_i$.

Taking the time derivative of $\sqrt{V(X)}$ along the above vector field we get

$$\frac{d}{dt}\sqrt{V(X)} = \frac{1}{2\sqrt{V(X)}}\Bigg[-X^\top Q X$$
$$\quad + \sum_{j=2}^n \frac{\partial V}{\partial v_{1j}} R_j \left((\mathbf{g}_j^j(X, R) \cdot e_3)e_3 - \mathbf{g}_j^j(X, R) \right)$$
$$\quad - \sum_{j=2}^n \frac{\partial V}{\partial v_{1j}} R_1 \left((\mathbf{g}_1^1(X, R) \cdot e_3)e_3 - \mathbf{g}_1^1(X, R) \right) \Bigg].$$

The first term in the bracket is the derivative of $V(X)$ along the nominal vector field (14), and $Q = Q^\top$ is a positive definite matrix. Letting $M_2 = \lambda_{\min}(Q)/(2\lambda_{\max}(P))$ and using the fact that the Euclidean norm is invariant under rotations, we have

$$\frac{d}{dt}\sqrt{V(X)} \leq -M_2\sqrt{V(X)} + \frac{1}{2\sqrt{V(X)}}\cdot$$
$$\left[\sum_{j=2}^n \left\| \frac{\partial V}{\partial v_{1j}} \right\| \left(\|(\mathbf{g}_j^j(X, R) \cdot e_3)e_3 - \mathbf{g}_j^j(X, R)\| \right.\right.$$
$$\left.\left. + \|(\mathbf{g}_1^1(X, R) \cdot e_3)e_3 - \mathbf{g}_1^1(X, R)\| \right) \right].$$

We claim that $\|(\mathbf{g}_i^i(X, R) \cdot e_3)e_3 - \mathbf{g}_i^i(X, R)\| = \|\mathbf{g}_i^i(X, R) \times e_3\|$. Indeed, writing $\mathbf{g}_i^i = (\mathbf{g}_i^i \cdot e_3)e_3 + \mathbf{g}_i^i - (\mathbf{g}_i^i \cdot e_3)e_3$, we have $\mathbf{g}_i^i \times e_3 = (\mathbf{g}_i^i - (\mathbf{g}_i^i \cdot e_3)e_3) \times e_3$. Since the vector $\mathbf{g}_i^i - (\mathbf{g}_i^i \cdot e_3)e_3$ is perpendicular to e_3, $\|(\mathbf{g}_i^i - (\mathbf{g}_i^i \cdot e_3)e_3) \times e_3\| = \|\mathbf{g}_i^i - (\mathbf{g}_i^i \cdot e_3)e_3\|$, so that $\|\mathbf{g}_i^i \times e_3\| = \|\mathbf{g}_i^i - (\mathbf{g}_i^i \cdot e_3)e_3\|$. This proves the claim. Using the identity just derived, we get

$$\frac{d}{dt}\sqrt{V(X)} \leq -M_2\sqrt{V(X)}$$
$$\quad + \frac{1}{2\sqrt{V(X)}}\Bigg[\sum_{j=2}^n \left\| \frac{\partial V}{\partial v_{1j}} \right\| \left(\|\mathbf{g}_j^j(X, R) \times e_3\| \right.$$
$$\left. + \|\mathbf{g}_1^1(X, R) \times e_3\| \right) \Bigg].$$

Using (20), we get

$$\dot{\rho} \leq -M_2\rho + \frac{1}{2\rho}\Bigg[\sum_{j=2}^n \left\| \frac{\partial V}{\partial v_{1j}}(\rho\theta) \right\| \left(\|\mathbf{g}_j^j(\rho\theta, R) \times e_3\| \right.$$
$$\left. + \|\mathbf{g}_1^1(\rho\theta, R) \times e_3\| \right) \Bigg].$$

Since the functions $\mathbf{g}_i^i$ are linear with respect to their first argument, and the partial derivatives of the quadratic form V are linear functions, by the homogeneity of the norm we have

$$\dot{\rho} \leq -M_2\rho + \frac{\rho}{2}\Bigg[\sum_{j=2}^n \left\| \frac{\partial V}{\partial v_{1j}}(\theta) \right\| \left(\|\mathbf{g}_j^j(\theta, R) \times e_3\| \right.$$
$$\left. + \|\mathbf{g}_1^1(\theta, R) \times e_3\| \right) \Bigg].$$

The functions $\|\partial V/\partial v_{1j}\|$ are continuous. The variable θ belongs to S_1, a compact set. Therefore $\|\partial V/\partial v_{1j}\|$ has a maximum,

$$\dot{\rho} \leq -M_2\rho + \max_{\substack{\theta \in S_1 \\ j \in \{2,\ldots,n\}}} \left\| \frac{\partial V}{\partial v_{1j}}(\theta) \right\| \frac{\rho}{2}\Bigg[\sum_{j=2}^n \left(\|\mathbf{g}_j^j(\theta, R) \times e_3\| \right)$$
$$\quad + (n-1)\|\mathbf{g}_1^1(\theta, R) \times e_3\| \Bigg]$$
$$\leq -M_2\rho + \max_{\substack{\theta \in S_1 \\ j \in \{2,\ldots,n\}}} \left\| \frac{\partial V}{\partial v_{1j}}(\theta) \right\| \frac{\rho}{2}(n-1)\sum_{j=1}^n \|\mathbf{g}_j^j(\theta, R) \times e_3\|.$$

Letting $M_1 := \max_{\substack{\theta \in S_1 \\ j \in \{2,\ldots,n\}}} \|\partial V/\partial v_{1j}\|(n-1)/2$, we get the first inequality in (17).

We now turn to the second inequality in (17). Recall the definition of W_{rot},

$$W_{\mathsf{rot}}(X, R, \omega) = \sum_{i=1}^n \mathbf{g}_i^i(X, R) \cdot e_3$$
$$\quad + \frac{1}{2}(\omega - \boldsymbol{\omega}(X, R))^\top \mathbb{J}(\omega - \boldsymbol{\omega}(X, R)).$$

The derivative of W_{rot} along the vector field in (10)-(11) is

$$\dot{W}_{\mathsf{rot}} = \sum_{i=1}^n \Bigg[\left(\frac{d}{dt}\mathbf{g}_i^i \right) \cdot e_3 + (\omega_i^i - \boldsymbol{\omega}_i^i(X, R)) \cdot$$
$$\quad \left(\tau_i - \omega_i^i \times J_i\omega_i^i - J_i\left(\frac{d}{dt}\boldsymbol{\omega}_i^i \right) \right) \Bigg].$$

To express $(d/dt)\mathbf{g}_i^i$, recall that $\mathbf{g}_i^i(X,R) = R_i^{-1}\mathbf{f}_i(h_i(X))$. Then,

$$\frac{d}{dt}\mathbf{g}_i^i = \left(\frac{d}{dt}R_i^{-1}\right)\mathbf{f}_i(h_i(X)) + R_i^{-1}\frac{d}{dt}\left(\mathbf{f}_i(h_i(X))\right).$$

The function $\mathbf{f}_i(h_i(X))$ is linear. Its derivative along the vector field (10)-(11) with feedback (6) is given by

$$\frac{d}{dt}\mathbf{f}_i(h_i(X)) = \sum_{j\in\mathcal{N}_i}\left(a_{ij}\frac{d}{dt}(x_{j1}-x_{i1}) + b_{ij}\frac{d}{dt}(v_{j1}-v_{i1})\right)$$
$$= \sum_{j\in\mathcal{N}_i}(a_{ij}(v_{j1}-v_{i1})$$
$$+ b_{ij}\left(-\frac{1}{m_j}R_j e_3 u_j + \frac{1}{m_i}R_i e_3 u_i\right)),$$

which is linear with respect to X because $u_i = -\mathbf{g}_i^i(X,R)\cdot e_3$ is such. We will denote it $\mathbf{h}_i(X,R)$, $\mathbf{h}_i(X,R) := (d/dt)\mathbf{f}_i(h_i(X))$. Consistently with our notational convention in Table I, we will let $\mathbf{h}_i^i(X,R) := R_i^{-1}\mathbf{h}_i(X,R)$. The function $\mathbf{h}_i^i(X,R)$ is linear with respect to X. Returning to the derivative of $\mathbf{g}_i^i$, we have

$$\frac{d}{dt}\mathbf{g}_i^i = -(\omega_i^i)^\times R_i^{-1}\mathbf{f}_i(h_i(X)) + R_i^{-1}\mathbf{h}_i(X,R)$$
$$= -\omega_i^i\times\mathbf{g}_i^i(X,R) + \mathbf{h}_i^i(X,R).$$

Similarly, since $\boldsymbol{\omega}_i^i(X,R) = k_1(\mathbf{g}_i^i(X,R)\times e_3)$, we have $\frac{d}{dt}\boldsymbol{\omega}_i^i = k_1(-\omega_i^i\times\mathbf{g}_i^i + \mathbf{h}_i^i)\times e_3$. Substituting the above identities in the expression for $\dot{W}_{\text{rot}}$ and since $\tau_i = \omega_i^i\times J_i\omega_i^i - k_1 J_i((\omega_i^i\times\mathbf{g}_i^i)\times e_3) - k_1^2 k_2(\omega_i^i-\boldsymbol{\omega}_i^i)$, we get

$$\dot{W}_{\text{rot}} = \sum_{i=1}^n\left[-(\omega_i^i\times\mathbf{g}_i^i)\cdot e_3 + \mathbf{h}_i^i\cdot e_3\right.$$
$$\left.-k_1(\omega_i^i-\boldsymbol{\omega}_i^i)\cdot J_i(\mathbf{h}_i^i\times e_3) - k_1^2 k_2\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2\right].$$

Using the property of the triple product that $(\omega_i^i\times\mathbf{g}_i^i)\cdot e_3 = (\mathbf{g}_i^i\times e_3)\cdot\omega_i^i$, we obtain

$$\dot{W}_{\text{rot}} = \sum_{i=1}^n\left[-(\mathbf{g}_i^i\times e_3)\cdot\omega_i^i + \mathbf{h}_i^i\cdot e_3\right.$$
$$\left.-k_1(\omega_i^i-\boldsymbol{\omega}_i^i)\cdot J_i(\mathbf{h}_i^i\times e_3) - k_1^2 k_2\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2\right].$$

Adding and subtracting the term $(\mathbf{g}_i^i\times e_3)\cdot\boldsymbol{\omega}_i^i$ and collecting the term $\omega_i^i-\boldsymbol{\omega}_i^i$, we have

$$\dot{W}_{\text{rot}} = \sum_{i=1}^n\left[-(\mathbf{g}_i^i\times e_3)\cdot\boldsymbol{\omega}_i^i + \mathbf{h}_i^i\cdot e_3 - ((\mathbf{g}_i^i\times e_3)\right.$$
$$\left.+ k_1 J_i(\mathbf{h}_i^i\times e_3))\cdot(\omega_i^i-\boldsymbol{\omega}_i^i) - k_1^2 k_2\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2\right].$$

Substituting in the first term inside the bracket $\boldsymbol{\omega}_i^i = -k_1(\mathbf{g}_i^i\times e_3)$, taking norms, and using the fact that $k_1\geq 1$, we arrive at the inequality

$$\dot{W}_{\text{rot}} \leq \sum_{i=1}^n\left[-k_1\|\mathbf{g}_i^i\times e_3\|^2 + \|\mathbf{h}_i^i\cdot e_3\|\right.$$
$$\left.+ k_1\mathbf{k}_i(X,R)\|\omega_i^i-\boldsymbol{\omega}_i^i\| - k_1^2 k_2\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2\right],$$

where $\mathbf{k}_i(X,R) := \|\mathbf{g}_i^i(X,R)\times e_3\| + \|J_i(\mathbf{h}_i^i(X,R)\times e_3)\|$. Note that $\mathbf{k}_i(X,R)$ is homogeneous with respect to X because

$\mathbf{g}_i^i$ and $\mathbf{h}_i$ are linear with respect to X and the norm is a homogeneous function.

Splitting the term $-k_1^2 k_2\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2$ into two parts and noticing that the function $k_1\mathbf{k}_i(X,R)\|\omega_i^i-\boldsymbol{\omega}_i^i\| - (k_1^2 k_2/2)\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2$ is quadratic in the variable $\|\omega_i^i-\boldsymbol{\omega}_i^i\|$ with maximum $\mathbf{k}_i^2(X,R)/(2k_2)$, we get

$$\dot{W}_{\text{rot}} \leq \sum_{i=1}^n\left[-k_1\|\mathbf{g}_i^i\times e_3\|^2 + \|\mathbf{h}_i^i\cdot e_3\| - \frac{k_1^2 k_2}{2}\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2\right.$$
$$\left.+ \frac{\mathbf{k}_i^2(X,R)}{2k_2}\right].$$

Now using (20) we get

$$\dot{\hat{W}}_{\text{rot}} \leq \sum_{i=1}^n\left[-k_1\|\mathbf{g}_i^i(\rho\theta,R)\times e_3\|^2 + \|\mathbf{h}_i^i(\rho\theta,R)\cdot e_3\|\right.$$
$$\left.- \frac{k_1^2 k_2}{2}\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2 + \frac{\mathbf{k}_i^2(\rho\theta,R)}{2k_2}\right].$$

Using the homogeneity with respect to X of $\|\mathbf{g}_i^i\times e_3\|$, $\|\mathbf{h}_i^i\cdot e_3\|$, and $\mathbf{k}_i$, we get

$$\dot{\hat{W}}_{\text{rot}} \leq \sum_{i=1}^n\left[-k_1\rho^2\|\mathbf{g}_i^i(\theta,R)\times e_3\|^2 + \rho\|\mathbf{h}_i^i(\theta,R)\cdot e_3\|\right.$$
$$\left.- \frac{k_1^2 k_2}{2}\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2 + \rho^2\frac{\mathbf{k}_i^2(\theta,R)}{2k_2}\right].$$

Since $\|\mathbf{h}_i^i(\theta,R)\cdot e_3\|$ and $\mathbf{k}_i^2(\theta,R)$ are continuous functions over the compact set $S_1\times\mathsf{R}$, they each have a maximum. Letting $M_3 = n\cdot\max_{(\theta,R)\in S_1\times\mathsf{R},\ i\in\{1,\dots,n\}}(\|\mathbf{h}_i^i(\theta,R)\cdot e_3\|)$, $M_4 = \max_{(\theta,R)\in S_1\times\mathsf{R},\ i\in\{1,\dots,n\}}\left(\frac{\mathbf{k}_i^2(\theta,R)}{2}\right)$, we conclude that

$$\dot{\hat{W}}_{\text{rot}} \leq \rho^2\sum_{i=1}^n\left[-k_1\|\mathbf{g}_i^i(\theta,R)\times e_3\|^2 + \frac{M_4}{k_2}\right] + \rho M_3$$
$$- \frac{k_1^2 k_2}{2}\sum_{i=1}^n\|\omega_i^i-\boldsymbol{\omega}_i^i\|^2,$$

as required. This concludes the proof of Lemma 2. $\qquad\square$

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